Nonlinear Gliding Stability and Control for Vehicles with Hydrodynamic Forcing

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Abstract

This paper presents Lyapunov functions for proving stability of steady gliding motions for vehicles with hydrodynamic or aerodynamic forces and moments. Because of lifting forces and moments, system energy cannot be used as a Lyapunov function candidate. A Lyapunov function is constructed using a conservation law discovered by Lanchester in his classical work on phugoid-mode dynamics of an airplane. The phugoid-mode dynamics, which are cast here as Hamiltonian dynamics, correspond to the slow dynamics in a multi-time-scale model of a hydro/aerodynamically forced vehicle in the longitudinal plane. Singular perturbation theory is used in the proof of stability of gliding motions. As an intermediate step, the simplifying assumptions of Lanchester are made rigorous. It is further shown how to design stabilizing control laws for gliding motions using the derived function as a control Lyapunov function and how to compute corresponding regions of attraction.

Key words: Singular perturbation methods; Nonlinear stability; Lyapunov based control; Composite Lyapunov function; Hamiltonian system; Underwater gliders; Gliders; Phugoid mode.

1 Introduction

Energy usually provides a natural Lyapunov function candidate for studying the stability of mechanical systems. Energy-based design methodologies, such as the method of controlled Lagrangians (Bloch, Leonard & Marsden 2000) and the method of interconnection and damping assignment (Ortega, Spong, Gomez-Estern & Blankenstein 2002), provide systematic procedures for choosing control laws for mechanical systems such that the resulting closed-loop dynamics also define a mechanical system. The energy of the closed-loop system serves as a control Lyapunov function. The gains of the control law can be adjusted such that the region of attraction for the closed-loop system is sufficiently large.

The energy-based analysis and control design approach falls short for steady motions when hydrodynamic or aerodynamic forces and moments act on the mechanical system. This is the case for vehicles with lifting surfaces such as airplanes, sailplanes and underwater gliders. Our objective is to provide an analysis and control design methodology for these kinds of hydro/aerodynamically-forced systems analogous to the approach for pure mechanical systems.

For the purposes of this paper, we refer to hydro/aerodynamically forced vehicles that can sustain steady gliding motions as *gliders*. This class includes airplanes, sailplanes, airships, air gliders and underwater gliders. It also includes inter-planetary gliding vehicles as proposed in (Morrow, Woolsey & Hangerman Jr. 2006). We consider the dynamics of gliders restricted to the longitudinal plane and use a multi-time-scale model. The slow variables define the translational glider dynamics and the fast variables define the rotational dynamics. We use singular perturbation theory (Saberi & Khalil 1984, Khalil 1987) to derive a composite Lyapunov function and prove stability of steady glides for the full dynamics.

A first step in the development is the identification of the reduced model (slow dynamics) with Lanchester's

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classical model of the phugoid-mode dynamics of an airplane (Lanchester 1908, von Mises 1959). A second and critical step is our casting of the phugoid-mode dynamics as a Hamiltonian system such that the lift force derives from the Hamiltonian. In this case, the Hamiltonian H is *not* the energy of the aircraft but rather an integral of motion discovered by Lanchester. We present this Hamiltonian formulation and use H to construct the Lyapunov function for proving the stability of the slow subsystem of glider dynamics. As an intermediate result, we use singular perturbation theory to make rigorous the simplifying assumptions of Lanchester.

In earlier studies on dynamics and control of underwater gliders (Leonard & Graver 2001, Bhatta & Leonard 2002), linearization was the basis for stability analysis and control design. In this paper we use Lyapunov functions to prove nonlinear stability of steady glides and to estimate the corresponding region of attraction. The Lyapunov function presented in this paper provides a new means to extend and systematize the underwater glider analysis and design work described in the earlier papers (Leonard & Graver 2001, Bhatta & Leonard 2002, Bhatta & Leonard 2004). In this paper we use our Lyapunov-based approach to design simple stabilizing controllers for gliders with different control configurations.

There is a significant literature addressing the tracking problem for aircraft using feedback linearization based methods, including (Tomlin, Lygeros, Benvenuti & Sastry 1995, Al-Hiddabi & McClamroch 1999). The non-minimum phase characteristic of controlled aircraft systems renders the tracking problem challenging and typically leads to large magnitude control inputs. Large inputs are usually undesirable in the context of gliders since low-energy control is an essential requirement of most glider operations. The results of this paper motivate further work utilizing Lyapunov functions for tracking certain classes of desired, unsteady glider trajectories that may be approximated by combining steady gliding motions. Preliminary results of applying such a tracking methodology for a conventional take-off and landing (CTOL) aircraft are presented in (Bhatta & Leonard 2006).

Although the model and framework in this paper are applicable to a variety of gliders, our motivating application is to autonomous underwater gliders. Underwater gliders are energy-efficient, buoyancy-driven, winged submersibles that are of great interest for ocean sampling tasks. For instance, a fleet of underwater gliders was utilized successfully as a mobile sensor network, gathering data on physical and biological fields in Monterey Bay, California during August 2003 as part of the Autonomous Ocean Sampling Network II (AOSN-II) experiment (Fiorelli, Leonard, Bhatta, Paley, Bachmayer & Fratantoni 2006, MBARI 2003) and during August 2006 as part of the Adaptive Sampling and Prediction (ASAP) experiment (Princeton University 2006). The Slocum glider (Webb, Simonetti & Jones 2001), the Spray glider (Sherman, Davis, Owens & Valdes 2001) and the Seaglider (Eriksen, Osse, Light, Wen, Lehmann, Sabin, Ballard & Chiodi 2001) are examples of underwater gliders that operate autonomously. They can spend several weeks continuously in the water and travel several hundreds of kilometers without the need to recharge batteries. Slocum and Spray gliders were employed in the AOSN-II and ASAP experiments.

This paper is arranged as follows. In Section 2 we provide a Hamiltonian description of the phugoid-mode model of an aircraft. In Section 3 we present a model describing underwater glider dynamics and nondimensionalize the equations of motion. We show how other types of gliders are special cases of the underwater glider. In Section 4 we apply singular perturbation theory to reduce the glider dynamics to the slow (phugoid-mode) dynamics. In Section 5 we apply the results of (Saberi & Khalil 1984, Khalil 1987) to construct a composite Lyapunov function to prove the stability of gliding for the full dynamics and estimate the region of attraction provided by the Lyapunov function. In Section 6 we use a Lyapunov-based approach to design stabilizing controllers for gliders with different control configurations. Final remarks are given in Section 7.

2 Phugoid-Mode Model: A Hamiltonian Description

In this section we present an integrable, Hamiltonian model of longitudinal aircraft dynamics, called the phugoid-mode model, studied by Lanchester (Lanchester 1908, von Mises 1959). The Hamiltonian function of this model is a main ingredient in the composite Lyapunov function that we construct in Section 5 for proving the stability of glider equilibria.

Lanchester developed a simple model of longitudinal aircraft dynamics by making the following assumptions:

(1) The thrust of the aircraft exactly cancels the drag.
(2) The small moment of inertia of the aircraft and the large angle of attack coefficient of the aerodynamic pitching moment cause the fast convergence of the angle of attack to a constant value.

Under the above assumptions, the equations describing Lanchester's phugoid-mode model of an aircraft are

$$\dot{V} = -mg\sin\gamma \tag{1}$$
$$\dot{\gamma} = \frac{1}{mV}\left(KV^2 - mg\cos\gamma\right), \tag{2}$$

where V, γ are the aircraft speed and flight path angle respectively, m is the aircraft mass, g is the gravitational acceleration and K is the constant lift coefficient.

The energy of the aircraft

$$E = \frac{1}{2}mV^2 + mg\int_0^t V(\tau)\sin(\gamma(\tau))d\tau$$

is conserved, since gravity is a conservative external force and lift, the only other external force always acts perpendicular to the aircraft's velocity. Lanchester further observed another conserved quantity

$$C = \left(\frac{V}{V_e}\right)\cos\gamma - \frac{1}{3}\left(\frac{V}{V_e}\right)^3,$$

where $V_e = \sqrt{\frac{mg}{K}}$ is the equilibrium aircraft speed. Lanchester noted that the value of C, which depends on initial conditions, determines the type of trajectory followed by the aircraft: $C = 1/3$ corresponds to the equilibrium steady level flight, $0 < C < 1/3$ to wavy flight paths, $C = 0$ to a singular trajectory with a sharp cusp and $C < 0$ to trajectories with loops.

