Distributed Nonuniform Coverage with Limited Scalar Measurements

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***Abstract*— In this paper we demonstrate a distributed coverage control method for a network of mobile agents moving in one dimension along a scalar information density field. The method requires each agent to take a finite number of measurements of the density field in the interval between its two neighbors and calculate its next position in order to drive the network nearer to the configuration for optimal coverage. We derive several results relating to the equilibrium properties of the sensor network and the convergence properties near fixed points. We illustrate with simulations of the algorithm.**

I. INTRODUCTION

As technological advances have improved the capabilities, reliability and cost of robotic sensing platforms, their potential for deployment in autonomous, cooperative networks has gained significant attention. The emergent capabilities of mobile sensor networks promise to revolutionize complex tasks such as surveillance, exploration and environmental monitoring. However, development of high-level capabilities requires solutions to lower-level problems such as formation control and coverage control, and these solutions should be distributed, adaptive to changing environments and robust to uncertainty and changes in network topology.

The present work focuses on coverage control, where the goal is to optimally locate the nodes, or agents, of the network to maximize the amount of information extracted from the environment or the likelihood of detection of an event of interest. More formally, for a domain with some measurable information density field, which could be a data stream from the environment or a measure of the probability of event occurence, the agents in the network seek to locate themselves to minimize a global coverage metric, which is determined by sensing capabilities and the goals of the network designer. If the information density field is uniform across the domain, optimal coverage is realized by a uniform spacing of agents, whereas for a nonuniform field, agents should be closer together in regions of high information density and spread out in regions of low information density.

In [1], [2] a distributed uniform coverage control law is developed which makes use of Voronoi partitions and gradient descent laws. The nonuniform case is treated in part by using density-dependent gradient descent laws with the Voronoi partitions computed for the uniform case. Coverage with communication constraints is treated in [3], [4]. The nonuniform coverage control problem is addressed in [5]; a density-dependent distance metric is defined that stretches and shrinks subregions of high and low density, respectively, and this determines Voronoi partitions of the nonuniform density field. A cartogram transformation is used to compute these partitions, and optimal nonuniform coverage is proved in the case of a static or slowly time-varying density field. When the density field is defined by mapping uncertainty, it changes with every measurement taken; nonuniform coverage for this reactively changing field is treated in the case of symmetric domains in [6].

To compute Voronoi partitions for a nonuniform density field in [5], agents use some global knowledge of the cartogram transformation. This is relaxed in the case of nonuniform coverage control in one dimension in [7] where two distributed nonuniform coverage algorithms are derived. In the first algorithm, each agent measures only the distance to its neighbors and the local density field; the control law drives the agents to their optimal coverage positions in a number of time steps that is quadratic in the number of agents N. The second algorithm requires greater agent capability but yields a convergence rate that is linear in N.

We derive a distributed, nonuniform coverage control law in one dimension that is a modification of the first algorithm of [7] with relaxed computational and sensing requirements: the agents do not measure local density but rather they take only a few local, scalar measurements of the density field. By recasting the problem as a nonlinear fixed point problem we make use of several results from numerical analysis that give insight into conditions under which a solution is unique, the rate at which the agents converge to their equilibrium positions and how far these positions are from optimal.

In Section II we summarize the first algorithm of [7]. We derive our modified distributed nonuniform coverage algorithm in Section III. In Section IV we derive several analytic results on existence and uniqueness of solutions, bounds on the transient dynamics, and equilibrium characteristics of the network. We illustrate with simulations in Section V.

II. BACKGROUND

We define our sensor network to consist of N mobile agents located on the interval $[-1, 1]$ of the real line. At discrete time step t the agents have positions $x_1^{(t)}, x_2^{(t)}, \ldots, x_N^{(t)}$. We assume that the labeling of agents from 1 to N matches their order along the line from left to right, with an agent's neighbors being the agents immediately to its left and right. By convention $x_0^{(t)}$ and $x_{N+1}^{(t)}$ are the stationary boundaries -1 and 1 respectively. An agent's sensing region is defined to be the interval between its two neighbors (or its neighbor and neighboring boundary for agents 1 and N).

