Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping

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Abstract— **We extend the method of controlled Lagrangians (CL) to include potential shaping, which achieves complete state-space asymptotic stabilization of mechanical systems. The CL method deals with mechanical systems with symmetry and provides symmetry-preserving kinetic shaping and feedback-controlled dissipation for state-space stabilization in all but the symmetry variables. Potential shaping complements the kinetic shaping by breaking symmetry and stabilizing the remaining state variables. The approach also extends the method of controlled Lagrangians to include a class of mechanical systems without symmetry such as the inverted pendulum on a cart that travels along an incline.**

I. Introduction

This paper continues the development in [13] of the *method of controlled Lagrangians* (CL), a constructive method for stabilizing mechanical systems. Various supplementary and additional results have appeared in [9], [10], [11], and [12]. Our main purpose is to introduce potential shaping into the CL method. This allows us to achieve complete state-space stabilization with large regions of attraction for underactuated systems such as the inverted pendulum on a cart. Preliminary tracking results are obtained. The class of mechanical systems considered, which includes balance systems, tend to be difficult to control; for example, they are often not feedback linearizable.

The CL Method. We consider a class of control laws for mechanical systems with symmetry, whose closed-loop dynamics is in Lagrangian form. This has the advantage that stabilization can be understood using energy-based Lyapunov functions. Correspondingly, one gets large and computable basins of stability, which become asymptotically stable when dissipative controls are added. [13] gives sufficient conditions, called *matching conditions* under which the CL method gives a control law that yields a closed-loop system in Lagrangian form. These *matching* conditions ensure that the Euler-Lagrange equations derived from the controlled Lagrangian are consistent with available control inputs, i.e., they match the controlled Euler-Lagrange equations for the given mechanical system.

The CL has a reshaped kinetic energy that retains the original symmetry. In [13], feedback-controlled dissipation was added to prove asymptotic stabilization in all state variables modulo the symmetry group variables. For the inverted pendulum on the cart, we drive the pendulum to the upright position and the cart to rest but not necessarily positioned at the origin. This limitation will be overcome in the present work.

History and Related Literature. The CL method has its origins in [8] and [16]. Our potential shaping approach is inspired by [12] and [28]. Other relevant work involving energy methods in control and stabilization includes [1], [3], [18], [26],[33], [34], [37], [38], and [41]. In [6], we relate the potential shaping approach here to that of [24], [25], and [40]. It would also be of interest to extend the methods here to more complex robotic systems, as in [21].

The work of [2], [22], and [23] studies the CL method from the point of view of matching Lagrangians defined in terms of general metric tensors. This has the advantage of generality and gives geometric insight into the problem, but it has the disadvantage that one is left with a rather general PDE to be solved in order to make the method effective in applications. We have focussed on techniques that give explicit and constructive matching conditions, control laws and stability criteria.

Nonlinear stabilization of the inverted pendulum on a cart has been studied elsewhere in the literature as it is a representative nonlinear problem not easily treated with traditional methods. For example, in [32] and [39], methods for stabilization of nonlinear systems in "feedforward" form are developed and applied to this example.

Main Results. As discussed above, in this paper we continue the strategy in [13] by augmenting the construction to include symmetry-breaking modifications to the potential energy. This provides the means to stabilize all state variables; for instance, in the cart–pendulum example, the cart position can be driven to the origin as well.

Following [12], we extend the class of mechanical systems considered to include those with an original potential energy that breaks symmetry. For example, the extended class includes the inverted pendulum on a cart that travels on an incline. The potential energy of this system does not have translational symmetry because it is a function of the cart position as well as the pendulum position. (The equations are translation invariant, but this symmetry does not lead to a conservation law in the naive sense.)

Finally, we also indicate in this paper how the results can be used for tracking problems. This topic is treated in a preliminary way here; much more needs to be done in this area, but our results indicate that the approach should be of interest in this area.

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Outline. In §II we outline the CL approach to stabilization and review matching and stabilization by kinetic shaping. In §III we introduce potential shaping and present sufficient conditions for matching. In §IV we provide sufficient conditions and the construction for complete state-space stabilization. In §V we prove the asymptotic stabilizability of the equilibria. In §VI we apply the construction to the inverted pendulum on a cart that travels on an incline. In §VII we examine the spherical pendulum on an inclined plane and in §VIII we use these methods to show that some interesting tracking problems can be handled. Finally, §IX presents some simulations of the techniques for the inverted pendulum to show their effectiveness.

II. Method of Controlled Lagrangians

We briefly review the CL approach to (partial state-space) stabilization by kinetic shaping as presented in [13] (see also [9], [10],[11],[12]). This section is a brief summary only of the key results of part I that are essential to the development in the rest of this paper. One begins with a mechanical system with an uncontrolled (free) Lagrangian L equal to kinetic energy minus potential energy. We modify the kinetic energy to produce a new CL, which describes the dynamics of the controlled closed-loop system.

Configuration Space and Symmetry Group. Suppose our system has configuration space Q and that a Lie group G acts freely and properly on Q. The goal of kinetic shaping is to control the variables lying in the *shape, or orbit space* $S = Q/G$ using controls that act directly on the variables lying in G (see [23] for a discussion of the geometric structure of actuation). *Throughout this paper we will assume that* G *is an abelian group.*

Lagrangian and the Metric Tensor. Assume that $L : TQ \to \mathbb{R}$ is invariant under the given action of G on Q. In many examples the invariance amounts to the L being cyclic in the G-variables, which gives a conservation law for the free system. The construction preserves the invariance of the Lagrangian, thus providing a modified or *controlled* conservation law. The essence of the modification of L involves changing the metric tensor $g(\cdot\,,\cdot)$ that defines the kinetic energy $\frac{1}{2}g(\dot q, \dot q)$. The tangent bundle TQ can be split into a sum of horizontal and vertical parts defined as follows: for each tangent vector v_q to Q at a point $q \in Q$, we can write a unique decomposition

$$v_q = \operatorname{Hor} v_q + \operatorname{Ver} v_q, \tag{1}$$

such that the vertical part is tangent to the orbits of the G-action and where the horizontal part is the metric orthogonal to the vertical space; that is, it is uniquely defined by requiring the identity

$$g(v_q, w_q) = g(\operatorname{Hor} v_q, \operatorname{Hor} w_q) + g(\operatorname{Ver} v_q, \operatorname{Ver} w_q) \tag{2}$$

where v_q and w_q are arbitrary tangent vectors to Q at the point $q \in Q$. This choice of horizontal space coincides with that given by the *mechanical connection* (see [30]).

Kinetic Shaping. The CL uses a modified kinetic energy, while the potential energy remains unchanged for the moment. Let ξ_Q denote the infinitesimal generator corresponding to a Lie algebra element $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G (see [30] or [31]). Thus, for each $\xi \in \mathfrak{g}$, ξ_Q is a vector field on the configuration manifold Q and its value at a point $q \in Q$ is denoted $\xi_Q(q)$.

Definition II.1: Let τ be a Lie algebra valued G equivariant horizontal one form on Q; that is, a one form with values in the Lie algebra $\mathfrak{g}$ of G that annihilates vertical vectors. The ***τ-horizontal space*** at $q \in Q$ consists of tangent vectors to Q at q of the form $\operatorname{Hor}_\tau v_q = \operatorname{Hor} v_q - [\tau(v)]_Q(q)$, which also defines $v_q \mapsto \operatorname{Hor}_\tau(v_q)$, called the ***$\tau$-horizontal projection***. The ***τ-vertical projection operator*** is defined by $\operatorname{Ver}_\tau(v_q) := \operatorname{Ver}(v_q) + [\tau(v)]_Q(q)$.

Definition II.2: Given g_σ, g_ρ and τ, the ***controlled Lagrangian (CL)*** is defined by $L_{\tau,\sigma,\rho} = K_{\tau,\sigma,\rho} - V$, where

$$K_{\tau,\sigma,\rho}(v) = \tfrac{1}{2}[g_\sigma(\operatorname{Hor}_\tau v_q, \operatorname{Hor}_\tau v_q) + g_\rho(\operatorname{Ver}_\tau v_q, \operatorname{Ver}_\tau v_q)]. \tag{3}$$

The equations corresponding to $L_{\tau,\sigma,\rho}(v)$ will be our closed-loop equations. The new terms appearing in those equations corresponding to the directly controlled variables are interpreted as control inputs. The modifications to the Lagrangian are chosen so that no new terms appear in the equations corresponding to the variables that are not directly controlled. We refer to this process as *matching*.

Once the control law is derived using the CL, the closed-loop stability of an equilibrium can be determined by energy methods, using any available freedom in the choice of τ, g_σ and g_ρ.

Structure of the CL. As shown in [13], the controlled Lagrangian $L_{\tau,\sigma,\rho}(v)$ has the following useful structure.

Theorem II.3: Assume that $g = g_\sigma$ on Hor and Hor and Ver are orthogonal for g_σ. Then

$$L_{\tau,\sigma,\rho}(v) = L(v + \tau(v)_Q) + \tfrac{1}{2}g_\sigma(\tau(v)_Q, \tau(v)_Q) + \tfrac{1}{2}\varpi(v)$$

where $v \in T_qQ$ and $\varpi(v) = (g_\rho - g)(\operatorname{Ver}_\tau(v), \operatorname{Ver}_\tau(v))$,

The coordinate formula for L is

$$L(x^\alpha, \dot x^\beta, \dot\theta^a) = \tfrac{1}{2}g_{\alpha\beta}\dot x^\alpha \dot x^\beta + g_{\alpha a}\dot x^\alpha \dot\theta^a + \tfrac{1}{2}g_{ab}\dot\theta^a\dot\theta^b - V(x^\alpha)$$

and the coordinate formula for $L_{\tau,\sigma,\rho}$ is

$$\begin{aligned} L_{\tau,\sigma,\rho}(v) &= L(x^\alpha, \dot x^\beta, \dot\theta^a + \tau^a_\alpha \dot x^\alpha) + \tfrac{1}{2}\sigma_{ab}\tau^a_\alpha\tau^b_\beta \dot x^\alpha \dot x^\beta + \\ &\tfrac{1}{2}\varpi_{ab}(\dot\theta^a + g^{ac}g_{\alpha c}\dot x^\alpha + \tau^a_\alpha \dot x^\alpha)(\dot\theta^b + g^{bd}g_{\beta d}\dot x^\beta + \tau^b_\beta \dot x^\beta). \end{aligned} \tag{4}$$

Here, θ^a are coordinates for the abelian symmetry group G and x^α are coordinates on the shape space Q/G; σ_{ab} and ϖ_{ab} are the coefficients for the last two terms, respectively, of the expression for $L_{\tau,\sigma,\rho}$ in Theorem II.3, and we let $\rho_{ab} = g_{ab} + \varpi_{ab}$.