The phugoid-mode equations (1)-(2) can be written as canonical Hamilton's equations in coordinates $q = V\cos\gamma$ and $p = -V\sin\gamma$:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q}, \end{aligned}$$

where $H = \frac{1}{3m}K\left(p^2 + q^2\right)^{3/2} - gq = -gV_eC$.

The slow subsystem of the glider model, derived in Section 4 is identical to the above phugoid-mode model, except for the presence of drag. Because of drag, C is not a conserved quantity for the slow subsystem of the glider dynamics. However, the stability of its equilibria can be proved using a variant of C as the Lyapunov function.

3 Underwater Glider Model

In this section we present a mathematical model describing the longitudinal dynamics of an underwater glider. The longitudinal plane is an invariant plane for underwater glider dynamics if we assume that the vehicle is symmetric about the longitudinal axis and neglect lateral disturbances. This kind of assumption is made often using slender body theory - see (Gertler & Hagen 1967) for example. Our model is derived from the more general model studied in (Leonard & Graver 2001, Graver 2005).

We consider an ellipsoidal rigid body with uniformly distributed mass, fixed wings and tail. The rigid body experiences forces due to gravity, lift and drag. It also experiences a hydrodynamic pitching moment. Since we consider the motion of the body underwater we also include buoyancy and added mass effects.

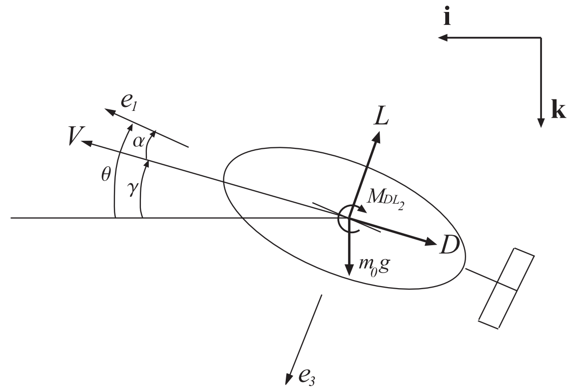

Fig. 1. Hydrodynamic forces and moment acting on the underwater glider.

The motion of an underwater glider is controlled by varying the total vehicle mass (and therefore its buoyancy) and by redistributing its internal mass. The internal mass redistribution is modelled using a point mass that moves within the rigid body, as described in (Leonard & Graver 2001). In this paper we fix this movable point mass at the center of buoyancy (CB) of the vehicle, making the CB coincident with the center of gravity (CG) of the vehicle. In Section 6 we consider a control torque as a proxy for regulation of internal mass position. We assume a constant buoyancy for deriving the stability results but consider buoyancy control in Section 6.

A schematic of the underwater glider is shown in Fig. 1. We fix a frame on the body with axis $\mathbf{e_1}$ along the body 1-axis. The axis $\mathbf{e_2}$ lies along the body 2-axis, and points into the page and $\mathbf{e_3}$ lies along the body 3-axis in the longitudinal plane, perpendicular to $\mathbf{e_1}$ with its direction defined by the right hand rule.

The mass of the vehicle plus the added mass in the body-1 direction is m_1 and the vehicle mass plus added mass in the body-3 direction is m_3. Let $\Delta m = m_3 - m_1$. We define m_0 to be the "heaviness" of the glider, i.e., the mass of the glider minus the mass of the displaced fluid. In the case of aircraft and sailplanes, added mass and buoyancy are negligible so that $m_1 = m_3 = m_0 = m$, where m is the mass of the vehicle. We let J_2 be the moment of inertia plus added moment of inertia about the axis in the $\mathbf{e_2}$ direction, passing through the CG of the glider. The pitch angle is θ, the glide path angle is γ and the angle of attack is α, where $\theta = \gamma + \alpha$.

We model lift L, drag D and a hydrodynamic moment M_{DL_2} as shown in Fig. 1. Lift acts perpendicular to the velocity and drag always acts in the direction opposite to the velocity. The forces depend on α and longitudinal speed V of the glider relative to the surrounding fluid. Following standard empirical studies on airfoils - discussed in (Abbott & von Doenhoff 1959, McCormick 1979) for example - we assume that the lift force varies linearly with α, and that the drag force has a parabolic dependence on α: $L = (K_{L0} + K_L\alpha) V^2$, $D = \left(K_{D0} + K_D\alpha^2\right) V^2$. The moment is a function of α, V and the pitch rate of the glider Ω_2: $M_{DL_2} = (K_{M0} + K_M\alpha + K_q\Omega_2) V^2$. The coefficients of lift, drag and moment depend on the shape of the glider and the design of its wings and tail. They can be estimated using empirical relations, and fine tuned by system identification (Graver, Bachmayer, Leonard & Fratantoni 2003). Typically K_{L0} is zero or a small positive number, K_L, K_{D0} and K_D are positive, K_M and K_q are negative. K_{M0} can be positive or negative.

Under the above assumptions the equations of motion of the underwater glider (derived in (Leonard & Graver 2001)) for $V > 0$, $\gamma, \alpha \in S^1$ are

$$
\begin{aligned}
\dot{V} = & -\frac{1}{m_3}\left(m_0 g \sin\gamma + D\right) && (3)\\
& -\frac{\Delta m \cos\alpha}{m_1 m_3}\{m_0 g \sin\theta - L\sin\alpha + D\cos\alpha \\
& \qquad + (m_1 + m_3) V\Omega_2 \sin\alpha\} \\
\dot{\gamma} = & \frac{1}{m_3 V}\left(-m_0 g\cos\gamma + L\right) && (4)\\
& -\frac{\Delta m}{m_1 m_3 V}\{(m_0 g\sin\theta - L\sin\alpha + D\cos\alpha)\sin\alpha \\
& \qquad + \left(m_3\sin^2\alpha - m_1\cos^2\alpha\right) V\Omega_2\} =: E_2 \\
\dot{\alpha} = & \ \Omega_2 - E_2 && (5)\\
\dot{\Omega}_2 = & \frac{1}{J_2}\left\{(K_{M0} + K_M\alpha + K_q\Omega_2) V^2\right. && (6)\\
& \left. + \Delta m V^2 \sin\alpha\cos\alpha\right\},
\end{aligned}
$$

where E_2 is the right-hand-side of equation (4).

We note that ocean currents do not feature in our model, because we only consider the motion of the underwater glider relative to the ocean currents. The length scales associated with ocean current dynamics are much larger compared to those relevant to vehicle dynamics. As a result, in most applications, it is not necessary to consider ocean currents explicitly in vehicle dynamics. Ocean currents will be relevant to navigating the vehicle in the inertial plane. For this purpose it is sufficient to superimpose the vehicle relative velocity with the estimated or measured ocean currents.

A steady glide is an isolated equilibrium of the system described by equations (3)-(6). The equilibrium angular velocity is $\Omega_{2e} = 0$, and the equilibrium angle of attack α_e is the solution of the nonlinear equation

$$K_{M0} + K_M\alpha_e + \Delta m \sin\alpha_e \cos\alpha_e = 0. \qquad (7)$$

The equilibrium values of V and γ for the case of $\Delta m = 0$ are

$$V_e = \left(\frac{|m_0| g}{\sqrt{K_{D_e}^2 + K_{L_e}^2}}\right)^{\frac{1}{2}}, \quad \gamma_e = \arctan\frac{-K_{D_e}/m_0}{K_{L_e}/m_0},$$

where $K_{D_e} = K_{D0} + K_D\alpha_e^2$, $K_{L_e} = K_{L0} + K_L\alpha_e$.

4 Singular Perturbation Reduction

In this section we reduce the underwater glider dynamics presented in Section 3 using singular perturbation theory. We also note the connection between the resulting slow dynamics and the phugoid-mode of aircraft longitudinal dynamics. For clarity of presentation we derive our results for the case of underwater gliders with equal added masses, i.e., $\Delta m = 0$. We note an extension of the results for the case of unequal added masses at the end of this section.