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The information density field, $\rho : [-1,1] \to (0,\infty)$, is a piecewise continuous mapping of the domain over which the agents move to finite, strictly positive scalar values. The distance between two points in the domain should be a function of ρ such that regions of high ρ are stretched relative to regions of lower ρ. Following [5], [7] we define the distance between $a, b \in [-1,1]$ as the non-Euclidean function

$$d_\rho(a,b) = \int_{\min(a,b)}^{\max(a,b)} \rho(z)dz.$$

Using this definition of distance, the coverage metric Φ is the largest distance from any point in the domain to the agent that is nearest to it:

$$\Phi(x_1,\dots,x_N,\rho) = \max_{y\in[-1,1]} \left[\min_{i=1,\dots,N} d_\rho(y,x_i) \right].$$

The optimum coverage Φ^* is the infimum of Φ over all possible agent configurations.

In [7] a distributed control law is developed that drives the coverage to Φ^* in time that is quadratic in N. In this framework, which is the starting point of the current approach, each agent must have knowledge of ρ over its entire sensing region in order to calculate the α-median of the region. For a sensing region given by the interval (a,b), the α-median, $m_\rho^\alpha(a,b)$, is the point $c \in (a,b)$ such that

$$\int_a^c \rho(z)dz = \alpha \int_c^b \rho(z)dz.$$

The agents update their positions using the control law

$$\begin{aligned} x_1^{(t+1)} &= m_\rho^{1/2}(-1, x_2^{(t)}) \\ x_k^{(t+1)} &= m_\rho^{1}(x_{k-1}^{(t)}, x_{k+1}^{(t)}) \\ x_N^{(t+1)} &= m_\rho^{2}(x_{N-1}^{(t)}, 1) \end{aligned}$$

for $k = 2,\dots,N-1$, where t is the current time step. The only additional assumptions are that each agent can determine whether it is on the boundary of the network or in the interior and can measure the distance to its neighbors.

The advantages of this algorithm are that it addresses nonuniformity in ρ without any need for global knowledge of the density field, and it is completely distributed in the sense that no information is passed from one agent to another. In addition, convergence is guaranteed for any starting configuration of the agents and performance (in terms of the number of time steps needed to reach a certain distance from optimal coverage) is bounded. There are two major drawbacks to this approach: i) at each time step each agent must have full knowledge of the information density in its sensing interval, and ii) the computational cost of calculating the α-median are high since there is no general closed form solution. The objective of the present work is to address these two drawbacks by introducing a modified coverage control algorithm for the same problem formulation.

III. CONTROL LAW WITH FINITE MEASUREMENTS

The primary modifications made in the proposed algorithm are i) each agent makes a finite number of measurements of ρ in its sensing region and ii) each agent approximates the location of the α-median using a simple algorithm. At the beginning of time step t, agent k measures ρ at its current location and at $M \geq 1$ locations on both its left and its right so that measurements are spaced evenly in each half of its sensing region. M is referred to as the measurement number.

Agent k then calculates the distance along the line it must travel, $\Delta x_k^{(t)}$, in order to approximately reach the appropriate α-median of its sensing region. The position update equations then become

$$x_k^{(t+1)} = x_k^{(t)} + \Delta x_k^{(t)} \tag{1}$$

for $k = 1,\dots,N$. To calculate this change in position, the agent first uses the measurements made of ρ to approximate the weighted distance to its neighbors (or neighboring boundary) using the trapezoid rule. Note that other numerical integration techniques could be used if desired. Since these weighted distances are not exact, we denote them with a bar. Using this method and the simplified notation $\rho(x_k^{(t)}) = \rho_k^{(t)}$ the approximate weighted distances from agent k to its left and right neighbors respectively are

$$\begin{aligned} \bar{d}_\rho(x_{k-1}^{(t)}, x_k^{(t)}) &= \frac{h_L}{2}\left(\rho_{k-1}^{(t)} + \rho_k^{(t)} + 2\sum_{j=1}^{M-1} \rho(x_{k-1}^{(t)} + jh_L) \right) \\ \bar{d}_\rho(x_k^{(t)}, x_{k+1}^{(t)}) &= \frac{h_R}{2}\left(\rho_k^{(t)} + \rho_{k+1}^{(t)} + 2\sum_{j=1}^{M-1} \rho(x_{k-1}^{(t)} + jh_R) \right) \end{aligned} \tag{2}$$

where h_L and h_R are the spacings on the line between measurements on the left and right defined by

$$h_L = \frac{x_k^{(t)} - x_{k-1}^{(t)}}{M}, \quad h_R = \frac{x_{k+1}^{(t)} - x_k^{(t)}}{M}.$$