Conserved Quantities. The ***controlled conserved quantity*** is given by

$$\tilde J_a := \frac{\partial L_{\tau,\sigma,\rho}}{\partial \dot\theta^a} = \rho_{ab}(\dot\theta^b + g^{bd}g_{\alpha d}\dot x^\alpha + \tau^b_\alpha \dot x^\alpha). \tag{5}$$

Matching. Consider the ***controlled Euler-Lagrange equations*** for the given Lagrangian:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}^a} = u_a.$$

where the controls are in the θ-directions only. ***Matching*** means that we seek controls and τ, σ, ρ such that these equations match the Euler-Lagrange equations for the Lagrangian $L_{\tau,\sigma,\rho}$. Sufficient conditions for matching were developed in [13] (see also [10], [11],[14]). We consider here simplified sufficient conditions for matching that are satisfied for a class of systems that includes the inverted (either planar or spherical) pendulum on a cart. A different perspective on matching is given in [2] and [23]. We give a summary of this perspective in Appendix A along with a discussion of the related paper [29].

For this section, we shall review the situation under the assumption that $g_\rho = g$, that is, $\varpi = 0$. This will be generalized to include nontrivial g_ρ in the next section. The ***simplified matching conditions*** are as follows

SM-1: $\sigma_{ab} = \sigma g_{ab}$ for a constant σ (this defines σ_{ab}),
SM-2: g_{ab} is independent of x^α,
SM-3: $\tau^b_\alpha = -(1/\sigma) g^{ab} g_{\alpha a}$ (this defines τ^b_α),
SM-4: $g_{\alpha a,\delta} = g_{\delta a,\alpha}$ (a second condition on the metric).

We use commas to denote partial differentiation with respect to x^α. The conditions SM-2 and SM-4 imply that the mechanical connection $g^{ab} g_{a\alpha}$ for the given system is flat, i.e., systems that satisfy the simplified matching conditions lack gyroscopic forces. This condition also plays a role in the work of [4], [5] in the context of flat inputs for systems controlled by oscillatory inputs.

Define $\kappa = -1/\sigma$. Under the simplified matching assumptions SM-1 – SM-4, the control law is

$$u_a = -\frac{d}{dt}(\kappa g_{\alpha a} \dot{x}^\alpha). \tag{6}$$

The acceleration terms can be eliminated using the equations themselves so that the control law becomes

$$u_a = -\kappa \left\{ g_{\beta a,\gamma} - g_{\delta a} A^{\delta\alpha} \left[g_{\alpha\beta,\gamma} - \tfrac{1}{2} g_{\beta\gamma,\alpha} - (1+\kappa)\, g_{\alpha d} g^{da} g_{\beta a,\gamma} \right] \right\} \dot{x}^\beta \dot{x}^\gamma + \kappa g_{\delta a} A^{\delta\alpha} \frac{\partial V}{\partial x^\alpha}, \tag{7}$$

where

$$A_{\alpha\beta} = g_{\alpha\beta} - (1+\kappa)\, g_{\alpha d} g^{da} g_{\beta a}. \tag{8}$$

Stabilization. An equilibrium for the controlled system corresponds to x^α_e, $\dot{x}^\alpha_e = 0$ and $\tilde{J}_a = \mu_a$. Let

$$V_\mu(x^\alpha) = V(x^\alpha) + \tfrac{1}{2} g^{ab} \mu_a \mu_b, \tag{9}$$

the amended potential. The following is proved in [13].

Theorem II.4: Assume SM-1 – SM-4 hold. Then, the given equilibrium is stabilized in the sense of Lyapunov (modulo the action of the group G) by the control law (7) provided that the second variation of

$$E_\mu := \tfrac{1}{2} A_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + V_\mu \tag{10}$$

(as a function of the variables x^α) evaluated at the equilibrium, is definite.

III. Matching with Symmetry-Breaking Potentials

In this section we extend the method of controlled Lagrangians to the class of Lagrangian mechanical systems with potential energy that may break symmetry, i.e., we still have a symmetry group G for the kinetic energy for the system but we now have a potential energy of the form $V = V(x^\alpha, \theta^a)$ that need not be G-invariant. Further, we consider a modification to the potential energy that also breaks symmetry in the G variables. Let the potential energy V' for the controlled Lagrangian be defined as

$$V'(x^\alpha, \theta^a) = V(x^\alpha, \theta^a) + V_\epsilon(x^\alpha, \theta^a) \tag{11}$$

where V_ϵ is the modification—to be determined—that depends on a new real parameter ϵ.

Our next goal is to relax the assumption that $g_\rho = g$. We consider the case of mechanical systems for which the simplified matching assumptions SM-1 – SM-4 hold. However, we retain the flexibility afforded by g_ρ.

We note that more general matching conditions are possible and indeed necessary in certain cases – see for example [11]. It is shown in that paper that one can achieve matching for systems where SM-2 does not hold, i.e. the inertial term g_{ab} depends on x^α. This is necessary for analyzing the pendulum on a rotor arm, for example. In this situation g_ρ is not taken be equal to g. A similar situation arises in the case of a system where the configuration space is a nonabelian group crossed with an abelian group – for example the satellite with momentum wheel – see [10],[14].

We consider $g_\rho = \rho g_{ab}$ where ρ is a scalar constant. The controlled Lagrangian takes the form

$$\begin{aligned} L_{\tau,\sigma,\rho,\epsilon}(v) &= L_{\tau,\sigma}(v) \\ &+ \tfrac{1}{2}(\rho - 1) g_{ab} (\dot{\theta}^a + g^{ac} g_{\alpha c} \dot{x}^\alpha + \tau^a_\alpha \dot{x}^\alpha) \\ &\times (\dot{\theta}^b + g^{bd} g_{\beta d} \dot{x}^\beta + \tau^b_\beta \dot{x}^\beta) - V_\epsilon(x^\alpha, \theta^a) \end{aligned} \tag{12}$$

where $L_{\tau,\sigma}(v) = L(x^\alpha, \dot{x}^\beta, \theta^a, \dot{\theta}^a + \tau^a_\alpha \dot{x}^\alpha) + \frac{1}{2}\sigma g_{ab} \tau^a_\alpha \tau^b_\beta \dot{x}^\alpha \dot{x}^\beta$.

The Conservation Law and Control Law. The conjugate momenta $\tilde{J}_a$ to θ^a is

$$\tilde{J}_a = \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{\theta}^a} = \rho g_{ab} (\dot{\theta}^b + g^{bd} g_{\alpha d} \dot{x}^\alpha + \tau^b_\alpha \dot{x}^\alpha). \tag{13}$$

The new Euler-Lagrange equations in the θ^a variables are

$$\frac{d}{dt}\left(\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{\theta}^a}\right) + \frac{\partial V}{\partial \theta^a} + \frac{\partial V_\epsilon}{\partial \theta^a} = 0.$$

Comparing this equation to our controlled θ^a equation, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^a}\right) + \frac{\partial V}{\partial \theta^a} = u_a,$$

the control law can be read off as

$$\begin{aligned} u_a &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^a} - \frac{1}{\rho}\frac{\partial L_{\tau,\sigma,\rho}}{\partial \dot{\theta}^a}\right) + \frac{\rho-1}{\rho}\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\epsilon}{\partial \theta^a} \\ &= -\frac{d}{dt}(g_{ab}\tau^b_\alpha \dot{x}^\alpha) + \frac{\rho-1}{\rho}\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\epsilon}{\partial \theta^a}. \end{aligned} \tag{14}$$

Matching the x-Euler-Lagrange Equations. The next step is to determine conditions so that the Euler-Lagrange equations in the x^α variables match.

Given a Lagrangian L in the variables $(x^\alpha, \dot{x}^\beta)$, we let

$$\mathcal{E}_x(L) := \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha}$$

denote the corresponding ***Euler-Lagrange operator***. We now seek conditions under which the controlled equations for the Lagrangian L imply that

$$\mathcal{E}_x(L_{\tau,\sigma,\rho,\epsilon}) = \frac{d}{dt}\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{x}^\alpha} - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial x^\alpha} = 0.$$

This is the condition we need for matching the complete set of controlled Euler-Lagrange equations with the Euler-Lagrange equations for the controlled Lagrangian.