First we non-dimensionalize the equations of motion and shift the equilibrium to the origin. We define $\bar{V} = \frac{V - V_e}{V_e}$, $\bar{\gamma} = \gamma - \gamma_e$, $\bar{\alpha} = \alpha - \alpha_e$, $\bar{\Omega}_2 = \frac{K_q}{K_M}\Omega_2$. We also define two non-dimensional, small parameters $\epsilon_1 = \left(\frac{K_q}{K_M}\right)\frac{1}{\tau_s}$ and $\epsilon_2 = -\left(\frac{J_2}{K_q V_e^2}\right)\frac{1}{\tau_s}$, where $\tau_s = \frac{m_3}{K_{D_e} V_e}$. The parameters ϵ_1 and ϵ_2 are small positive numbers for a typical underwater glider (such as the Slocum (Webb et al. 2001)) i.e., $0 < \epsilon_1 \ll 1$, $0 < \epsilon_2 \ll 1$.

We define a non-dimensional time variable $t_n = t/\tau_s$ and rewrite the dynamics (3)-(6) in terms of the non-dimensional state variables:

$$
\begin{aligned}
\frac{d\bar{V}}{dt_n} &= -\frac{1}{K_{D_e} V_e^2}\left(m_0 g \sin(\bar{\gamma} + \gamma_e) + D\right) && (8)\\
\frac{d\bar{\gamma}}{dt_n} &= \frac{1}{K_{D_e} V_e^2 (1 + \bar{V})}\left(-m_0 g\cos(\bar{\gamma} + \gamma_e) + L\right) && (9)\\
&=: \bar{E}_2 \\
\epsilon_1 \frac{d\bar{\alpha}}{dt_n} &= \bar{\Omega}_2 - \epsilon_1 \bar{E}_2 && (10)\\
\epsilon_2 \frac{d\bar{\Omega}_2}{dt_n} &= -\left(\bar{\alpha} + \bar{\Omega}_2\right)\left(1 + \bar{V}\right)^2, && (11)
\end{aligned}
$$

where $\bar{E}_2$ is the right-hand-side of equation (9).

In the above equations

$$
\begin{aligned}
D &= \left(K_{D0} + K_D(\bar{\alpha} + \alpha_e)^2\right) V_e^2 \left(1 + \bar{V}\right)^2 \\
L &= \left(K_{L0} + K_L(\bar{\alpha} + \alpha_e)\right) V_e^2 \left(1 + \bar{V}\right)^2.
\end{aligned}
$$

We define $\mu = \max\{\epsilon_1, \epsilon_2\}$, as in (Khalil 1987), and assume that $\frac{\mu}{\epsilon_1}$, $\frac{\mu}{\epsilon_2}$ are $O(1)$.[1] It does not matter whether ϵ_1 is less than, equal to, or greater than ϵ_2.

In the remainder of this paper we consider the dynamics represented by equations (8)-(11) in a domain defined as follows: $(x, y, \epsilon_i) \in B_x \times B_y \times [0, \epsilon_i^*]$, where $x := (\bar{V}, \bar{\gamma})$, $y := (\bar{\alpha}, \bar{\Omega}_2)$, $B_x \in \mathbb{R}^2$ is a neighborhood of $(\bar{V}, \bar{\gamma}) = (0,0)$ such that $-1 < \bar{V}_{min} < \bar{V} < \bar{V}_{max}$, $-\pi \le \bar{\gamma} < \pi$, and $B_y \in \mathbb{R}^2$ is a neighborhood of $(\bar{\alpha}, \bar{\Omega}_2) = (0,0)$ such that $\bar{\alpha}_{min} < \bar{\alpha} < \bar{\alpha}_{max}$.

We rewrite equations (8)-(11) in the compact form

$$\frac{dx}{dt_n} = f(x, y) \tag{12}$$

$$\mu \frac{dy}{dt_n} = A g(x, y, \epsilon) \tag{13}$$

where, $x = \begin{bmatrix} \bar{V} \\ \bar{\gamma} \end{bmatrix}$ $y = \begin{bmatrix} \bar{\alpha} \\ \bar{\Omega}_2 \end{bmatrix}$, $\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$,

$$A = \begin{bmatrix} \frac{\mu}{\epsilon_1} & 0 \\ 0 & \frac{\mu}{\epsilon_2} \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

4.1 Boundary-Layer Model

The boundary-layer model of the underwater glider is obtained from equation (13) by setting μ to zero, yet allowing $\mu \frac{dy}{dt_n}$ to be nonzero. This is done by writing equation (13) using a "stretched" time scale $\tau = \frac{t_n}{\mu}$:

$$\frac{dy}{d\tau} = A g(x, y, 0). \tag{14}$$

In the boundary-layer model we treat x as a fixed parameter. The following proposition proves uniform (in x) exponential stability of the origin for the boundary-layer model.

Proposition 1 *The origin is an exponentially stable equilibrium of the boundary-layer model (14).*

Proof This proposition follows by Theorem 4.10 of (Khalil 2002) if there is a Lyapunov function $\hat{W}$ that satisfies

(1) $q_3 \|y\|^2 \le \hat{W}(x, y) \le q_4 \|y\|^2$ for some $q_3, q_4 > 0$.
(2) $\frac{\partial \hat{W}}{\partial y} A g(x, y) \le -a_2 \sigma^2(y) \le -b_2 \|y\|^2$ where $\sigma(.)$ is a positive definite function on $\mathbb{R}^2$, that vanishes only at $y = 0$, and a_2, b_2 are positive constants.

Consider the quadratic Lyapunov function

$$\hat{W} = \frac{1}{2} y^T C_W y = \frac{1}{2} \begin{bmatrix} \bar{\alpha} & \bar{\Omega}_2 \end{bmatrix} \begin{bmatrix} c_1 & c_3 \\ c_3 & c_2 \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\Omega}_2 \end{bmatrix}$$

with $c_1 = r_1 + r_1 a + r_2 a$, $c_2 = 1 + \frac{r_1}{r_2 a}$ and $c_3 = 1$, where $a = \left(1 + \bar{V}\right)^2$, $r_1 = \frac{\mu}{\epsilon_1}$, $r_2 = \frac{\mu}{\epsilon_2}$. The matrix C_W is positive definite because $c_1 > 0$, $c_1 c_2 - c_3^2 > 0$. We satisfy condition (1) of with q_3 and q_4 equal to the minimum and maximum eigenvalues of C_W respectively. We also have

$$\frac{d\hat{W}}{d\tau} = -r_2 a \left(\bar{\alpha}^2 + \bar{\Omega}_2^2\right),$$

thereby satisfying condition (2) with $a_2 = r_2$, $\sigma = \left(1 + \bar{V}\right) \sqrt{\bar{\alpha}^2 + \bar{\Omega}_2^2}$ and $b_2 = a_2 \left(1 + \bar{V}_{min}\right)^2$. $\square$

4.2 Reduced Model

The reduced model of the underwater glider is obtained from equation (12) by setting $\mu = 0$ and assuming the boundary-layer model to have reached its equilibrium, the origin:

$$\frac{dx}{dt_n} = f(x, 0). \tag{15}$$

For the reduced model (15) to approximate the evolution of the corresponding states of the full model (12)-(13), its equilibrium (the origin) has to be exponentially stable.

Proposition 2 *The origin is an exponentially stable equilibrium point of the reduced model (15).*

Proof This proposition follows by Theorem 4.10 of (Khalil 2002) if there is a Lyapunov function Φ that satisfies

(1) $q_1 \|x\|^2 < \Phi(x) < q_2 \|x\|^2$ for some $q_1, q_2 > 0$.
(2) $\frac{\partial \Phi}{\partial x} f(x, 0) \le -a_1 \psi^2(x) \le -b_1 \|x\|^2$ where $\psi(.)$ is a scalar positive-definite function of x that vanishes only at $x = 0$, and a_1, b_1 are positive constants.