Next the agent finds the weighted distance that it must travel to the right, Δd_ρ, such that

$$\bar{d}_\rho(x_{k-1}^{(t)}, x_k^{(t)}) + \Delta d_\rho = \alpha(\bar{d}_\rho(x_k^{(t)}, x_{k+1}^{(t)}) - \Delta d_\rho). \tag{3}$$

In order to translate this approximate change in weighted distance to a change in position along the real line, the agent interpolates ρ to order zero using the measurement available at its current position. With this approximation we have $\Delta d_\rho \approx \rho(x_k^{(t)})\Delta x_k^{(t)}$, which we can then substitute into Eq (3) and solve to get

$$\Delta x_k^{(t)} = \frac{\alpha \bar{d}_\rho(x_k^{(t)}, x_{k+1}^{(t)}) - \bar{d}_\rho(x_{k-1}^{(t)}, x_k^{(t)})}{(1+\alpha)\rho(x_k^{(t)})}. \tag{4}$$

This gives a relatively simple expression for the position update equation (1); the value of α depends on whether or not agent k is a boundary agent, $x_0^{(t)} = -1$ and $x_{N+1}^{(t)} = 1$.

If after several time steps, $\Delta x_k^{(t)} = 0$, agent k will have stopped moving and is said to be in equilibrium. If $\Delta x_k^{(t)} = 0$ $\forall k = 1, \ldots, N$ then the network is said to be in equilibrium.

It is important to note that no information is carried from one time step to the next and that the algorithm remains distributed since the actions of each agent depend only on the relative positions of its neighbors and not on information passed from one agent to another. The designer can choose M and a deployment that determines the agents' initial positions, $x_k^{(0)}$. There are a number of other combinations of distance estimation methods and interpolation schemes to find the approximate α-median that could have been chosen; the methods chosen here makes performance analysis tractable as we will show in the next section.

IV. PERFORMANCE ANALYSIS

To make the statement of the problem more compact let $\mathbf{x}^{(t)}$ be the vector of agent positions at time step t:

$$\mathbf{x}^{(t)} = \begin{bmatrix} x_1^{(t)} & x_2^{(t)} & \ldots & x_N^{(t)} \end{bmatrix}^T.$$

Let the update equation for agent k from one time step to the next be represented by the nonlinear map g_k:

$$x_k^{(t+1)} = g_k(\mathbf{x}^{(t)}) = x_k^{(t)} + \Delta x_k^{(t)}.$$

To describe the time update for all agents as a single equation, let $\mathbf{G} : \mathbb{R}^N \to \mathbb{R}^N$ be the vector of g_k's:

$$\mathbf{x}^{(t+1)} = \mathbf{G}(\mathbf{x}^{(t)}) = \begin{bmatrix} g_1(\mathbf{x}^{(t)}) & \ldots & g_N(\mathbf{x}^{(t)}) \end{bmatrix}^T. \quad (5)$$

The coverage problem is a nonlinear fixed point problem: the solution is the agent positions $\mathbf{x}^*$ such that $\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*)$. We call $\mathbf{x}^*$ a fixed point or an equilibrium configuration.

A. Equilibrium Behavior

In [7] equilibrium existence and uniqueness is guaranteed, and the equilibrium is the agent configuration that yields optimal coverage. For the current study of agents making finite measurements, uniqueness of a fixed point is not guaranteed, and in general a fixed point will not be the agent configuration for optimal coverage. Sufficient conditions for the existence of at least one fixed point for a single agent in an interval (a, b) is given in the following theorem.

Theorem 1: If $\alpha > 0$ and $\rho(z)$ is continuous and strictly positive for $z \in [a, b]$, then there exists at least one point $x^* \in (a, b)$ such that $\bar{d}_\rho(a, x^*) = \alpha \bar{d}_\rho(x^*, b)$.

Proof: Define the function $R(x) = \alpha \bar{d}_\rho(x, b) - \bar{d}_\rho(a, x)$ for $x \in [a, b]$. Since $\bar{d}_\rho$ is made up of continuous functions, both terms in $R(x)$ are continuous, which means that $R(x)$ itself is continuous. From the expression for $\bar{d}_\rho$ we know $\bar{d}_\rho(a, b) > 0$ and $\bar{d}_\rho(a, a) = \bar{d}_\rho(b, b) = 0$ so $R(a) > 0$ and $R(b) < 0$. By the Intermediate Value Theorem there must exist at least one $x^* \in (a, b)$ such that $R(x^*) = 0$, which implies that $\bar{d}_\rho(a, x^*) = \alpha \bar{d}_\rho(x^*, b)$. ■

Generalizing this result for a single agent on an arbitrary interval (a, b) to the entire network over the whole domain is trivial, since we need only replace the boundaries a and b with the positions of neighboring agents or boundaries to see that each agent has an equilibrium position in the interval regardless of the position of its neighbors.