Using (12) and (13), we compute that

$$\begin{aligned}
\mathcal{E}_x(L_{\tau,\sigma,\rho,\epsilon}) &= \mathcal{E}_x(L_{\tau,\sigma}) + \frac{\rho-1}{\rho}\frac{d}{dt}\left(\tilde{J}_b(g^{bd}g_{\alpha d} + \tau^b_\alpha)\right) \\
&\quad - \frac{\rho-1}{\rho}\tilde{J}_b\left(g^{bd}g_{\beta d,\alpha} + \tau^b_{\beta,\alpha}\right)\dot{x}^\beta + \frac{\partial V_\epsilon}{\partial x^\alpha} \\
&= \mathcal{E}_x(L_{\tau,\sigma}) + \frac{\rho-1}{\rho}(g^{bd}g_{\alpha d} + \tau^b_\alpha)\dot{\tilde{J}}_b \\
&\quad + \frac{\rho-1}{\rho}\tilde{J}_b(g^{bd}g_{\alpha d,\beta} + \tau^b_{\alpha,\beta})\dot{x}^\beta \\
&\quad - \frac{\rho-1}{\rho}\tilde{J}_b\left(g^{bd}g_{\beta d,\alpha} + \tau^b_{\beta,\alpha}\right)\dot{x}^\beta + \frac{\partial V_\epsilon}{\partial x^\alpha} \\
&= \mathcal{E}_x(L_{\tau,\sigma}) + \frac{\rho-1}{\rho}(g^{bd}g_{\alpha d} + \tau^b_\alpha)\dot{\tilde{J}}_b + \frac{\partial V_\epsilon}{\partial x^\alpha}
\end{aligned}$$

where the last equality follows by the simplified matching assumptions. Using the calculation of $\mathcal{E}_x(L_{\tau,\sigma})$ from [13], we compute

$$\begin{aligned}
\mathcal{E}_x(L_{\tau,\sigma,\rho,\epsilon}) &= \frac{1}{\rho}\dot{\tilde{J}}_a\tau^a_\alpha + \frac{\partial V_\epsilon}{\partial x^\alpha} + \frac{\rho-1}{\rho}(g^{ad}g_{\alpha d} + \tau^a_\alpha)\dot{\tilde{J}}_a \\
&= -\frac{\partial V'}{\partial \theta^a}\left(\frac{1}{\rho}\tau^a_\alpha + \frac{\rho-1}{\rho}(g^{ad}g_{\alpha d} + \tau^a_\alpha)\right) + \frac{\partial V_\epsilon}{\partial x^\alpha} \\
&= -\frac{\partial V'}{\partial \theta^a}\left(\tau^a_\alpha + \frac{\rho-1}{\rho}g^{ad}g_{\alpha d}\right) + \frac{\partial V_\epsilon}{\partial x^\alpha} \\
&= -\frac{\partial V'}{\partial \theta^a}\left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right)g^{ad}g_{\alpha d} + \frac{\partial V_\epsilon}{\partial x^\alpha} \qquad (15)
\end{aligned}$$

We define a new matching condition as follows:

SM-5. The potential V satisfies

$$\frac{\partial^2 V}{\partial x^\alpha \partial \theta^a}g^{ad}g_{\beta d} = \frac{\partial^2 V}{\partial x^\beta \partial \theta^a}g^{ad}g_{\alpha d}.$$

In §V, it is shown that SM-5 is the necessary and sufficient condition for the existence of the solution V_ϵ to the following PDE

$$-\left(\frac{\partial V}{\partial \theta^a} + \frac{\partial V_\epsilon}{\partial \theta^a}\right)\left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right)g^{ad}g_{\alpha d} + \frac{\partial V_\epsilon}{\partial x^\alpha} = 0 \quad (16)$$

which makes $\mathcal{E}_x(L_{\tau,\sigma,\rho,\epsilon}) = 0$ in (15).

With respect to equation (16) we note that for $V_\epsilon = 0$ and V independent of θ, there is no condition on ρ. This is because in the special matching situation discussed here ρ is not needed when there is no symmetry breaking. As discussed at the beginning of the section, however, for more general inertia matrices, ρ is needed for matching even in the presence of symmetry (see [11]). In this case condition (16) will need to be modified. A more general matching condition in the presence of a potential was given in [2] and [23]. These computations prove the following theorem, which gives sufficient conditions for matching with symmetry-breaking potentials.

Theorem III.1 (Matching with Potential Shaping) Under Assumptions SM-1, SM-2, SM-3, SM-4, SM-5 the Euler-Lagrange equations for the controlled Lagrangian $L_{\tau,\sigma,\rho,\epsilon}$ coincide with the controlled Euler-Lagrange equations.

Next, we consider stabilization and recompute the stabilizing control law given by (14) as a function of positions and velocities only (i.e., we eliminate acceleration terms).

IV. Stabilization with Symmetry-Breaking Potentials

In the case that the conditions for Theorem III.1 are satisfied, the energy function $E_{\tau,\sigma,\rho,\epsilon}$ for the controlled Lagrangian $L_{\tau,\sigma,\rho,\epsilon}$, that is, the energy function associated to the closed-loop system, can be used as a Lyapunov function. In particular, we use it to assign the remaining freedom in σ, ρ and ϵ to guarantee stability of an equilibrium of interest. Notice that any equilibrium necessarily has the form $(x^\alpha, \theta^a, \dot{x}^\alpha, \dot{\theta}^a) = (x^\alpha_e, \theta^a_e, 0, 0)$.

We note that in this paper we achieve stabilization of an equilibrium for the system, i.e., a fixed point for the flow in the full phase space. This is in contrast to the situation in [13] where we considered stabilization of systems modulo the symmetry group, i.e. stabilization of a *relative equilibrium.*

Conditions for Stabilization. We compute $E_{\tau,\sigma,\rho,\epsilon}$:

$$\begin{aligned}
E_{\tau,\sigma,\rho,\epsilon} &= \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{x}^\alpha}\dot{x}^\alpha + \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{\theta}^a}\dot{\theta}^a - L_{\tau,\sigma,\rho,\epsilon} \\
&= \tfrac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha(\dot{\theta}^a + \tau^a_\beta\dot{x}^\beta) \\
&\quad + \tfrac{1}{2}g_{ab}(\dot{\theta}^a + \tau^a_\alpha\dot{x}^\alpha)(\dot{\theta}^b + \tau^b_\beta\dot{x}^\beta) + \tfrac{1}{2}\sigma g_{ab}\tau^a_\alpha\tau^b_\beta\dot{x}^\alpha\dot{x}^\beta \\
&\quad + \tfrac{1}{2}(\rho-1)g_{ab}(\dot{\theta}^a + g^{ac}g_{\alpha c}\dot{x}^\alpha + \tau^a_\alpha\dot{x}^\alpha) \\
&\qquad \times (\dot{\theta}^b + g^{bd}g_{\beta d}\dot{x}^\beta + \tau^b_\beta\dot{x}^\beta) + V'(x^\alpha, \theta^a). \quad (17)
\end{aligned}$$

The Lagrange-Dirichlet Theorem then gives the following sufficient conditions for Lyapunov stability.

Theorem IV.1 (Lyapunov Stability & Potential Shaping) Assume SM-1 – SM-5 hold. The equilibrium defined by $(x^\alpha_e, \theta^a_e, 0, 0)$ is Lyapunov stable if it is a critical point of V' and if the second derivative of $E_{\tau,\sigma,\rho,\epsilon}$ evaluated at the equilibrium is definite.

Conditions for Asymptotic Stabilization. To achieve asymptotic stability, we add a dissipative control term, i.e.,

$$u_a = u_a^{\rm cons} + \frac{1}{\rho} u_a^{\rm diss}$$

where

$$u_a^{\rm cons} = -\frac{d}{dt}(g_{ab}\tau^b_\alpha \dot{x}^\alpha) + \frac{\rho-1}{\rho}\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\epsilon}{\partial \theta^a}.$$

The Euler-Lagrange equations in terms of the CL are:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{x}^\alpha} - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial x^\alpha} &= \left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right) g^{ad} g_{\alpha d} u_a^{\rm diss} \\
\frac{d}{dt}\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{\theta}^a} - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \theta^a} &= u_a^{\rm diss}. \qquad (18)
\end{aligned}$$

Note that the first of equations (18) is identically zero in the absence of a dissipative control as it should be by matching. The parameters in the controlled Lagrangian are chosen to achieve nonlinear (but not asymptotic) stability. One computes that

$$\begin{aligned}
\frac{d}{dt}E_{\tau,\sigma,\rho,\epsilon} &= \left(\frac{d}{dt}\left(\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{\theta}^a}\right) - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \theta^a}\right)\dot{\theta}^a \\
&\quad + \left(\frac{d}{dt}\left(\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{x}^\alpha}\right) - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial x^\alpha}\right)\dot{x}^\alpha \\
&= \left(\dot{\theta}^a + \left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right) g^{ad} g_{\alpha d}\dot{x}^\alpha\right) u_a^{\rm diss}. \qquad (19)
\end{aligned}$$

Therefore, we can choose

$$u_a^{\rm diss} = c_a^d g_{bd}\left(\dot{\theta}^b + \left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right) g^{be} g_{\alpha e}\dot{x}^\alpha\right). \qquad (20)$$

Here, (c_a^d) is a control gain matrix, which is chosen to be positive (resp. negative) definite if the equilibrium is a maximum (resp. minimum) of $E_{\tau,\sigma,\rho,\epsilon}$; the matrix (c_a^d) may depend on (x^α, θ^a). This choice of control gives

$$\begin{aligned}
\frac{d}{dt}E_{\tau,\sigma,\rho,\epsilon} &= c_a^d g_{bd}\left(\dot{\theta}^a + \left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right) g^{ae} g_{\alpha e}\dot{x}^\alpha\right) \\
&\quad \times \left(\dot{\theta}^b + \left(-\frac{1}{\sigma} + \frac{\rho-1}{\rho}\right) g^{be} g_{\beta e}\dot{x}^\beta\right).
\end{aligned}$$

To get asymptotic stability of the equilibrium, we will use LaSalle's invariance principle. From the above, we see that $d(E_{\tau,\sigma,\rho,\epsilon})/dt$ vanishes on the set $\mathcal{M}$ defined by

$$u_a^{\rm diss} = c_a^d \frac{1}{\rho}(\tilde{J}_d - g_{\alpha d}\dot{x}^\alpha) = 0.$$

Theorem IV.2 (Asymptotic Stabilization) Assume that the hypotheses of the Stabilization Theorem IV.1 as well as the assumptions SM-1 – SM-5 hold. In addition, assume that $\mathcal{M}$ consists only of equilibria and that the dissipative control law is chosen as in (20). Then, the given equilibrium is asymptotically stable.