We note that the reduced model without the drag term has the same form as Lanchester's phugoid-mode model described in Section 2. We use a variant of the conserved quantity C as the Lyapunov function for proving the exponential stability of the reduced model:

$$\Phi = \frac{2}{3} - (1 + \bar{V}) \cos(\bar{\gamma}) + \frac{1}{3} \left(1 + \bar{V}\right)^3.$$

We note that $\Phi = 0$ and $\nabla \Phi = 0$ only at $(0,0)$, and the Hessian of Φ at $(0,0)$ is positive definite. Using the

[1] $f_1(\delta) = O(f_2(\delta))$ if $\exists$ positive constants k, c such that $|f_1(\delta)| \le k\,|f_2(\delta)|$ $\forall$ $|\delta| < c$. (Khalil 2002)

power series expansions of $\cos\bar{\gamma}$ we satisfy condition (1) with $q_1 = \min[1+\frac{\bar{V}_{min}}{3}, \left(\frac{1+\bar{V}_{min}}{2}\right)\left(1-\frac{\pi^2}{12}\right)]$ and $q_2 = \max[\frac{1+\bar{V}_{max}}{2}, 1+\frac{\bar{V}_{max}}{3}]$. We also have

$$\begin{aligned}\frac{d\Phi}{dt_n} &= -\left\{\left(\bar{V}(\bar{V}+2)\right)^2 + 4(1+\bar{V})^2\sin^2\left(\frac{\bar{\gamma}}{2}\right)\right\} \\ &\leq -e^2\left(1+\bar{V}_{min}\right)^2\left(\bar{V}^2+\bar{\gamma}^2\right)\end{aligned}$$

where $e = \left(1-\frac{\pi^2}{24}\right) > 0$. In the above calculation we have used $K_{D_e}V_e^2 = -m_0 g\sin\gamma_e$, and $K_{L_e}V_e^2 = m_0 g\cos\gamma_e$.

Thus, we satisfy the condition (2) with $a_1 = 1$, $\psi = \sqrt{(\bar{V}^2+2\bar{V})^2 + 4(1+\bar{V})^2\sin^2\left(\frac{\bar{\gamma}}{2}\right)}$ and $b_1 = a_1 e^2\left(1+\bar{V}_{min}\right)^2$. $\square$

4.3 Reduction of Dynamics

By Propositions 1 and 2 we meet all conditions and assumptions of Theorem 11.2 of (Khalil 2002). Thus we are able to reduce the full underwater glider dynamics to the slow, phugoid-mode dynamics. More precisely we can conclude the following.

Theorem 3 *Let $R_y^A \subset B_y$ be the region of attraction of (14) and Λ_y be a compact subset of R_y^A. Let the set $\{\|x\|^2 \leq q_5\}$, where $q_5 > 0$, be a compact subset of B_x. For each compact set $\Lambda_x \subset \{\|x\|^2 \leq \rho q_5, 0 < \rho < 1\}$ there is a positive constant μ^* such that for all $(\bar{V}_0, \bar{\gamma}_0) \in \Lambda_x$, $(\bar{\alpha}_0, \bar{\Omega}_{2,0}) \in \Lambda_y$, $0 < \mu < \mu^*$ and $t_n \in [t_{n,0}, \infty)$,*

$$\begin{aligned}x(t_n,\mu) - x_r(t_n) &= O(\mu) \\ y(t_n,\mu) - \hat{y}(t_n/\mu) &= O(\mu)\end{aligned}$$

where $x_r(t_n)$ and $\hat{y}(\tau)$ are the solutions of the reduced (15) and boundary-layer (14) systems, respectively.

Proof This theorem follows by applying Theorem 11.2 of (Khalil 2002) to equations (12)-(13). $\square$

Remark: Theorem 3 proves that Lanchester's simplifying assumptions presented in Section 2 are valid if ϵ_1 and ϵ_2 for the aircraft are sufficiently small.

4.4 Reduction of Dynamics for Unequal Added Masses

Singular perturbation reduction of underwater glider dynamics in the case of unequal added masses presents additional technical difficulties due to presence of Ω_2 in the V, γ dynamics (equations (3)-(4)). Recall that $\bar{\Omega}_2 = \frac{K_q}{K_M}\Omega_2$. As in the case of equal added masses, the reduced model is derived by setting the states of the boundary-layer model to their equilibrium values, i.e. $\bar{\Omega}_2 = 0$, $\bar{\alpha} = 0$, and setting $\mu = 0$. But $\bar{\Omega}_2 = 0$ and $\mu = 0$ do not necessarily imply $\Omega_2 = \frac{\bar{\Omega}_2}{\tau_s\epsilon_1} = \frac{\bar{\Omega}_2 r_1}{\tau_s\mu} = 0$. However, $\bar{\alpha} = 0$ implies $\Omega_2 = \dot{\gamma} = \frac{1}{\tau_s}\frac{d\bar{\gamma}}{dt_n}$. Thus, the reduced model is obtained from equations (3)-(4) by setting $\Omega_2 = \frac{1}{\tau_s}\frac{d\bar{\gamma}}{dt_n}$, $\alpha = \alpha_e$ and $\theta = \bar{\gamma} + \gamma_e + \alpha_e$ and then solving for $\frac{d\bar{V}}{dt_n}$ and $\frac{d\bar{\gamma}}{dt_n}$. The exponential stability of the equilibria for the boundary-layer and reduced models can be proven for a small enough Δm. This allows the reduction of underwater glider dynamics to the reduced model, as in the case of equal added masses. The proofs are omitted for the sake of brevity.

5 Composite Lyapunov Function for Proving Gliding Stability

Stability of the boundary-layer and reduced models guarantees the local stability of the equilibrium of the full dynamics. Furthermore we can construct a composite Lyapunov function for the full dynamics using the methods of (Saberi & Khalil 1984, Khalil 1987). The Lyapunov function allows us to calculate estimates of the region of attraction for the equilibrium of the full glider dynamics.

Theorem 4 *The origin of the singularly perturbed system (12)-(13) is an asymptotically stable equilibrium point for sufficiently small ϵ_i. Moreover,*

$$\begin{aligned}\nu = (1-d)&\left\{\frac{2}{3} - (1+\bar{V})\cos(\bar{\gamma}) + \frac{1}{3}\left(1+\bar{V}\right)^3\right\} \\ &+ \frac{d}{2}\left[\left\{r_1 + \left(1+\bar{V}\right)^2(r_1+r_2)\right\}\bar{\alpha}^2\right. \\ &\left.+\left\{1+\frac{r_1}{r_2\left(1+\bar{V}\right)^2}\right\}\bar{\Omega}_2^2 + 2\bar{\alpha}\bar{\Omega}_2\right], \qquad (16)\end{aligned}$$

where $0 < d < 1$, is a Lyapunov function candidate for proving the asymptotic stability of the the origin of (12)-(13).

Proof Note that the origin is the unique equilibrium of (12)-(13) in the neighborhood $B_x \times B_y$, and $y_{be} = 0$ is the unique equilibrium for the boundary-layer system (14). Suppose Propositions 1 and 2 hold. Then, the proof follows from Theorem 1 of (Khalil 1987) if the following inteconnection conditions hold:

(1) $\frac{\partial\Phi}{\partial x}\left[f(x,y) - f(x,0)\right] \leq \beta_1\psi(x)\sigma(y) + \mu\gamma_1\psi^2(x)$

(2) $\frac{\partial\hat{W}}{\partial y}A\left[g(x,y,\epsilon) - g(x,y,0)\right]$
$$\leq \mu\left(\gamma_2'\sigma^2(y) + \beta_2'\psi(x)\sigma(y)\right)$$

(3) $\left|\frac{\partial \hat{W}}{\partial x} f(x,y)\right| \le \gamma_2^{''}\sigma^2(y) + \beta_2^{''}\psi(x)\sigma(y)$

where $\beta_1,\ \beta_2^{'},\ \beta_2^{''},\ \gamma_1,\ \gamma_2^{'},\ \gamma_2^{''}$ are nonnegative constants. The derivation of the coefficients in the interconnection conditions, satisfied for the underwater glider model with equal added masses, is outlined in Appendix A.