Uniqueness of a fixed point is less straightforward since, depending on the location of the measurements, the agents can find multiple fixed points even if ρ is continuous. We leave a derivation of uniqueness conditions to future work.

For an agent configuration that has reached a fixed point, an important question is how far this configuration is from optimal coverage. Again restricting to a single agent on an interval (a, b), we define the position for which the coverage metric over $[a, b]$ is minimized as x_{opt}. The following theorem bounds the distance between the fixed point and x_{opt}.

Theorem 2: Let $\rho(z) > 0$ be continuous on the interval $[a, b]$. Further, let the magnitude of its second derivative be bounded above by Q and let $\rho(z)$ have no local minima in (a, b). If there is an agent with measurement number M located at a fixed point x^* in (a, b), then the distance to the optimal position given by $x_{opt} = m_\rho^\alpha(a, b)$ is bounded by

$$|x_{opt} - x^*| \leq \frac{Q}{12M^2} \frac{\alpha(b - x^*)^3 + (x^* - a)^3}{\min[\rho(a), \rho(b)](1 + \alpha)}. \quad (6)$$

Proof: Optimal coverage occurs when

$$\int_a^{x_{opt}} \rho(z) dz = \alpha \int_{x_{opt}}^b \rho(z) dz.$$

Without loss of generality assume $x^* < x_{opt}$, i.e., that the optimal location is to the right of the fixed point. Then,

$$\int_a^{x^*} \rho(z) dz + \int_{x^*}^{x_{opt}} \rho(z) dz = \alpha \int_{x^*}^b \rho(z) dz - \alpha \int_{x^*}^{x_{opt}} \rho(z) dz.$$

The integrals from the boundaries to the fixed point can be rewritten as the estimated distances plus some error introduced by using the trapezoid rule:

$$\bar{d}_\rho(a, x^*) + E_L + \int_{x^*}^{x_{opt}} \rho(z) dz = \alpha \bar{d}_\rho(x^*, b) + \alpha E_R - \int_{x^*}^{x_{opt}} \rho(z) dz.$$

Rewriting to isolate the remaining integral gives

$$\int_{x^*}^{x_{opt}} \rho(z) dz = \frac{\alpha E_R - E_L}{(1 + \alpha)}. \quad (7)$$

The two error terms have simple bounds given in [8]. Including these bounds and using the triangle inequality we can bound the left side of Eq (7) as

$$\left| \int_{x^*}^{x_{opt}} \rho(z) dz \right| \leq \frac{Q}{12M^2} \frac{\alpha(b - x^*)^3 + (x^* - a)^3}{1 + \alpha}.$$

The left side of Eq (7) is bounded below by the minimum value of ρ on the interval multiplied by the interval length:

$$\left| \int_{x^*}^{x_{opt}} \rho(z) dz \right| \geq \left| x_{opt} - x^* \right| \min[\rho(a), \rho(b)].$$

Combining these two results gives the bound in Eq (6). ■

Although this theorem does not tell us how far the entire configuration is from optimal coverage, it does give the expected result that the more measurements an agent takes, the closer its equilibrium will be to its optimal location.

B. Transient Dynamics

Since the update equations given in Eq (4) which comprise $\mathbf{G}$ are highly nonlinear, many of the relatively simple analysis techniques for linear iterative systems cannot be applied to study sensor network convergence. Instead we make use of the Contraction Mapping Theorem [9] of nonlinear analysis to derive a bound on the error for a subset of density fields and initial conditions.

We begin by defining the Jacobian of $\mathbf{G}$ as $J_G \in \mathbb{R}^{n\times n}$, where the (j,k) element is $\frac{\partial g_j}{\partial x_k}$. For the update scheme in Eq (4), an agent's updated position depends only on the relative position of its neighbors, so J_G is tridiagonal.