We investigate specific conditions under which the hypotheses of this theorem can be verified in the next section.

We again define $\kappa = -1/\sigma$. The total control u_a is

$$u_a = u_a^{\rm cons} + \frac{1}{\rho}u_a^{\rm diss} = -\frac{d}{dt}(\kappa g_{\alpha a}\dot{x}^\alpha) + w_a \qquad (21)$$

where

$$w_a = \frac{\rho-1}{\rho}\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\epsilon}{\partial \theta^a} + \frac{1}{\rho}u_a^{\rm diss}.$$

This control law is the sum of our original stabilizing control law without symmetry breaking (6) plus the potential modification and the dissipation term.

Using the same procedure as in [13], we can eliminate accelerations in the control law expression. We compute

$$\begin{aligned}
u_a &= (\text{rhs of (7)}) + \kappa g_{\delta a} A^{\delta\alpha}\frac{1}{\rho} g_{\alpha d} g^{db}\left(-\frac{\partial V'}{\partial \theta^b} + u_b^{\rm diss}\right) + w_a \\
&= -\kappa\left\{g_{\beta a,\gamma} - g_{\delta a}A^{\delta\alpha}\left[g_{\alpha\beta,\gamma} - \tfrac{1}{2}g_{\beta\gamma,\alpha}\right.\right. \\
&\qquad\qquad \left.\left. - (1+\kappa)\, g_{\alpha d} g^{da} g_{\beta a,\gamma}\right]\right\}\dot{x}^\beta \dot{x}^\gamma \\
&\quad + \kappa g_{\delta a}A^{\delta\alpha}\frac{\partial V}{\partial x^\alpha} + \frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\left(1 + \kappa g_{\delta a}A^{\delta\alpha} g_{\alpha d} g^{db}\right)\frac{\partial V'}{\partial \theta^a} \\
&\quad + \frac{1}{\rho}\left(1 + \kappa g_{\delta a}A^{\delta\alpha} g_{\alpha d} g^{db}\right) u_b^{\rm diss}. \qquad (22)
\end{aligned}$$

V. Asymptotic Stabilization with Symmetry-Breaking Potentials

In the previous section we derived a general result on stability, which depends on the invariant set $\mathcal{M}$ in Theorem IV.2 consisting only of equilibria. In this section we give sufficient conditions for this to hold.

Notation. When we say a function f has a maximum or a minimum at x, we will mean that it is a local maximum or a local minimum and that x is a non-degenerate critical point of f.

We begin by deriving an integrability condition for the PDE in (16). Let $(x_e^\alpha, \theta_e^a, 0, 0) \in TQ$ be the equilibrium of interest. When there is no confusion, we sometimes omit the indices $\alpha, \beta, \ldots$ or $a, b, \ldots$ in the coordinate expression of points. By SM-2, SM-4, and the Poincaré Lemma, for each a the one form $g^{ac}g_{\alpha c}dx^\alpha$ is closed and hence locally exact. (Recall that local exactness of this form is equivalent to the fact that the mechanical connection $g^{ab}g_{a\alpha}$ is flat.)

Therefore, there is a function $h : U \to \mathfrak{g}$ for an open subset U in S such that

$$\frac{\partial h^a}{\partial x^\alpha} = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) g^{ac} g_{\alpha c} \quad \text{with} \quad h^a(x_e) = 0. \qquad (23)$$

We introduce a new coordinate chart for Q as follows:

$$(x^\alpha, y^a) = (x^\alpha, \theta^a + h^a(x^\alpha)). \qquad (24)$$

This coordinate change induces the following new local coordinates for TQ:

$$(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) = \left(x^\alpha, \theta^a + h^a(x^\alpha), \dot{x}^\alpha, \dot{\theta}^a + \frac{\partial h^a}{\partial x^\beta}\dot{x}^\beta\right). \qquad (25)$$

Notice that this change of coordinates fixes the equilibrium $(x_e, \theta_e, 0, 0)$, i.e., $(x_e^\alpha, \theta_e^a, 0, 0) = (x_e^\alpha, y_e^a, 0, 0)$.

In the new coordinates, the PDE (16) becomes

$$\frac{\partial V_\epsilon}{\partial x^\alpha} = \frac{\partial V}{\partial y^a}\frac{\partial h^a}{\partial x^\alpha}. \tag{26}$$

Assume that we have a solution V_ϵ to this PDE. Then, the mixed partials of V_ϵ should be equal, i.e.,

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial V}{\partial y^a}\frac{\partial h^a}{\partial x^\beta}\right) = \frac{\partial}{\partial x^\beta}\left(\frac{\partial V}{\partial y^a}\frac{\partial h^a}{\partial x^\alpha}\right). \tag{27}$$

Therefore, (27) becomes a necessary condition for the integrability of the PDE (16).

Now assume that (27) holds. Then, by using the vector calculus, we derive the following solution to the PDE in (26):

$$V_\epsilon(x^\alpha, y^a) = \int_C \frac{\partial V}{\partial y^a}\frac{\partial h^a}{\partial x^\alpha} dx^\alpha + \tilde{V}_\epsilon(y^a)$$

where $\tilde{V}_\epsilon$ is an arbitrary function. We define the curve C as follows: Fix $(x_e^\alpha) \in S$. For each $(x^\alpha, y^a) \in S \times G$, we choose any curve $C \in S \times \{(y^a)\}$ joining (x_e^α, y^a) and (x^α, y^a). Then, the integration is path-independent by (27) and Stokes' Theorem (we regard $\frac{\partial V}{\partial y^a}\frac{\partial h^a}{\partial x^\alpha}dx^\alpha$ as a y^a-dependent one-form on S). In the old coordinates, (27) is expressed as

$$\frac{\partial^2 V}{\partial x^\alpha \partial \theta^a} g^{ad} g_{\beta d} = \frac{\partial^2 V}{\partial x^\beta \partial \theta^a} g^{ad} g_{\alpha d} \tag{28}$$

which is Assumption SM-5. Thus, Assumption SM-5 is a necessary and sufficient condition for the integrability of the PDE (16). In particular, when V is of the form

$$V(x^\alpha, \theta^a) = V_1(x^\alpha) + V_2(\theta^a) = V_1(x^\alpha) + V_2(y^a - h^a(x^\alpha)),$$

(28) is satisfied and the solution V_ϵ is given by

$$V_\epsilon(x^\alpha, y^a) = -V_2(y^a - h^a(x^\alpha)) + \tilde{V}_\epsilon(y^a) \tag{29}$$

where $\tilde{V}_\epsilon$ is an *arbitrary* function.

Kinetic and Potential Shaping. First, we consider the kinetic shaping. By definition of the new metric, we can express the kinetic energy as follows (see [13] for additional details):

$$K_{\tau,\sigma,\rho}(v_q) = \tfrac{1}{2}A_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \tfrac{1}{2}\rho g_{ab}\zeta^a\zeta^b, \tag{30}$$

where $\zeta^a = \dot{y}^a + \frac{1}{\rho}g^{ab}g_{\alpha b}\dot{x}^\alpha$ and $A_{\alpha\beta} = g_{\alpha\beta} - (1 - 1/\sigma)g_{\alpha d}g^{da}g_{a\beta}$ where the latter is the same as in (8). Notice that the vertical part of the kinetic energy can be made negative definite or positive definite in the new vertical space Ver_τ depending on the sign of ρ since (g_{ab}) is a positive definite matrix. To have control of the horizontal part of the kinetic energy, we make the following assumption:

SM-6 The matrix $(g_{a\alpha}(x_e^\alpha))$ is one-to-one.

SM-6 requires that the mechanical connection as a map be injective. It is equivalent to the (*locally*) *strong inertial coupling property* in [36] and the *internal/external convertible system* in [20].

Note that SM-6 requires that $\dim G \geq \dim S$. That is, the number of actuated directions is larger than or equal to the number of unactuated directions. By positive definiteness of the matrix (g_{ab}) and SM-6, the matrix $(g_{\alpha d}g^{da}g_{\beta a})$ is positive definite at x_e^α. Using the standard simultaneous diagonalization technique in linear algebra, one sees that the matrix $A_{\alpha\beta}$ becomes negative definite at x_e^α if

$$1 - \frac{1}{\sigma} > \max\{\,\lambda \mid \det\left(g_{\alpha\beta} - \lambda g_{\alpha a}g^{ab}g_{b\beta}\right)\big|_{x=x_e} = 0\,\}. \tag{31}$$

Then, by continuity the matrix $A_{\alpha\beta}$ is negative definite in a neighborhood of x_e^α. Also, it can be made positive definite if σ satisfies

$$1 - \frac{1}{\sigma} < \min\{\,\lambda \mid \det\left(g_{\alpha\beta} - \lambda g_{\alpha a}g^{ab}g_{b\beta}\right)\big|_{x=x_e} = 0\,\}.$$

Thus, we have complete control over the shape of the kinetic energy under condition SM-6.

In this section we are interested in the system whose potential energy is of the following form:

SM-5′ The potential $V(x^\alpha, \theta^a)$ is of the form,

$$V(x^\alpha, \theta^a) = V_1(x^\alpha) + V_2(\theta^a) \tag{32}$$

where V_1 has a maximum at $(x^\alpha) = (x_e^\alpha)$.

As shown above, this form of potential V satisfies SM-5 with V_ϵ given by (29). The potential V' for the controlled Lagrangian is given in the new coordinates by

$$V'(x^\alpha, y^a) = V_1(x^\alpha) + \tilde{V}_\epsilon(y^a) \tag{33}$$

where $\tilde{V}_\epsilon$ is an arbitrary function on G. When the given potential is of the form SM-5′, then potential shaping alone cannot handle this problem (see [24], [25] and [40] for an account of the potential shaping approach).