Thus, the origin is an asymptotically stable equilibrium of (12)-(13) for all $0 < \epsilon_i < \min\{\mu^*, \epsilon_i^*\}$ where $\mu^* = \frac{a_1 a_2}{a_1\gamma_2 + a_2\gamma_1 + \beta_1\beta_2}$, $\beta_2 = \beta_2^{'} + \beta_2^{''}$, $\gamma_2 = \gamma_2^{'} + \gamma_2^{''}$. For every $0 < d < 1$, ν given by equation (16) is a Lyapunov function for all $0 < \epsilon_i < \min\{\mu_d, \epsilon_i^*\}$, where $\mu_d = \frac{a_1 a_2}{a_1\gamma_2 + a_2\gamma_1 + \frac{1}{4(1-d)d}[(1-d)\beta_1 + d\beta_2]^2}$. $\square$

Note that the proof of Theorem 4 makes explicit the bound on ϵ_1 over which the origin is asymptotically stable and the Lyapunov function ν proves stability.

5.1 Region of Attraction

Following (Saberi & Khalil 1984), if $L_R = \{(\bar{V}, \bar{\gamma}) \in B_x \,|\, \Phi \le v_0\}$ is in the region of attraction of the reduced system and $L_B = \{(\bar{V}, \bar{\gamma}, \bar{\alpha}, \bar{\Omega}_2) \in B_x \times B_y \,|\, \hat{W} \le w_0\}$ is in the region of attraction of the boundary-layer system, then $d = \frac{v_0}{v_0 + w_0}$ yields the largest estimate of the region of attraction, L^*, provided by the Lyapunov function ν:

$$L^* = \left\{ (\bar{V}, \bar{\gamma}, \bar{\alpha}, \bar{\Omega}_2) \in B_x \times B_y \,\middle|\, \nu \le \frac{v_0 w_0}{v_0 + w_0} \right\}.$$

6 Lyapunov-Based Control Design

In this section we provide three examples of designing control for vehicles with lift and drag, using the Lyapunov functions presented in this paper.

6.1 Pure Torque Control for a Glider With Equal Added Masses

We consider a winged underwater vehicle with equal added masses, equipped with pitching moment control. The pitching moment control can be realized by the motion of an internal mass or by the use of an elevator. Both control methods induce additional dynamics that we ignore here. We model the pitching moment control as a pure torque. In Subsection 6.3 we consider pitching moment control using an elevator and include additional dynamics due to moment-to-force coupling.

The corresponding equations of motion of the vehicle are

$$\begin{aligned}
\dot{V} &= -\frac{1}{m_3}\left(m_0 g \sin\gamma + D\right) \\
\dot{\gamma} &= \frac{1}{m_3}\left(-m_0 g \cos\gamma + L\right) \\
\dot{\alpha} &= \Omega_2 - \frac{1}{m_3}\left(-m_0 g \cos\gamma + L\right) \\
\dot{\Omega}_2 &= \frac{1}{J_2}\left(K_{M0} + K_M\alpha + K_q\Omega_2\right)V^2 + u.
\end{aligned}$$

We can choose u to modify the closed-loop equilibrium angle of attack α_e and parameters ϵ_1, ϵ_2 in order to change the equilibrium speed V_e and flight path angle γ_e. We recall from Section 3 that α_e affects both V_e and γ_e. Thus, we cannot change V_e and γ_e independently using pure torque control. However, we can choose u such that we reach a desired closed-loop V_e *or* γ_e as well as set the region of attraction of the resulting steady glide to desired limits. If we set

$$u = \frac{1}{J_2}\left(K_{M0u} + K_{Mu}\alpha + K_{qu}\Omega_2\right)V^2 \tag{17}$$

we achieve the following parameters for the closed-loop system: $\alpha_e = \frac{K_{M0} + K_{M0u}}{K_M + K_{Mu}}$, $\epsilon_1 = \frac{m_3}{K_{D_e} V_e}\frac{(K_q + K_{qu})}{(K_M + K_{Mu})}$ and $\epsilon_2 = \frac{m_3}{K_{D_e} V_e^3}\frac{J_2}{(K_q + K_{qu})}$, where V_e denotes the closed-loop equilibrium speed. Clearly we can select control gains K_{qu} and K_{Mu} to obtain small enough ϵ_1 and ϵ_2. We note that $\mu = \max\{\epsilon_1, \epsilon_2\}$ directly affects the size of the region of attraction. A smaller μ provides a larger region of attraction. We also note that we can select control gain K_{M0u} after selecting K_{Mu} and K_{qu} to achieve any desirable closed-loop α_e, which is determined by the desired V_e or α_e.

In order to simplify the adjustment of closed-loop region of attraction we can choose the control gains such that ϵ_1/ϵ_2 remains unaffected. This can be done by imposing the following constraint between K_{Mu} and K_{qu};

$$\frac{(K_q + K_{qu})^2}{(K_M + K_{Mu})} V_e^2 = \frac{K_q^2}{K_M} V_{e,ol}^2, \tag{18}$$

where $V_{e,ol}$ is the open-loop equilibrium speed. This constraint ensures invariance of r_1 and r_2 to changes in the control gains. This way μ_d, which is a function of r_1 and r_2, also remains unaffected by the control gains and will depend only on B_x and B_y that are determined by the desired region of attraction. Then, given a desired region of attraction, we can select K_{qu} and K_{Mu} (bounded by the constraint given by equation (18)) such that $\mu < \mu_d$.

The discussion of this subsection is summarized by the following corollary:

Corollary 5 *The pure torque control law (17) stabilizes the glider to a closed-loop equilibrium with a desired glider* *speed or a desired flight path angle. Furthermore, the coefficients of the control law can be adjusted to provide desired region of attraction guarantees.*

6.2 Buoyancy Control for an Underwater Glider

Underwater gliders are equipped with mechanisms for changing their buoyancy. In this subsection we use buoyancy control for adjusting the desired equilibrium speed of the underwater glider.

Given a desired equilibrium speed V_e we determine the corresponding equilibrium buoyancy m_{0e} as

$$m_{0e} = \pm \frac{\sqrt{K_{D_e}^2 + K_{L_e}^2} V_e^2}{g}, \tag{19}$$

where the sign of m_{0e} is determined by γ_e. m_{0e} is positive for negative γ_e and vice-versa.

The buoyancy control is modelled as follows:

$$\frac{d\bar{m}_0}{dt_n} = u, \tag{20}$$

where $\bar{m}_0 = \frac{m_0 - m_{0e}}{m_e}$ and m_e is the total equilibrium mass of the glider. Extensions of our results to more complex buoyancy control models should be straightforward.

We choose the control law

$$u = -K_b \bar{m}_0, \tag{21}$$

K_b a scalar control gain, which is positive.

Equations (3)-(6) (with $\Delta m = 0$) along with equation (20) describe the (somewhat idealized) dynamics of the underwater glider with changing buoyancy and equal added masses. Using the Lyapunov function

$$\nu_b = \nu + \frac{1}{2}\bar{m}_0^2, \tag{22}$$

where ν is the composite Lyapunov function derived for the constant buoyancy case in Section 5, we can prove the following:

Corollary 6 *The control law (21) renders the closed-loop equilibrium corresponding to a desired glider speed V_e locally asymptotically stable for all*

$$K_b > \frac{1}{2}\left(\frac{m_e g}{K_{D_e} V_e^2}\right)^2. \tag{23}$$

A sketch of the proof of the above result follows.