To apply the Contraction Mapping Theorem, the mapping $\mathbf{G}$ must be a contraction mapping on some subset D of the set of all possible agent configurations. As defined in [10], in order for $\mathbf{G}$ to be a contraction mapping on D, there must exist a contraction constant $K<1$ such that $\forall \mathbf{x},\mathbf{y}\in D$

$$\|\mathbf{G}\mathbf{x}-\mathbf{G}\mathbf{y}\| \le K\|\mathbf{x}-\mathbf{y}\| \tag{8}$$

where the norm used is free to be chosen. Since (8) is not useful in the current form, we make use of the result given in [8] that $\mathbf{G}$ is a contraction on D with contraction constant $K \le \bar{K}$ if for all elements of D we have

$$\|J_G\| \le \bar{K} < 1. \tag{9}$$

In summary if some norm of J_G is bounded by a constant less than 1 for some set of configurations, then $\mathbf{G}$ is a contraction over those configurations. Combining this result with a form of the Contraction Mapping Theorem given below (proof given in [9]), we can bound the 'distance' of the network from its equilibrium configuration as well as guarantee uniqueness of this equilibrium.

Theorem 3: Suppose that $\mathbf{G}$ is a contraction on a set D with contraction constant K. Then $\mathbf{G}$ has a unique fixed point $\mathbf{x}^*\in D$ and for any initial condition $\mathbf{x}^{(0)}\in D$ the iteration in Eq (5) converges to $\mathbf{x}^*$. In addition

$$\|\mathbf{x}^{(t)}-\mathbf{x}^*\| \le \frac{K^t}{1-K}\|\mathbf{x}^{(1)}-\mathbf{x}^{(0)}\|. \tag{10}$$

It is important to note that the norm in (10) is the same as the one used to bound the contraction constant in (9).

In the following corollary we apply Theorem 3 to the case of a network of $N=3$ agents with $M=1$ on a uniform information density field given by $\rho(z)=1$.

Corollary 1: For a mobile sensor network with $N=3$, $M=1$ and a uniform information density field $\rho(z)=1$, the agents converge to their optimal coverage locations from any initial positions. In addition, the transient error is bounded by

$$\|\mathbf{x}^{(t)}-\mathbf{x}^*\|_1 \le 3\left(\frac{2}{3}\right)^t\|\mathbf{x}^{(1)}-\mathbf{x}^{(0)}\|_1. \tag{11}$$

Proof: For $\rho=1$ the weighted distance between two points is the same as the Euclidean distance between them. The Jacobian can be computed directly to be

$$J_G = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 \end{bmatrix}. \tag{12}$$

The 1-norm of J_G in this case is equal to $2/3$, independent of agent positions, which means that $\mathbf{G}$ is a contraction for all agent configurations. Thus, by Theorem 3, the network converges to the same, unique fixed point for all initial agent configurations and the error is bounded by (11). In addition, since the second derivative of ρ is identically zero, Theorem 2 tells us that each agent's equilibrium position is the same as its position for optimal coverage. ■

In general the approach taken above cannot be used to determine the Jacobian prior to running the simulation since the elements depend in a nonlinear way on both the agents' positions and ρ. A simpler alternative to give some insight into agent convergence is explored in the following theorem.

Theorem 4: Assume that $\|J_G(\mathbf{x}^*)\|<1$ for a fixed point of the update equation $\mathbf{x}^*=\mathbf{G}(\mathbf{x}^*)$ (i.e., $\mathbf{G}$ is a contraction at its fixed point). Let $\mathbf{x}^{(0)}$, the positions of the agents at $t=0$, satisfy $\|\mathbf{x}^{(0)}-\mathbf{x}^*\|\le\delta$. Then for δ sufficiently small, the iterates of $\mathbf{x}^{(0)}$ given by Eq (5) will converge to $\mathbf{x}^*$.

Proof: By Theorem 9.3 in [11] we know that a matrix norm is continuous in the elements of the matrix. Therefore, if we make the variations in the elements of J_G small enough we can get an arbitrarily small change in the value of $\|J_G\|$. In addition, the elements of J_G are continuous in agent positions, so we can adjust the positions to achieve an arbitrarily small change in J_G. Therefore $\|J_G\|$ is continuous with respect to agent positions.