The controlled Lagrangian $L_{\tau,\sigma,\rho,\epsilon}$ has the following form in the new coordinates,

$$\begin{aligned} L_{\tau,\sigma,\rho,\epsilon} = \tfrac{1}{2}\left(g_{\alpha\beta} - \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right)g^{ab}g_{\alpha a}g_{\beta b}\right)\dot{x}^\alpha\dot{x}^\beta \\ + g_{\alpha a}\dot{x}^\alpha\dot{y}^a + \tfrac{1}{2}\rho g_{ab}\dot{y}^a\dot{y}^b - V_1(x^\alpha) - \tilde{V}_\epsilon(y^a), \end{aligned} \tag{34}$$

while the Euler-Lagrange equations (18) take the form

$$\begin{aligned} \frac{d}{dt}\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{x}^\alpha} - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial x^\alpha} &= 0, \\ \frac{d}{dt}\frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial \dot{y}^a} - \frac{\partial L_{\tau,\sigma,\rho,\epsilon}}{\partial y^a} &= u_a^{\mathrm{diss}}. \end{aligned} \tag{35}$$

This shows that the coordinate change makes the controlled Lagrangian problem with the dissipative input look exactly like the original Lagrangian problem with a general input. That is, the two Lagrangian systems (L, u) and $(L_{\tau,\sigma,\rho,\epsilon}, u^{\mathrm{diss}})$ are feedback equivalent.

The controlled energy, $E_{\tau,\sigma,\rho,\epsilon}$, may be written as

$$E_{\tau,\sigma,\rho,\epsilon} = K_{\tau,\sigma,\rho} + V_1(x^\alpha) + \tilde{V}_\epsilon(y^a). \tag{36}$$

We want to use $E_{\tau,\sigma,\rho,\epsilon}$ as a Lyapunov function. Because $V_1(x^\alpha)$ has a maximum at $x^\alpha = x_e^\alpha$, it is appropriate to make $E_{\tau,\sigma,\rho,\epsilon}$ have a maximum at $(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) = (x_e^\alpha, y_e^a, 0, 0)$. Choose any $\tilde{V}_\epsilon(y^a)$ with a maximum at $y^a = y_e^a$. Usually a negative definite quadratic function will do. Then $(x_e^\alpha, y_e^a, 0, 0)$ becomes a critical point of $E_{\tau,\sigma,\rho,\epsilon}$. Next, we seek to make the second derivative of $E_{\tau,\sigma,\rho,\epsilon}$ negative definite at $(x_e^\alpha, y_e^a, 0, 0)$. This second derivative is

$$\begin{aligned} &D^2 E_{\tau,\sigma,\rho,\epsilon}(x_e, y_e, 0, 0) \\ &\quad = \begin{pmatrix} \frac{\partial^2 V_1}{\partial x^\alpha \partial x^\beta}(x_e) & 0 & 0 \\ 0 & \frac{\partial^2 \tilde{V}_\epsilon}{\partial y^a \partial y^b}(y_e) & 0 \\ 0 & 0 & D^2 K(x_e, y_e, 0, 0) \end{pmatrix}, \end{aligned}$$

where $D^2 K$ denotes the second derivative of the kinetic energy part of the controlled energy in (36) with respect to $(\dot{x}^\alpha, \dot{y}^a)$. The first two diagonal blocks are already negative definite and by kinetic shaping we can make the last block $D^2 K(x_e^\alpha, y_e^a, 0, 0)$ negative definite by choosing $\rho < 0$ and σ satisfying (31). Therefore, $E_{\tau,\sigma,\rho,\epsilon}$ has a maximum at $(x_e^\alpha, y_e^a, 0, 0)$. Using (35) and (36),

$$\frac{d}{dt} E_{\tau,\sigma,\rho,\epsilon} = u_a^{\mathrm{diss}} \dot{y}^a.$$

Define u^{diss} as follows:

$$u_a^{\mathrm{diss}} = c_a^d g_{bd} \dot{y}^b \tag{37}$$

where (c_a^d) is a positive definite matrix in the (g_{ab}) metric. (This definition is identical to (20)). Then, $(x_e^\alpha, y_e^a, 0, 0)$ is an equilibrium of the closed-loop system and the time derivative of the controlled energy is given by

$$\frac{d}{dt} E_{\tau,\sigma,\rho,\epsilon} = c_a^d g_{bd} \dot{y}^a \dot{y}^b \geq 0. \tag{38}$$

Thus, $(x_e^\alpha, y_e^a, 0, 0)$ becomes a Lyapunov stable equilibrium of the closed-loop system.

Asymptotic Stabilization. Now we show that the equilibrium $(x_e, y_e, 0, 0)$ is asymptotically stable. Since $E_{\tau,\sigma,\rho,\epsilon}$ has a maximum at $(x_e, y_e, 0, 0)$ and it is non-decreasing along the solution curve by (38), there is $c \in \mathbb{R}$ such that the set

$$\Omega_c = \{ z = (x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) \in TQ \mid E_{\tau,\sigma,\rho,\epsilon}(z) \geq c \}$$

is a nonempty, compact and positively invariant set. By compactness and positive invariance, integral curves starting in Ω_c are defined and stay in Ω_c for all $t \geq 0$.

Define

$$\begin{aligned} \mathcal{E} &= \left\{ z = (x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) \in \Omega_c \,\middle|\, \frac{d}{dt} E_{\tau,\sigma,\rho,\epsilon}(z) = 0 \right\} \\ &= \{ z = (x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) \in \Omega_c \mid \dot{y}^a = 0 \} \end{aligned}$$

$\mathcal{M}$ = the largest invariant subset of $\mathcal{E}$.

As in §V, there is a function $l : U \subset S \to \mathfrak{g}$ for an open subset U of S satisfying $\partial l^a / \partial x^\alpha = g^{ac} g_{\alpha c}$. Endow the Lie algebra $\mathfrak{g}$ of the group G with the metric (g_{ab}). By shrinking Ω_c, we may assume that U contains $K_c := \tau_s \circ T\pi(\Omega_c)$ where $\pi : Q \to S = Q/G$ is the G-principal bundle projection and $\tau_s : TS \to S$ is the tangent bundle projection. Note that K_c is also compact in S since it is a continuous image of the compact set Ω_c. Since l is a continuous function and K_c is compact, there is an $M > 0$ such that

$$\| l(x^\alpha) \| \leq M \tag{39}$$

for all $(x^\alpha) \in K_c$. Suppose $z(t) = (x^\alpha(t), y^a(t), \dot{x}^\alpha(t), \dot{y}^a(t))$ is contained in $\mathcal{M}$ for all $t \geq 0$. Then we have

$$y^a(t) = y^a(0), \quad \dot{y}^a(t) = 0 \tag{40}$$

for all $t \geq 0$ and

$$x^\alpha(t) = \tau_s \circ T\pi(z(t)) \in K_c \tag{41}$$

for all $t \geq 0$. Using (34), (35), and (37), we get the following Euler-Lagrange equation for the y^a variables:

$$\frac{d}{dt}(g_{\alpha a} \dot{x}^\alpha) + \rho g_{ab} \ddot{y}^b + \frac{\partial \tilde{V}_\epsilon}{\partial y^a} = c_a^d g_{bd} \dot{y}^b.$$

By (40), this becomes

$$\frac{d}{dt}(g^{ac} g_{\alpha c} \dot{x}^\alpha) = -g^{ac} \frac{\partial \tilde{V}_\epsilon}{\partial y^c}(y^b(0)) = -(\operatorname{grad} \tilde{V}_\epsilon)^a (y^b(0)).$$

Integrating this twice with respect to t with use of the definition of l^a, we get

$$l^a(x^\alpha(t)) = -\tfrac{1}{2} (\operatorname{grad} \tilde{V}_\epsilon)^a (y^b(0)) t^2 + \mu^a t + \nu^a \tag{42}$$

for some constants μ^a and ν^a. Thus,

$$\begin{aligned} &\| l(x^\alpha(t)) \|^2 \\ &\quad = \tfrac{1}{4} \| (\operatorname{grad} \tilde{V}_\epsilon)(y^a(0)) \|^2 t^4 - \langle (\operatorname{grad} \tilde{V}_\epsilon)(y^a(0)), \mu \rangle t^3 \\ &\qquad + \left(\|\mu\|^2 - \langle (\operatorname{grad} \tilde{V}_\epsilon)(y^a(0)), \nu \rangle \right) t^2 + 2 \langle \mu, \nu \rangle t + \|\nu\|^2 \end{aligned}$$

where $\mu = (\mu^a)$, and $\nu = (\nu^a)$. If $\| (\operatorname{grad} \tilde{V}_\epsilon)(y^a(0)) \| \neq 0$ or $\|\mu\| \neq 0$, then $\| l(x^\alpha(t)) \|$ will eventually get unbounded, which contradicts (39) and (41). Thus, it follows that $\| (\operatorname{grad} \tilde{V}_\epsilon)(y^a(0)) \| = 0$ and $\|\mu\| = 0$. Since $y^a = y_e^a$ is an isolated critical point of $\tilde{V}_\epsilon$, it follows $y^a(0) = y_e^a$. Using the above arguments, (42) becomes $l^a(x^\alpha(t)) = \nu^a =$ constant. Differentiate with respect to t, getting

$$g^{ac} g_{\alpha c} \dot{x}^\alpha = 0 \tag{43}$$

for all $t \geq 0$. So far we have shown that the trajectory $z(t) \in \mathcal{M}$ for all $t \geq 0$ is of the form, $z(t) = (x^\alpha(t), y_e^a, \dot{x}^\alpha(t), 0)$ for all $t \geq 0$. Using (34) and (35) we get the following Euler-Lagrange equation for the x^α variables:

$$\begin{aligned} &\frac{d}{dt} \left[\left(g_{\alpha\beta} - \left(\frac{\rho - 1}{\rho} - \frac{1}{\sigma} \right) g^{ab} g_{\alpha a} g_{\beta b} \right) \dot{x}^\beta + g_{\alpha a} \dot{y}^a \right] \\ &\quad - \tfrac{1}{2} \left(g_{\gamma\beta,\alpha} - \left(\frac{\rho - 1}{\rho} - \frac{1}{\sigma} \right) g^{ab} \left(g_{\gamma a,\alpha} g_{\beta b} + g_{\gamma a} g_{\beta b,\alpha} \right) \right) \\ &\qquad \times \dot{x}^\gamma \dot{x}^\beta + \frac{\partial V_1}{\partial x^\alpha} = 0. \end{aligned} \tag{44}$$