6.2.1 Lyapunov Function for a Buoyancy Controlled Underwater Glider

Just like for an underwater glider with constant buoyancy, we first non-dimensionalize the equations of motion such that the desired steady glide corresponds to the origin. The boundary-layer model is identically equal to the model represented by equation (14) and the proof of its stability follows from Proposition 1. The reduced system is given by equation (15) along with equation (20). We consider the Lyapunov function ν_b given by equation (22). The derivative of ν_b is

$$\frac{d\nu_b}{dt_n} = \frac{d\nu}{dt_n} + \frac{1}{2}\frac{d\bar{m}_0^2}{dt_n}.$$

We calculate

$$\frac{d\nu}{dt_n} = -\left(A^2 + B^2\right) - \frac{m_e \bar{m}_0 g}{K_{D_e} V_e^2}\Big\{A \sin\left(\bar{\gamma} + \gamma_e\right) - B\cos\left(\bar{\gamma} + \gamma_e\right)\Big\}, \tag{24}$$

where $A = \left(1 + \bar{V}\right)^2 - \cos\bar{\gamma}$ and $B = \sin\bar{\gamma}$. We consider the initial value of $|\bar{m}_0|$ to be less than some positive number η. Then $|\bar{m}_0(t)| \leq \eta \; \forall \, t \geq 0$. We derive

$$-\frac{m_e \bar{m}_0 g}{K_{D_e} V_e^2}\left\{A \sin\left(\bar{\gamma} + \gamma_e\right) - B\cos\left(\bar{\gamma} + \gamma_e\right)\right\} \leq \frac{m_e g \eta \sqrt{A^2 + B^2}}{K_{D_e} V_e^2}. \tag{25}$$

We have

$$\frac{1}{2}\frac{d\bar{m}_0^2}{dt_n} = -K_b \bar{m}_0^2. \tag{26}$$

Substituting relations (25)-(26) in equation (24) we find

$$\frac{d\nu_b}{dt_n} \leq \frac{m_e g |m_0| \sqrt{A^2 + B^2}}{K_{D_e} V_e^2} - \left(A^2 + B^2\right) - K_b |m_0|^2. \tag{27}$$

Since

$$\left(\frac{m_e g}{K_{D_e} V_e^2}\right) |m_0| \sqrt{A^2 + B^2} \leq \frac{1}{2}(c_{wd}^2 |m_0|^2 + \left(A^2 + B^2\right),$$

where $c_{wd} = \frac{m_e g}{K_{D_e} V_e^2}$, we can compute the following from inequality (27):

$$\frac{d\nu_b}{dt_n} \leq -\frac{1}{2}\left(A^2 + B^2\right) - \left(K_b - \frac{c_{wd}}{2}\right)|m_0|^2.$$

We note that

$$A^2 + B^2 = \left\{\left(\bar{V}(\bar{V} + 2)\right)^2 + 4(1 + \bar{V})^2 \sin^2\left(\frac{\bar{\gamma}}{2}\right)\right\}.$$

Using relation (16), we can conclude that, if $K_b > c_{wd}^2/2$, then

$$\frac{d\nu_b}{dt_n} \leq -\min\left(e^2\left(1+\bar{V}_{min}\right)^2, K_b - (c_{wd}/2)\right) \cdot \left(\bar{V}^2 + \bar{\gamma}^2 + \bar{m}_0^2\right).$$

Furthermore $\frac{d\nu_b}{dt_n} = 0$ only at equilibrium. Thus, we can conclude the exponential stability of the origin of the reduced system.

The interconnection conditions hold since the vector field describing our system is bounded. We omit the calculation of the coefficients pertaining to the interconnection conditions for the sake of brevity. We note that for a given set of underwater glider parameters we can always choose a small enough B_x and B_y where the interconnection conditions are satisfied.

Now, invoking the result of Theorem 4 we conclude local asymptotic stability of the desired equilibrium for the buoyancy controlled underwater glider.

6.3 Elevator Control for an Underwater Glider in the Presence of Moment-to-Force Coupling

In this subsection we consider an underwater glider equipped with buoyancy control as well as an elevator that controls the pitching moment. The elevator is modeled as changing the effective angle of attack of the underwater glider by an amount u_2. We also consider a coupling factor δ that describes an additional force induced by elevator action. This force is given by $\delta u_2 V^2$ and it acts along the 3-axis of the underwater glider as shown in Fig. 2. We note that this model of pitching moment control is similar to the model presented in (Tomlin et al. 1995, Al-Hiddabi & McClamroch 1999) for a Conventional Takeoff and Landing Aircraft.

The non-dimensionalized equations of motion describing the dynamics of the underwater glider having equal added masses with buoyancy and elevator controls are

$$\frac{d\bar{V}}{dt_n} = -\frac{1}{K_{D_e}V_e^2}\left\{m_0 g\sin(\bar{\gamma}+\gamma_e) + D - f\sin\left(\bar{\alpha}+\alpha_e\right)\right\} \tag{28}$$

$$\frac{d\bar{\gamma}}{dt_n} = \frac{1}{K_{D_e}V_e^2(1+\bar{V})}\left\{-m_0 g\cos(\bar{\gamma}+\gamma_e) + L + f\cos\left(\bar{\alpha}+\alpha_e\right)\right\} =: \bar{E}_2' \tag{29}$$

$$\frac{d\bar{m}_0}{dt_n} = u_1, \tag{30}$$

$$\epsilon_1\frac{d\bar{\alpha}}{dt_n} = \bar{\Omega}_2 - \epsilon_1\bar{E}_2' \tag{31}$$

$$\epsilon_2\frac{d\bar{\Omega}_2}{dt_n} = -\left(\bar{\alpha}+\bar{\Omega}_2-\bar{u}_2\right)\left(1+\bar{V}\right)^2, \tag{32}$$

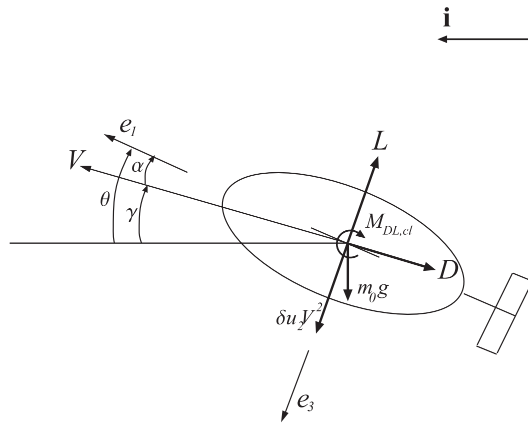

Fig. 2. All external forces and moments acting on underwater glider. Forces D and L are the hydrodynamic lift and drag. $M_{DL,cl} = M_{DL} + K_M u_2 V^2$ is the closed-loop hydrodynamic pitching moment, $\delta u_2 V^2$ is the elevator induced force, and $m_0 g$ is the net force due to gravity.

where $\bar{E}_2'$ is the right-hand-side of equation (29), $f = \delta u_2 V_e^2 (1+\bar{V})^2$, $\bar{u}_2 = u_2 - u_{2e}$; u_{2e} being a reference value of u_2. Since we are interested in steady motions we take u_{2e} to be the equilibrium value of u_2. We further note that in the presence of elevator control the equilibrium angle of attack is $\alpha_e = -\frac{K_{M0}}{K_M} + u_{2e}$.

Given a desired steady speed and flight path angle we first calculate the corresponding equilibrium buoyancy and angle of attack from the translational dynamics (equations (28)-(29)). Then we choose

$$u_1 = -K\bar{m}_0 \tag{33}$$

$$u_2 = u_{2e}. \tag{34}$$

Corollary 7 *The control law (33)-(34) provides an asymptotically stable, desired steady glide if the effective equilibrium drag constant*

$$K_{D_e}' = K_{D0} + K_D\alpha_e^2 - \delta u_{2e}V_e^2\sin\alpha_e > 0, \tag{35}$$

and K_b satisfies the inequality (23).

A sketch of the proof of the above result is given in Subsection 6.3.1. We note that the condition (35) holds provided the coupling factor δ is small enough and/or the elevator control is small enough. For nominal parameter values (35) holds for a wide range of desired equilibria.

6.3.1 Lyapunov Function for an Underwater Glider with Buoyancy and Elevator Controls

We note that since $u_2 = u_{2e}$, $\bar{u}_2 = 0$. Now, the boundary-layer model is identically equal to the model represented by equation (14) and the proof of its stability follows from Proposition 1.