By continuity, if $\|J_G\|<1$ at the fixed point, then we can perturb the agents some small distance away from $\mathbf{x}^*$ so that $\|J_G\|$ remains less than 1. If we start them at this perturbed configuration, which is no longer a fixed point, then Theorem 3 tells us that the positions will converge to $\mathbf{x}^*$. ■

The problem now becomes one of determining when some norm of the Jacobian is less than 1 at a fixed point, which greatly simplifies the expression of the Jacobian. The following corollary to Theorem 4 enumerates a subset of linear information density fields for which a network converges within some neighborhood of its fixed point configuration.

Corollary 2: For a linear information density field of the form $\rho(z)=Az+B$, a mobile sensor network with measurement number $M=1$ following the position update law in Eq (1) will converge to its fixed point for all initial configurations arbitrarily close to its fixed point if

$$B > \frac{1+\sqrt{2}}{2}|A|.$$

Proof: We proceed by showing that the condition above implies that the ∞-norm of the Jacobian at the fixed point is less than 1. We define the notation $x_k=x_k^{(t)}$, $\rho_k=\rho(x_k)$, $\bar{d}_L=\bar{d}_\rho(x_{k-1},x_k)$, $\bar{d}_R=\bar{d}_\rho(x_k,x_{k+1})$, $\Delta_L=x_k-x_{k-1}$ and $\Delta_R=x_{k+1}-x_k$. We compute

$$\frac{\partial g_k}{\partial x_k} = 1+\frac{\left[\alpha\frac{\partial\bar{d}_R}{\partial x_k}-\frac{\partial\bar{d}_L}{\partial x_k}\right]\rho_k-\left[\alpha\bar{d}_R-\bar{d}_L\right]\rho_k'}{(1+\alpha)\rho_k^2}, \tag{13}$$

$$\frac{\partial g_k}{\partial x_{k-1}} = \frac{-1}{(1+\alpha)\rho_k}\frac{\partial\bar{d}_L}{\partial x_{k-1}}, \tag{14}$$

$$\frac{\partial g_k}{\partial x_{k+1}} = \frac{\alpha}{(1+\alpha)\rho_k}\frac{\partial\bar{d}_R}{\partial x_{k+1}}. \tag{15}$$

The derivative terms in Eqs (13)-(15) with the simplification $M = 1$ are

$$\frac{\partial \bar{d}_L}{\partial x_k} = \frac{\bar{d}_L}{\Delta_L} + \frac{\Delta_L}{2}\rho'_k \tag{16}$$

$$\frac{\partial \bar{d}_L}{\partial x_{k-1}} = -\frac{\bar{d}_L}{\Delta_L} + \frac{\Delta_L}{2}\rho'_{k-1} \tag{17}$$

$$\frac{\partial \bar{d}_R}{\partial x_k} = -\frac{\bar{d}_R}{\Delta_R} + \frac{\Delta_R}{2}\rho'_k \tag{18}$$

$$\frac{\partial \bar{d}_R}{\partial x_{k+1}} = \frac{\bar{d}_R}{\Delta_R} + \frac{\Delta_R}{2}\rho'_{k+1}. \tag{19}$$

The ∞-norm can be written as the maximum over k of the sum of Eqs (13)-(15). At a fixed point, by definition $\alpha\bar{d}_R = \bar{d}_L$, so the second term in the numerator of Eq (13) is zero. For a linear function we can write the value of ρ at x_{k+1} and x_{k-1} as $\rho_{k+1} = \rho_k + A\Delta_R$ and $\rho_{k-1} = \rho_k - A\Delta_L$ respectively.

Substituting these forms into the distance equations (2) with $M = 1$, we can rewrite Eqs (13)-(15) as

$$\frac{\partial g_k}{\partial x_k} = 0$$

$$\frac{\partial g_k}{\partial x_{k-1}} = \frac{1}{1+\alpha}\frac{\rho_{k-1}}{\rho_k}$$

$$\frac{\partial g_k}{\partial x_{k+1}} = \frac{\alpha}{1+\alpha}\frac{\rho_{k+1}}{\rho_k}.$$

We can then sum these equations to calculate the row sum of J_G for each value of k:

$$\frac{1}{3}\frac{\rho_{k+1}}{\rho_k}, \quad k = 1 \tag{20}$$

$$\frac{\rho_{k-1}+\rho_{k+1}}{2\rho_k}, \quad k = 2, \dots, N \tag{21}$$

$$\frac{1}{3}\frac{\rho_{k-1}}{\rho_k}, \quad k = N. \tag{22}$$

The maximum of terms (20)-(22) is $\|J_G\|_\infty$, which we want to show is less than 1. We note that (21) must always be less than 1 because $\frac{\rho_{k-1}+\rho_{k+1}}{2}$ is the average of ρ_{k-1} and ρ_{k+1}, and ρ_k will always be closer to the larger of the two values at equilibrium, and therefore greater than the average.