Substituting (40) and (43) into (44), we see that $z(t) = (x^\alpha(t), y_e^a, \dot{x}^\alpha(t), 0) \in \mathcal{M}$ obeys the following equation:

$$\frac{d}{dt}(g_{\alpha\beta}\dot{x}^\beta) - \tfrac{1}{2}g_{\gamma\beta,\alpha}\dot{x}^\gamma\dot{x}^\beta + \frac{\partial V_1}{\partial x^\alpha} = 0. \tag{45}$$

Notice that $(x_e^\alpha, 0) \in TS$ is an equilibrium of (45), that $(g_{\alpha\beta}(x_e^\alpha))$ is positive definite and that $\frac{\partial^2 V_1}{\partial x^2}(x_e^\alpha)$ is negative definite since V_1 has a maximum at $x^\alpha = x_e^\alpha$. The linearization of (45) at $(x^\alpha, \dot{x}^\alpha) = (x_e^\alpha, 0)$ shows that $(x_e^\alpha, 0)$ is a saddle equilibrium of (45) with $\dim S$ real positive and $\dim S$ real negative eigenvalues. Since Ω_c is an invariant set, $T\pi(z(t)) = (x^\alpha(t), \dot{x}^\alpha(t))$ remains in $T\pi(\Omega_c)$.

Therefore, after shrinking Ω_c if necessary, $(x^\alpha(t), \dot{x}^\alpha(t))$ must converge to the equilibrium $(x_e^\alpha, 0)$ of the dynamics in (45). Otherwise, it will leave $T\pi(\Omega_c)$, contradicting the invariance of Ω_c.[1]

Note that $z(t) = (x^\alpha(t), y_e^a, \dot{x}^\alpha(t), 0)$ is an *iso-energy* trajectory since $z(t) \in \mathcal{M} \subset \mathcal{E}$ and so $E_{\tau,\sigma,\rho,\epsilon}(z(t))$ is constant. Since $(x_e^\alpha, y_e^a, 0, 0)$ is an isolated maximum of $E_{\tau,\sigma,\rho,\epsilon}$, no iso-energy flows except the equilibrium at $(x_e^\alpha, y_e^a, 0, 0)$ can converge to $(x_e^\alpha, y_e^a, 0, 0)$. However, the fact that $(x^\alpha(t), \dot{x}^\alpha(t))$ converges to $(x_e^\alpha, 0)$ implies that the iso-energy flow $z(t) = (x^\alpha(t), y_e^a, \dot{x}^\alpha(t), 0)$ will converge to $(x_e^\alpha, y_e^a, 0, 0)$. Therefore, the only possibility is that $z(t) = (x_e^\alpha, y_e^a, 0, 0)$ for all $t \geq 0$. Hence, $\mathcal{M}$ consists only of the equilibrium $(x_e^\alpha, y_e^a, 0, 0)$. Thus, by LaSalle's invariance principle, $(x_e^\alpha, y_e^a, 0, 0)$ is an asymptotically stable equilibrium of the closed-loop system, and Ω_c is a region of attraction. Recall that $(x_e^\alpha, y_e^a, 0, 0)$ in the new coordinates corresponds to $(x_e^\alpha, \theta_e^a, 0, 0)$ in the old coordinates. Therefore, we have proven

Theorem V.1 (Asymptotic Stabilization–Specific Case) Assume that conditions SM-1 to SM-4, SM-$5'$ and SM-6 hold. Let (x_e^α) be the maximum point of V_1 of interest. Then, there is an explicit feedback control such that $(x_e^\alpha, \theta_e^a, 0, 0)$ becomes an asymptotically stable equilibrium. The control is given in (22) and (20) with parameters chosen to satisfy the following three conditions:

1. $\tilde{V}_\epsilon(y^a)$ should be chosen to have a maximum at $y_e^a = \theta_e^a$.
2. $\rho < 0$
3. $\kappa := -1/\sigma$
   $> \max\left\{ \lambda \,\middle|\, \det\left(g_{\alpha\beta} - \lambda g_{\alpha a} g^{ab} g_{b\beta}\right)\big|_{x=x_e} = 0 \right\} - 1$

Remarks.

1. Note that Ω_c here is not the best estimate of a region of attraction. We used Ω_c as an invariant set above to obtain a rigorous proof. In some instances there may be a larger invariant set and hence larger region of attraction.

2. The results here, as described earlier, are applied to a restricted class of systems satisfying our so-called special matching conditions. We intend to consider other systems in forthcoming work.

3. The fact that that the energy $E_{\tau,\sigma,\rho,\epsilon}$ of the controlled Lagrangian may have a maximum at the equilibrium rather than a minimum does not necessarily imply that the controlled Lagrangian system is fictitious or unphysical. Notice that in (34), (35) and (36), we can use $(-1)L_{\tau,\sigma,\rho,\epsilon}$ and $(-1)u^{\text{diss}}$ as a new controlled Lagrangian and new input to the controlled Lagrangian so that the resultant controlled energy $(-1)E_{\tau,\sigma,\rho,\epsilon}$ has a minimum at the equilibrium. This operation does not affect the matching conditions. Furthermore, investigation has been made of the effect of friction on the stabilization of an equilibrium that is a maximum for the controlled system (see [42] and [43]). In this work it is shown (analytically and experimentally) that friction contributes to stabilization in the unactuated directions and can be compensated for in the actuated directions. This was verified on an experimental inverted pendulum with fulcrum attached to a rotating link. More recent work has shown evidence of robustness to unmodelled dynamics, namely the presence of an unmodelled extra link attached to the end of an inverted pendulum.

[1] This may be proved by appealing to the Hartman Grobman theorem or to the fact that any trajectory that remains in a neighborhood of an equilibrium indefinitely must lie on the center-stable manifold and in this case the center-stable manifold equals the stable manifold.

VI. Inverted Pendulum on an Inclined Plane

We apply the above result to stabilize the inverted planar pendulum on a cart that travels on an incline of angle ψ. Let s denote the position of the cart along the incline and let ϕ denote the angle of the pendulum with the upright vertical as shown in Fig. 1. This example generalizes the pendulum on a cart example considered in [13] to the case of stabilization in the full phase space as well as putting the pendulum on an incline.

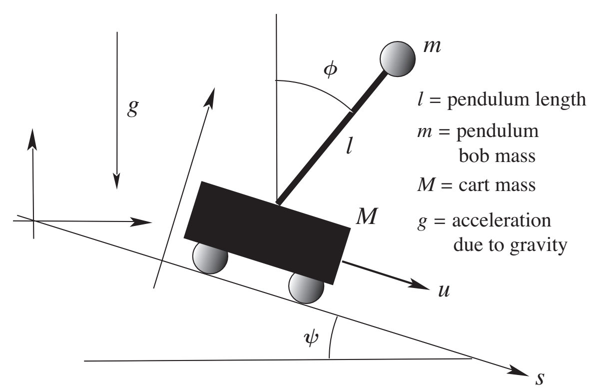

Fig. 1. The cart-pendulum on an inclined plane.

Configuration Space and Lagrangian. The configuration space for this system is $Q = S \times G = S^1 \times \mathbb{R}$, with the first factor being the pendulum angle ϕ and the second factor being the cart position s. The velocity phase space TQ has coordinates $z = (\phi, s, \dot{\phi}, \dot{s})$. We seek to asymptotically stabilize the origin, i.e., $z = 0$.

The velocity of the cart relative to the lab frame is $\dot{s}$, while the velocity of the pendulum relative to the lab frame is the vector

$$v_{\text{pend}} = (\dot{s}\cos\psi + l\cos\phi\,\dot{\phi}, -\dot{s}\sin\psi - l\sin\phi\,\dot{\phi}).$$

The system kinetic energy is the sum of the kinetic energies of the cart and the pendulum:

$$K(\phi, s, \dot{\phi}, \dot{s}) = \tfrac{1}{2}\,[\dot{\phi}, \dot{s}] \begin{bmatrix} ml^2 & ml\cos(\phi-\psi) \\ ml\cos(\phi-\psi) & M+m \end{bmatrix} \cdot \begin{bmatrix} \dot{\phi} \\ \dot{s} \end{bmatrix}.$$

The potential energy is given by $V(\phi, s) = V_1(\phi) + V_2(s)$ where $V_1(\phi) = mgl\cos\phi$ and $V_2(s) = -(m+M)gs\sin\psi$. The Lagrangian is the kinetic minus potential energy, so we get

$$L(\phi, s, \dot{\phi}, \dot{s}) = K(\phi, s, \dot{\phi}, \dot{s}) - V(\phi, s).$$

Notice that the potential energy breaks symmetry in the cart translation s. For notational convenience we rewrite the Lagrangian as

$$L(\phi, s, \dot{\phi}, \dot{s}) = \tfrac{1}{2}(\alpha\dot{\phi}^2 + 2\beta\cos(\phi-\psi)\dot{s}\dot{\phi} + \gamma\dot{s}^2) + D\cos\phi + \gamma g s\sin\psi, \tag{46}$$

where $\alpha = ml^2, \beta = ml, \gamma = M+m$ and $D = -mgl$.