The equations describing the reduced model are

$$\frac{d\bar{V}}{dt_n} = -\frac{1}{K_{D_e} V_e^2}\left\{(\bar{m}_0 + m_{0e})\, g \sin(\bar{\gamma} + \gamma_e) + K'_{D_e}\left(1+\bar{V}\right)^2\right\}$$

$$\frac{d\bar{\gamma}}{dt_n} = \frac{1}{K_{D_e} V_e^2 \left(1+\bar{V}\right)^2}\left\{-(\bar{m}_0 + m_{0e})\, g \cos(\bar{\gamma} + \gamma_e) + K'_{L_e}\left(1+\bar{V}\right)^2\right\}$$

$$\frac{d\bar{m}_0}{dt_n} = -K_b \bar{m}_0,$$

where $K'_{D_e} = K_{D0} + K_D \alpha_e^2 - \delta u_{2e} V_e^2 \sin\alpha_e$ and $K'_{L_e} = K_{L0} + K_L \alpha_e + \delta u_{2e} V_e^2 \cos\alpha_e$ are the effective drag and lift coefficients respectively. The above equations are identical to the equations of the reduced model considered in Section 6.2.1, except that we now have K'_{D_e} and K'_{L_e} instead of K_{D_e} and K_{L_e}. Thus, the proof of the stability of the equilibrium of the reduced system follows. However, we need to ensure that $K'_{D_e} > 0$. Recall that while non-dimensionalizing the equations of motion in Section 4 we defined $t_n = t/\tau_s$ where $\tau_s = \frac{m_3}{K_{D_e} V_e}$. For the present problem we use $\tau_s = \frac{m_3}{K'_{D_e} V_e}$. We need K'_{D_e} to be greater than zero so that the non-dimensional time t_n always has the same sign as t. Then stability of the system with respect to the time variable t_n would imply stability with respect to t. Thus, condition (35) along with condition (23) ensures the stability of the equilibrium of the reduced system.

The interconnection conditions hold since the vector field describing our system is bounded, as discussed in Section 6.2.1. Thus we can invoke the result of Theorem 4 to conclude local asymptotic stability of the desired equilibrium for the underwater glider with buoyancy and elevator controls.

6.3.2 Numerical Example

We illustrate the stability of the closed-loop system by way of a numerical simulation for an underwater glider with the following parameters: $m = m_1 - m_0 = m_3 - m_0 = 28$ kg, $J_2 = 0.1$ kgm^2, $K_{L0} = 0$ N(s/m)^2, $K_L = 300$ N(s/m)^2, $K_{D0} = 10$ N(s/m)^2, $K_D = 100$ N(s/m)^2, $K_q = -5$ Nms(s/m)^2, $K_{M0} = 1$ Nm(s/m)^2, $K_M = -40$ Nm(s/m)^2, $\delta = 0.3$. The goal is to stabilize the glider to an equilibrium speed $V_e = 1$ m/s and a flight path angle γ_e = -45^o. This corresponds to an equilibrium angle of attack $\alpha_e = 1.93^o$, buoyancy $m_{0e} = 1.460$ kg, calculated using equation (19) and equation (7). The time-scaling parameters are $\epsilon_1 = 0.0429$, $\epsilon_2 = 0.0069$. Figure 3 shows the motion of the glider in the longitudinal plane and the evolution of the system states for 20 s.

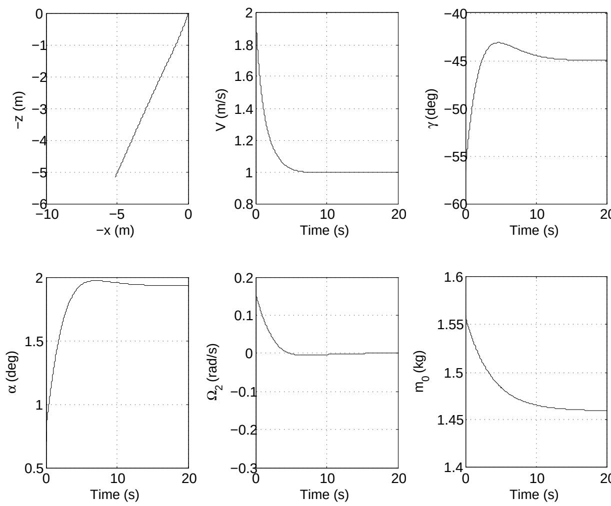

Fig. 3. Elevator control simulation

7 Final Remarks

In this paper we have derived a Lyapunov function to prove the stability of steady glides for vehicles in air or the water, subject to aerodynamic forcing. A key ingredient is the conservation law derived by Lanchester in his original analysis of the aircraft phugoid mode model. We have also presented a Lyapunov-based approach to derive stabilizing control laws for underwater gliders with different control configurations. Future work will focus on extending this approach to vehicles with greater degrees of control actuation, as well as utilizing the results presented here in designing control laws for tracking desired position trajectories.

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A Interconnection Conditions

In this section we calculate the coefficients of the interconnection conditions of the proof of Theorem 4 for the underwater glider model (8)-(11) restricted to $(x, y, \epsilon_i) \in B_x \times B_y \times [0, \epsilon_i^*]$, where $B_x \in \mathbb{R}^2$ is a neighborhood of $(\bar{V}, \bar{\gamma}) = (0, 0)$ such that $-1 < \bar{V}_{min} < \bar{V} < \bar{V}_{max}$, $-\pi \leq \bar{\gamma} < \pi$, and $B_y \in \mathbb{R}^2$ is a neighborhood of $(\bar{\alpha}, \bar{\Omega}_2) = (0, 0)$ such that $\bar{\alpha}_{min} < \bar{\alpha} < \bar{\alpha}_{max}$.

A.1 Condition (1)

We denote the left hand side of this condition by C_1. We calculate

$$\begin{aligned} C_1 = \frac{1}{K_{D_e}} \Bigg[& K_D \left\{ (\bar{\alpha} + \alpha_e)^2 - \alpha_e^2 \right\} \left\{ \left(1 + \bar{V}\right)^2 - \cos(\bar{\gamma}) \right\} \\ & + K_L \bar{\alpha} \sin \bar{\gamma} \Bigg] \left(1 + \bar{V}\right)^2 \\ \leq \frac{1}{K_{D_e}} \Bigg[& K_D \Big\{ (|\bar{\alpha}|_{max} + 2|\alpha_e|) \left(1 + \bar{V}\right) |\bar{\alpha}| \Big\} \\ & \times \left\{ \left(1 + \bar{V}\right) \left| \bar{V}^2 + 2\bar{V} + 2\sin^2(\frac{\bar{\gamma}}{2}) \right| \right\} \\ & + K_L \left(1 + \bar{V}\right) |\bar{\alpha}| \left| \left(1 + \bar{V}\right) 2 \sin(\frac{\bar{\gamma}}{2}) \right| \Bigg], \end{aligned} \tag{A.1}$$

where $|\bar{\alpha}|_{max} = \max\{-\bar{\alpha}_{min}, \bar{\alpha}_{max}\}$. We observe that $0 \leq \left(1 + \bar{V}\right) |\bar{\alpha}| \leq \left(1 + \bar{V}\right) \sqrt{\bar{\alpha}^2 + \bar{\Omega}_2^2} = \sigma$. Furthermore, by straightforward computation one can see that $\left(1 + \bar{V}\right) \left|\bar{V}^2 + 2\bar{V} + 2\sin^2(\frac{\bar{\gamma}}{2})\right| \leq \left(1 + (\sqrt{2} + 1)\bar{V}_{max}\right) \psi$, and that $\left|\left(1 + \bar{V}\right) 2 \sin(\frac{\bar{\gamma}}{2})\right| \leq \psi$.

Using the above results in (A.1), we find that $C_1 \leq \frac{1}{K_{D_e}} \left[K_D \left(|\bar{\alpha}|_{max} + 2|\alpha_e|\right) \left(1 + (\sqrt{2} + 1)\bar{V}_{max}\right) + K_L \right] \sigma\psi = \beta_1 \sigma \psi + \mu \gamma_1 \psi^2$, with $\gamma_1 = 0$ and $\beta_1 = \frac{1}{K_{D_e}} \Big[K_L + K_D \left(|\bar{\alpha}|_{max} + 2|\alpha_e|\right) \left(1 + (\sqrt{2} + 1)\bar{V}_{max}\right) \Big]$.