For the case that $A > 0$, $\rho_N > \rho_{N-1}$, so (22) will always be less than 1. The condition for (20) to be less than 1 can be written as $\rho_1 - \frac{1}{3}\rho_2 > 0$. Since $\rho_k = Ax_k + B$ we can again rewrite this condition as

$$Ax_1 + B - \frac{1}{3}[Ax_2 + B] > 0. \tag{23}$$

At equilibrium (which is the same as optimal for linear ρ) we know that the weighted distance from the boundary to agent 1 is half the weighted distance from agent 1 to agent 2. This is the same as writing

$$\frac{1}{3}\int_{-1}^{x_2}\rho(z)dz = \int_{-1}^{x_1}\rho(z)dz.$$

We can evaluate this integral and rearrange terms to find

$$Ax_1 + B - \frac{1}{3}[Ax_2 + B] = \frac{2}{3}[B - A] + \frac{A^2}{3B} + \frac{A^2}{2B}\left(\frac{x_2^2}{3} - x_1^2\right).$$

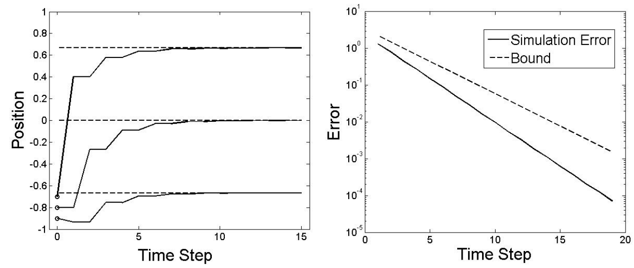

Fig. 1. Parameters: $\rho(z)$=1, N=3, M=1, $\mathbf{x}^{(0)}$=$[-0.9 - 0.8 - 0.7]^T$. The left plot shows the trajectory of the three agents as solid lines and the optimal coverage locations as dashed lines. The position of each agent is given on the y-axis, with the time step given on the x-axis. The right plot shows the error relative to the fixed point configuration at each time step as $\log\|\mathbf{x}^{(t)} - \mathbf{x}^*\|_1$ along with the bound given by Corollary 1.

Note that the right side of this equation is minimized when $x_2 = 0$ and $x_1 = -1$ and that the left side of this equation is the same as the left side of our rewritten condition (23). This means that we can again rewrite (23) as

$$B - A - \frac{A^2}{4B} > 0. \tag{24}$$

By multiplying (24) by B, we can factor the left side into $(B - \frac{1+\sqrt{2}}{2}A)(B - \frac{1-\sqrt{2}}{2}A)$. For $A > 0$, the second term will always be positive as long as $B > 0$ (which is required for ρ to be strictly positive). This means that if $B > \frac{1+\sqrt{2}}{2}A$ then $\|J_G\|_\infty < 1$ for $A > 0$.

Performing the same analysis for the case that $A < 0$, we find that the bound is $B > -\frac{1+\sqrt{2}}{2}A$. These two bounds combine to form the general bound

$$B > \frac{1+\sqrt{2}}{2}|A|. \tag{25}$$

Therefore, according to Theorem 4, whenever (25) is satisfied for a linear information density field, a network with $M = 1$ will converge to its fixed point for initial configurations in a neighborhood of the fixed point. ■

V. SIMULATIONS

We show here a number of simulations of the coverage control method developed in Section III. The simulation shown in Figure 1 illustrates the result given in Corollary 1 for three agents with $M = 1$ and $\rho = 1$. The depiction of the simulation on the left shows the trajectory of the three agents over 15 time steps and convergence to their optimal coverage positions. We note that in the optimal configuration the distance between two neighboring agents is twice the distance from a boundary agent to the boundary. The rate of convergence of the network to its fixed point is shown in the right subfigure as the logarithm of the 1-norm of $\mathbf{x}^{(t)} - \mathbf{x}^*$. The bound derived in Corollary 1 is also shown; the plot confirms that the contraction constant K is less than $\|J_G\|_1$.