The Controlled Cart. The equations of motion for the cart-pendulum system with a control force u acting on the cart (and no direct forces acting on the pendulum) are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0; \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{s}} - \frac{\partial L}{\partial s} = u.$$

By inspection we see that SM-2 and SM-4 hold. To satisfy SM-1 and SM-3, we take $\sigma_{ab} = \sigma g_{ab} = \sigma\gamma$ and $\tau^b_\alpha = -(1/\sigma)g^{ab}g_{\alpha a} = (\kappa/\gamma)\beta\cos(\phi-\psi)$, where σ is a scalar constant and $\kappa = -1/\sigma$. It is easy to see that the potential in V satisfies SM-5′ with $V_1(\phi) = mgl\cos\phi$ having a maximum at $\phi = 0$ and that $g_{\alpha a}(\phi) = ml\cos(\phi-\psi) \neq 0$ is clearly one-to-one for $-\frac{\pi}{2} + \psi < \phi < \frac{\pi}{2} + \psi$, satisfying SM-6 unless the incline is vertical. From (11) and (33), the potential energy for the controlled system is

$$V'(\phi, s) = V_1(\phi) + \tilde{V}_\epsilon(y) = mgl\cos\phi + \tilde{V}_\epsilon(y),$$

where from (23) and (24)

$$y = s + \left(\kappa + \frac{\rho-1}{\rho}\right)\frac{\beta}{\gamma}(\sin(\phi-\psi) + \sin\psi).$$

Following Theorem V.1, we choose $\tilde{V}_\epsilon$ to be $\tilde{V}_\epsilon = \epsilon D\gamma^2 y^2/(2\beta^2)$ with $\epsilon > 0$ so that $\tilde{V}_\epsilon$ has a maximum at $y = 0$. Note that the modification to the original potential energy V_ϵ is therefore given by $V_\epsilon = V' - V = \tilde{V}_\epsilon + \gamma g s\sin\psi$. Thus, by Theorem V.1, if $\rho < 0$ and κ satisfies

$$\kappa > \frac{ml^2(M+m)}{m^2l^2\cos^2\psi} - 1 = \frac{m\sin^2\psi + M}{m\cos^2\psi},$$

then the vertical position with the cart at the origin is asymptotically stabilizable.

The controlled energy $E_{\tau,\sigma,\rho,\epsilon}$ is given by

$$\begin{aligned} E_{\tau,\sigma,\rho,\epsilon} &= \tfrac{1}{2}\alpha\dot{\phi}^2 + \beta\cos(\phi-\psi)\left(\dot{s} + \frac{\kappa}{\gamma}\beta\cos(\phi-\psi)\dot{\phi}\right)\dot{\phi} \\ &\quad + \tfrac{1}{2}\gamma\left(\dot{s} + \frac{\kappa}{\gamma}\beta\cos(\phi-\psi)\dot{\phi}\right)^2 - \tfrac{1}{2}\frac{\kappa}{\gamma}\beta^2\cos^2(\phi-\psi)\dot{\phi}^2 \\ &\quad + \tfrac{1}{2}(\rho-1)\gamma\left(\dot{s} + (\kappa+1)\frac{\beta}{\gamma}\cos(\phi-\psi)\dot{\phi}\right)^2 + V'. \end{aligned} \tag{47}$$

The dissipation term following (37) is

$$u^{\text{diss}} = c\gamma\dot{y} = c\gamma\left(\dot{s} + \left(\kappa + \frac{\rho-1}{\rho}\right)\frac{\beta}{\gamma}\cos(\phi-\psi)\dot{\phi}\right)$$

with $c > 0$. The complete control law (22) becomes

$$\begin{aligned} u &= \frac{1}{\alpha - \frac{\beta^2}{\gamma}(\kappa+1)\cos^2(\phi-\psi)} \\ &\quad \times \left\{\kappa\beta\left(\alpha\sin(\phi-\psi)\dot{\phi}^2 + \cos(\phi-\psi)D\sin\phi\right)\right. \\ &\quad \left. -B\frac{\partial V'}{\partial s} + Bu^{\text{diss}}\right\} - \gamma g\sin\psi, \end{aligned} \tag{48}$$

where $B = (\alpha - \frac{\beta^2}{\gamma}\cos^2(\phi-\psi))/\rho$. This control law is finite if the denominator is strictly negative, i.e., if

$$\sin^2(\phi-\psi) < \frac{\beta^2(\kappa+1) - \alpha\gamma}{\beta^2(\kappa+1)}. \tag{49}$$

This range of ϕ tends to the range $-\pi/2 + \psi < \phi < \pi/2 + \psi$ for large κ.

Region of Attraction. Consider the case that the inclination angle ψ is zero for simplicity. The function $h : U \to \mathbb{R}$ defined in §V by (23) is given by $h(\phi) = (1 + \kappa - 1/\rho)(\beta/\gamma)\sin\phi$ and $U = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, we can use $U \times \mathbb{R} \subset Q$ as a domain of a local chart on Q. We use the following as a local chart on TQ:

$$(q, v) = (\phi, s, \dot{\phi}, \dot{s}) \mapsto (\phi, y, \dot{\phi}, \zeta) \in (U \times \mathbb{R}) \times \mathbb{R}^2$$

where $y = s + h(\phi)$ and $\zeta = \dot{s} + \dot{\phi}(1+\kappa)(\beta/\gamma)\cos\phi$. Notice that $\dot{\phi}$ and ζ are coordinates for Hor_τ and Ver_τ, respectively.

In this chart, the controlled energy $E_{\tau,\sigma,\rho,\epsilon}$ is given by

$$\begin{aligned} E_{\tau,\sigma,\rho,\epsilon}(\phi, y, \dot{\phi}, \zeta) &= V_1(\phi) + \tilde{V}_\epsilon(y) + K_{\tau,\sigma,\rho}(v) \\ &= a_1\cos\phi + a_2 y^2 + a_3(\phi)\dot{\phi}^2 + a_4\zeta^2, \end{aligned}$$

where $K_{\tau,\sigma,\rho}$ is defined in (30) and

$$a_1 = -D > 0; \quad a_2 = \tfrac{1}{2}\epsilon D\frac{\gamma^2}{\beta^2} < 0 \tag{50}$$

$$a_3(\phi) = \tfrac{1}{2}\left(\alpha - \frac{\beta^2}{\gamma}(\kappa+1)\cos^2\phi\right); \quad a_4 = \tfrac{1}{2}\rho\gamma < 0.$$

Let W be the subset of U satisfying $a_3(\phi) < 0$. Then we can check that the controlled energy $E_{\tau,\sigma,\rho,\epsilon}$ has a maximum at $(0,0,0,0)$ in $(W \times \mathbb{R}) \times \mathbb{R}^2$. As can be seen from (49), W *converges to* U as κ goes to infinity.

There are several points in §V to be checked. First, take Ω_c in $(W \times \mathbb{R}) \times \mathbb{R}^2$ as large as possible. Then, it follows that $K_c \subset W$. In §V we said that we could shrink Ω_c to study the dynamics in (45) since we had to rely on the linearized dynamics to deal with a general case. But here we directly study the nonlinear dynamics. In this specific case of the inverted pendulum on a cart, (45) is given by

$$\ddot{\phi} - \frac{g}{l}\sin\phi = 0.$$

Since the system is planar, it can be checked that any trajectory $(\phi, \dot{\phi})$ starting in $W \times \mathbb{R}$ will escape from $W \times \mathbb{R}$ except when the trajectory is the equilibrium $(0,0)$. It follows that shrinking Ω_c is unnecessary. As discussed in the first remark following Theorem V.1, Ω_c is not necessarily the best estimate of a region of attraction. In §IX, we show with simulation examples that we can get a large region of attraction.

Suppose that the initial position of the pendulum is close to the horizontal position. Then, regardless of the control methods we use, since actuation is available only through the translational motion of the cart, it is physically obvious that we need a large initial force to prevent the pendulum from falling past 90 degrees. Hence, it is difficult to achieve a large region of attraction with a control force of limited magnitude irrespective of control methods. We mention, however, that in the "swing-up" problem where we swing up the pendulum from the downward pointing state, large forces are not needed to initialize the pendulum motion. We intend to consider the swing-up problem in a future publication.

VII. Spherical Pendulum on an Inclined Plane

We apply the above results to the spherical pendulum on a cart that travels on an incline of angle ψ. This generalizes the spherical pendulum on a plane considered by [10], [13]. This example is important for illustrating the results of the present paper since it has *two* unactuated degrees of freedom. The system is shown in Fig. 2.

The configuration space for this system is $Q = S \times G = S^2 \times \mathbb{R}^2$. We denote by (x, y) the Cartesian coordinates of the cart on the incline and assume that we have independent controls that can move the cart in the x and y directions. Let P be the plane whose origin is attached to the cart and which is parallel to the incline. We will use the projection onto the plane P for a local chart for S^2. Let (X, Y) be the Cartesian coordinates of the bob in the plane P under the local chart. Let $q = (X, Y, x, y)$ be the local coordiantes for Q.