A.2 Condition (2)

We denote the left hand side of this condition by C_2. We calculate

$$\begin{aligned} C_2 = \frac{\mu \left(c_1 \bar{\alpha} + c_3 \bar{\Omega}_2\right)}{K_{D_e} V_e^2 \left(1 + \bar{V}\right)} \Big\{ & m_0 g \cos(\bar{\gamma} + \gamma_e) \\ & - (K_{L0} + K_L(\bar{\alpha} + \alpha_e)) V_e^2 \left(1 + \bar{V}\right)^2 \Big\}. \end{aligned}$$

Using the fact that $m_0 g \cos \gamma_e = K_{L_e} V_e^2$, we find

$$C_2 = -p_1 s_1 \bar{\Omega}_2 - p_2 \bar{\alpha} \bar{\Omega}_2 - p_3 s_1 \bar{\alpha} - p_4 \bar{\alpha}^2,$$

where $p_1 = \frac{\mu}{(1+\bar{V})}\left(\frac{K_{L_e}}{K_{D_e}\cos\gamma_e}\right)$, $p_2 = \frac{\mu K_L\left(1+\bar{V}\right)}{K_{D_e}} > 0$, $p_3 = p_1\left\{r_1 + (r_1+r_2)\left(1+\bar{V}\right)^2\right\}$, $p_4 = \frac{p_2 p_3}{p_1} > 0$, $s_1 = \left(\bar{V}^2+2\bar{V}\right)\cos\gamma_e + 2\sin\frac{\bar{\gamma}}{2}\sin\left(\frac{\bar{\gamma}+\gamma_e}{2}\right)$. Since the last term in the expression of C_2 is nonpositive, we can write

$$C_2 \leq |p_1||s_1|\left|\bar{\Omega}_2\right| + p_2\left|\bar{\alpha}\right|\left|\bar{\Omega}_2\right| + |p_3||s_1|\left|\bar{\alpha}\right|. \qquad \text{(A.2)}$$

We closely examine $|s_1|$: $|s_1| \leq \left|\left(\bar{V}^2+2\bar{V}\right)\right| + 2\left|\sin\frac{\bar{\gamma}}{2}\right| = s_1^1 + s_1^2$, where $s_1^1 = \left|\left(\bar{V}^2+2\bar{V}\right)\right| + 2\left(1+\bar{V}\right)\left|\sin\frac{\bar{\gamma}}{2}\right|$ and $s_1^2 = -2\bar{V}\left|\sin\frac{\bar{\gamma}}{2}\right|$. We can easily calculate that $s_1^1 \leq \sqrt{2}\psi$. We have $s_1^2 \leq 2\left|\bar{V}\right|\left|\sin\frac{\bar{\gamma}}{2}\right| = \frac{1}{(1+\bar{V})}\left|\bar{V}\right|\left|\left(1+\bar{V}\right)2\sin\frac{\bar{\gamma}}{2}\right| \leq \frac{|\bar{V}|}{(1+\bar{V})}\psi$. Thus, $|s_1| \leq \sqrt{2}\psi + \frac{|\bar{V}|}{(1+\bar{V})}\psi$. We also have $p_2\left|\bar{\alpha}\right|\left|\bar{\Omega}_2\right| = \frac{p_2}{(1+\bar{V})}\left(1+\bar{V}\right)\left|\bar{\alpha}\right|\left|\bar{\Omega}_2\right| \leq \frac{p_2}{2(1+\bar{V})}\left(1+\bar{V}\right)\left[\left|\bar{\alpha}\right|^2 + \left|\bar{\Omega}_2\right|^2\right] = \frac{p_2}{2(1+\bar{V})}\sigma^2$. Moreover, $\left|\bar{\alpha}\right| = \frac{1}{(1+\bar{V})}\left(1+\bar{V}\right)\sqrt{\bar{\alpha}^2} \leq \frac{1}{(1+\bar{V})}\sigma$. Similarly $\left|\bar{\Omega}_2\right| \leq \frac{1}{(1+\bar{V})}\sigma$.

Substituting the above results and the expressions for $|p_1|$, $|p_3|$ in (A.2) we get

$$\begin{aligned} C_2 \leq{}& \frac{\mu|K_{L_e}|}{K_{D_e}|\cos(\gamma_e)|}\left\{\sqrt{2} + \frac{|\bar{V}|}{(1+\bar{V})}\right\} \\ &\times\left\{\frac{(1+r_1)}{\left(1+\bar{V}\right)^2} + r_1 + r_2\right\}\sigma\psi + \frac{\mu K_L}{2K_{D_e}}\sigma^2 \\ \leq{}& \mu\beta_2^{'}\psi\sigma + \mu\gamma_2^{'}\sigma^2, \end{aligned}$$

with $\beta_2^{'} = \frac{|K_{L_e}|}{K_{D_e}|\cos(\gamma_e)|}\left\{\sqrt{2} + \frac{|\bar{V}|_{max}}{(1+\bar{V}_{min})}\right\} \times \left\{\frac{(1+r_1)}{(1+\bar{V}_{min})^2} + r_1 + r_2\right\}$, $\gamma_2^{'} = \frac{K_L}{2K_{D_e}}$; $|\bar{V}|_{max} = \max\{-\bar{V}_{min}, \bar{V}_{max}\}$.

A.3 Condition (3)

We denote the left hand side of this condition by C_3. We calculate

$$\begin{aligned} C_3 ={}& \Bigg|\left\{\frac{K_D\left(\bar{\alpha}+2\alpha_e\right)\bar{\alpha}V_e^2}{\left(1+\bar{V}\right)} + \frac{m_0 g\left(\sin\gamma - \sin\gamma_e\right)}{\left(1+\bar{V}\right)^3}\right. \\ &\left. + \frac{K_{D_e}V_e^2\left(\bar{V}^2+2\bar{V}\right)}{\left(1+\bar{V}\right)^3}\right\}\frac{r_1\bar{\Omega}_2^2}{r_2} \\ &-\left\{K_D\left(\bar{\alpha}+2\alpha_e\right)\bar{\alpha}V_e^2\left(1+\bar{V}\right)^2 + K_{D_e}V_e^2\left(\bar{V}^2+2\bar{V}\right)\right. \\ &\left. + m_0 g\left(\sin\gamma - \sin\gamma_e\right)\right\}\left(1+\bar{V}\right)\left(r_1+r_2\right)\bar{\alpha}^2\Bigg|\frac{1}{K_{D_e}V_e^2} \\ \leq{}& \left[\left\{\frac{K_D\left(\bar{\alpha}^2+2\bar{\alpha}\alpha_e\right)_{max}V_e^2}{\left(1+\bar{V}_{min}\right)} + \frac{2\left|m_0 g\right|}{\left(1+\bar{V}_{min}\right)^3}\right\}\frac{r_1\bar{\Omega}_2^2}{r_2}\right. \\ &+\left(r_1+r_2\right)\left\{K_D\left(\bar{\alpha}^2+2\bar{\alpha}\alpha_e\right)_{max}V_e^2\left(1+\bar{V}_{max}\right)^2\right. \\ &\left.\left. + 2\left|m_0 g\right|\right\}\left(1+\bar{V}_{max}\right)\bar{\alpha}^2\right]\frac{1}{K_{D_e}V_e^2} \\ \leq{}& \beta_2^{''}\psi\sigma + \gamma_2^{''}\sigma^2, \end{aligned}$$

with $\gamma_2^{''} = \frac{1}{K_{D_e}V_e^2}\max\left[\left\{\frac{2|m_0 g|}{(1+\bar{V}_{min})^3} + \frac{K_{D_e}V_e^2\left(\bar{V}^2+2\bar{V}\right)}{\left(1+\bar{V}_{min}\right)^3} + \frac{K_D\left(\bar{\alpha}^2+2\bar{\alpha}\alpha_e\right)_{max}V_e^2}{\left(1+\bar{V}_{min}\right)}\right\}, \left\{2|m_0 g| + K_{D_e}V_e^2\left(\bar{V}^2+2\bar{V}\right)_{max} + K_D\left(\bar{\alpha}^2+2\bar{\alpha}\alpha_e\right)_{max}\left(1+\bar{V}_{max}\right)^2V_e^2\right\}\left(1+\bar{V}_{max}\right)\right]$

and $\beta_2^{''} = 0$; $(\bar{\alpha}^2 + 2\bar{\alpha}\alpha_e)_{max}$ is the maximum value of $(\bar{\alpha}^2 + 2\bar{\alpha}\alpha_e)$ in B_y. In the first inequality of the above calculation we have made use of the fact that $|\sin\gamma - \sin\gamma_e| \leq 2$.

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