Figure 2 shows simulation results for the linear information density field $\rho(z) = 2z + 5$ with $N = 4$ and $M = 1$ meant to illustrate Theorem 4. The left plot shows

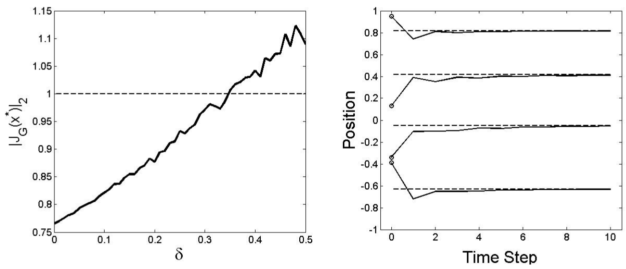

Fig. 2. Parameters: $\rho(z)=2z+5$, $N=4$, $M=1$. The left plot shows the 2-norm of the Jacobian as a function of δ, which is a metric of the starting configuration's distance from the fixed point. The right plot shows a sample simulation for an initial configuration with $\delta = 0.3$.

the behavior of the 2-norm of the Jacobian with respect to δ, which is defined by

$$\delta \geq \|\mathbf{x}^{(0)} - \mathbf{x}^*\|_\infty. \tag{26}$$

For each value of δ on the x-axis $100,000$ initial configurations that satisfied (26) were randomly selected and the 2-norm of the Jacobian at each of those configurations was calculated. The maximum of those norms is the value plotted on the y-axis. We can see that at the fixed point ($\delta = 0$), the Jacobian norm is less than 1. It is clear that the further from the fixed point the network starts the larger the Jacobian norm, until at some value of δ the norm exceeds 1. It is important to note that this does not imply that the network diverges at initial configurations with δ larger than this cutoff value, it simply no longer guarantees that the network will converge. In the case shown here, we can see that initial configurations quite far from the fixed point are guaranteed to converge.

On the right plot in Figure 2 the trajectories of the four agents for a sample initial configuration with $\delta = 0.3$ are plotted. Due to the non-uniformity of ρ, the agents are no longer uniformly spaced through the region. Instead they are more tightly grouped closer to $+1$ since ρ is higher in that region.

The simulation shown in Figure 3 illustrates a more pronounced grouping induced by non-uniformity in the information density field. The dotted line plotted on the left is an overlay of $\rho(z)$ showing that it contains two information 'peaks', while the solid and dashed lines are the familiar agent trajectories and optimal coverage locations respectively. At time step zero the agents are uniformly spaced, but after only a few steps they cluster into a group of 3 and a group of 2 according to the relative sizes of the information peaks.

VI. CONCLUSIONS

In this paper we have developed, analyzed and implemented a distributed coverage control law for mobile sensor networks moving in one dimension. The primary advantage of this control law compared to other distributed coverage methods is its simplicity in terms of the number of measurements of the information field agents must make and

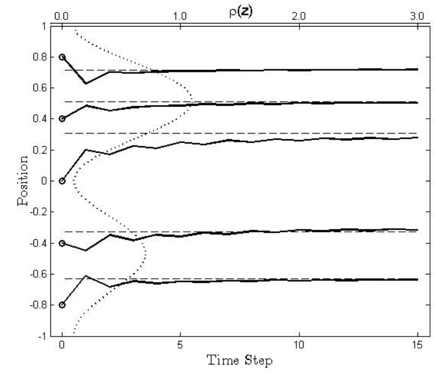

Fig. 3. Parameters: $\rho(z)=\frac{1}{2}+(z+2)(1-\cos(2\pi z))$, $N=4$, $M=2$. Agents are uniformly spaced at $t=0$. Solid lines are agent trajectories with time step on the lower x-axis and position on the y-axis. The dashed lines are the optimal coverage positions. The dotted line is an overlay of $\rho(z)$ showing the dual peak structure, with the scale on the upper x-axis.

the computations they must perform. The primary shortfall is that convergence of the network to its fixed point is not guaranteed for arbitrary agent configurations, so behavior of the control law is unpredictable for a rapidly varying ρ or an initial configuration far from the fixed point.

Future research into this coverage method should focus on either gaining a better understanding of the convergence properties by applying Theorems 3 and 4 to other classes of ρ or making the control law itself more robust by changing the interpolation scheme or introducing gain constants to tune the dynamics. In addition, generalizing the method presented here to two or more dimensions promises a richer and more challenging problem.

VII. ACKNOWLEDGMENTS

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