Let M and m be the masses of the cart and the bob, respectively and r be the length of the pendulum. The position R of the bob in the inertial frame is given by

$$R = (x + X, y + Y, \sqrt{r^2 - X^2 - Y^2}).$$

The total kinetic energy is given by $K(q, \dot{q}) = 1/2g(q)(\dot{q}, \dot{q})$

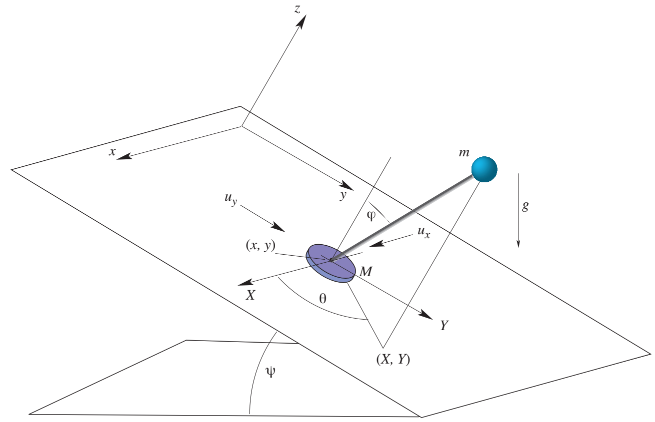

Fig. 2. Spherical pendulum moving on an incline.

where the metric $g(q)$ is given by

$$\begin{pmatrix} m\left(\frac{r^2-Y^2}{r^2-X^2-Y^2}\right) & m\left(\frac{XY}{r^2-X^2-Y^2}\right) & m & 0 \\ m\left(\frac{XY}{r^2-X^2-Y^2}\right) & m\left(\frac{r^2-X^2}{r^2-X^2-Y^2}\right) & 0 & m \\ m & 0 & m+M & 0 \\ 0 & m & 0 & m+M \end{pmatrix}$$

The total potential energy is given by $V(X, Y, x, y) = V_1(X, Y) + V_2(x, y)$, where

$$V_1(X, Y) = mg\left(\cos\psi\sqrt{r^2 - X^2 - Y^2} - \sin\psi Y\right),$$
$$V_2(x, y) = -(m + M)gy\sin\psi.$$

The Lagrangian for this system is $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$.

It is easy to check that SM-1 – SM-4 are satisfied. In this case, we have

$$\sigma_{ab} = \sigma g_{ab} = \sigma(m + M)\delta_{ab},$$
$$\tau_X^x = \tau_Y^y = \frac{m}{\sigma(m + M)}; \quad \tau_Y^x = \tau_X^y = 0$$

where δ_{ab} is the Kronecker delta. The form of the potential V satisfies SM-5′. Physically, it is obvious that $V_1(X, Y)$ has a maximum at $(X, Y) = (0, -r\sin\psi)$ which is, as it should be, the position of the pendulum vertical to the ground, not to the incline. The matrix

$$(g_{a\alpha}(0, -r\sin\psi)) = m\mathrm{I}_{2\times 2}$$

is clearly one-to-one, so SM-6 holds. By Theorem V.1, the vertical position (*relative to the ground*) of the pendulum and any fixed position for the cart on the incline is asymptotically stabilizable.

VIII. Tracking

Here we consider one of the simplest nontrivial tracking problems, namely we make the θ^a variables track a constant acceleration curve in $G = \mathbb{R}^k$, while regulating the x^α variables at a fixed point x_e^α in S.

We assume that the given Lagrangian L satisfies SM-1 to SM-4, SM-5′ and SM-6. Let $r(t) \in G$ be the reference signal satisfying $\ddot{r}(t) = c =$ constant. Consider a moving frame which moves along $(0, r^a(t))$. Let (x^α, y^a) be the coordinates in the moving frame satisfying

$$y^a = \theta^a - r^a(t).$$

Let $L_m : TQ \times \mathbb{R} \to \mathbb{R}$ be the Lagrangian in the moving frame defined by

$$L_m(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a, t) = L(x^\alpha, y^a + r^a(t), \dot{x}^\alpha, \dot{y}^a + \dot{r}^a(t)).$$

In coordinates,

$$\begin{aligned} &L_m(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a, t) \\ &\quad = \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{y}^a + \tfrac{1}{2} g_{ab}\dot{y}^a\dot{y}^b + g_{\alpha a}\dot{x}^\alpha \dot{r}^a(t) \\ &\qquad + g_{ab}\dot{y}^a\dot{r}^b(t) + \tfrac{1}{2} g_{ab}\dot{r}^a(t)\dot{r}^b(t) - V_1(x^\alpha) \\ &\qquad - V_2(y^a + r^a(t)). \end{aligned} \tag{51}$$

By SM-4 and the Poincaré Lemma, there exists a function $l : U \subset S \to \mathbb{R}^k$ such that $\partial l_a/\partial x^\alpha = g_{\alpha a}$. Hence (51) can be written as

$$\begin{aligned} &L_m(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a, t) \\ &\quad = \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{y}^a + \tfrac{1}{2} g_{ab}\dot{y}^a\dot{y}^b - l_a(x^\alpha)\ddot{r}^a(t) \\ &\qquad - g_{ab}y^a\ddot{r}^b(t) + \frac{d}{dt}(l_a(x^\alpha)\dot{r}^a(t)) + \frac{d}{dt}(g_{ab}y^a\dot{r}^b(t)) \\ &\qquad + \tfrac{1}{2} g_{ab}\dot{r}^a(t)\dot{r}^b(t) - V_1(x^\alpha) - V_2(y^a + r^a(t)). \end{aligned} \tag{52}$$

Since exact time derivatives do not affect the variational principle, we can ignore the following three terms:

$$\frac{d}{dt}(l_a(x^\alpha)\dot{r}^a(t)), \quad \frac{d}{dt}(g_{ab}y^a\dot{r}^b(t)), \quad \tfrac{1}{2} g_{ab}\dot{r}^a(t)\dot{r}^b(t).$$

Hence the Lagrangian L_m in (52) can be replaced by the following Lagrangian:

$$\begin{aligned} L_m(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a, t) &= \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{y}^a + \tfrac{1}{2} g_{ab}\dot{y}^a\dot{y}^b \\ &\quad - l_a(x^\alpha)c^a - g_{ab}y^a c^b - V_1(x^\alpha) - V_2(y^a + r^a(t)) \end{aligned}$$

where $\ddot{r}(t) =$ constant was used. The Euler-Lagrange equations in the moving frame are given by

$$\frac{d}{dt}\frac{\partial L_m}{\partial \dot{x}^\alpha} - \frac{\partial L_m}{\partial x^\alpha} = 0; \quad \frac{d}{dt}\frac{\partial L_m}{\partial \dot{y}^a} - \frac{\partial L_m}{\partial y^a} = v_a$$

where the input v in the moving frame has the following relationship with the input u in the fixed frame:

$$v(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) = u(x^\alpha, y^a + r^a(t), \dot{x}^\alpha, \dot{y}^a + \dot{r}^a(t)). \tag{53}$$

General discussions about the relationship between the Lagrangian system with forces in the fixed frame and that in the moving frame are given in Appendix B.

Here we perform potential shaping first by choosing the input of the following form:

$$v_a = g_{ab}c^b + \frac{\partial}{\partial y^a}V_2(y^a + r^a(t)) + w_a. \tag{54}$$

Define $\tilde{L}_m : TQ \to \mathbb{R}$ by

$$\begin{aligned} \tilde{L}_m(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a) &= \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{y}^a \\ &\quad + \tfrac{1}{2} g_{ab}\dot{y}^a\dot{y}^b - \tilde{V}_1(x^\alpha) \end{aligned} \tag{55}$$

where $\tilde{V}_1(x^\alpha) = V_1(x^\alpha) + l_a(x^\alpha)c^a$. Then, the Euler-Lagrange equations from the Lagrangian $\tilde{L}_m$ with the input w are equal to those from the Lagrangian L_m with the input v.

Notice that $\tilde{L}_m$ is time-independent and its kinetic energy is of the same form as that of L. We can check that $\tilde{L}_m$ satisfies SM-1 to SM-4, SM-5′ and SM-6. Let x_e be a maximum of $\tilde{V}_1$. By Theorem V.1, we can design a controller w so that $(x_e, 0, 0, 0)$ becomes an asymptotically stable equilibrium in the moving frame. From w we can derive the input u by (53) and (54). The asymptotic stabilization in the moving frame is equal to the tracking in the fixed frame. Thus u becomes a tracking controller such that $(x(t), \phi(t), \dot{x}(t), \dot{\phi}(t))$ asymptotically converges to $(x_e, r(t), 0, \dot{r}(t))$.

Example. Consider again the inverted pendulum on a cart. In this case, $\tilde{V}_1$ is given by

$$\tilde{V}_1(\phi) = mgl\cos\phi + mlc\sin\phi$$

where c is the constant acceleration of the reference curve. $\tilde{V}_1$ has a maximum at $\phi_o = \arctan(c/g)$. This means that the cart will move at the acceleration c with the pendulum slanted by the angle ϕ_o which agrees with physical intuition.

Remark. We note that in tracking problems on general manifolds, we should be cautious in comparing two points or two vectors at different base points since a naive subtraction does not make sense on manifolds in general. An error function and a transport map are employed in [17] to deal with this . The problem of tracking a general reference signal is an important problem that remains to be tackled by the methods of this paper.

IX. Simulations

In this section, we give some simulations using the inverted pendulum on a cart. First, we look at the case when the cart is on an inclined plane to show that our controller works well when there is no symmetry. Second, by using the analysis in §VI, we show that we can achieve a large region of attraction in the sense that our method can handle the case when the initial position of the pendulum is close to the horizontal position. Third, we do simulations of a tracking problem.

Inverted Pendulum on an Inclined Plane. We designed an asymptotically stabilizing control law in the case of an inverted pendulum on an inclined cart. Here, we show a MATLAB simulation using the control law in (48). Here $m = 0.14$ kg, $M = 0.44$ kg, $l = 0.215$ m, and $\psi = \frac{\pi}{9}$ radians $= 20°$. Our goal is to regulate the cart at $s = 0$ and the pendulum at $\phi = 0$. We choose control gains to be $\kappa = 20$, $\rho = -0.02$, $\epsilon = 0.00001$ and $c = 0.015$. Fig. 3 shows plots of pendulum angle and velocity and cart position and velocity for the system subject to our asymptotically stabilizing controller. The pendulum starts from $(\phi(0), s(0), \dot{\phi}(0), \dot{s}(0)) = (\pi/6, 3, 0, 0)$. Note that the cart comes to rest at the origin with the pendulum upright and vertical to the ground.

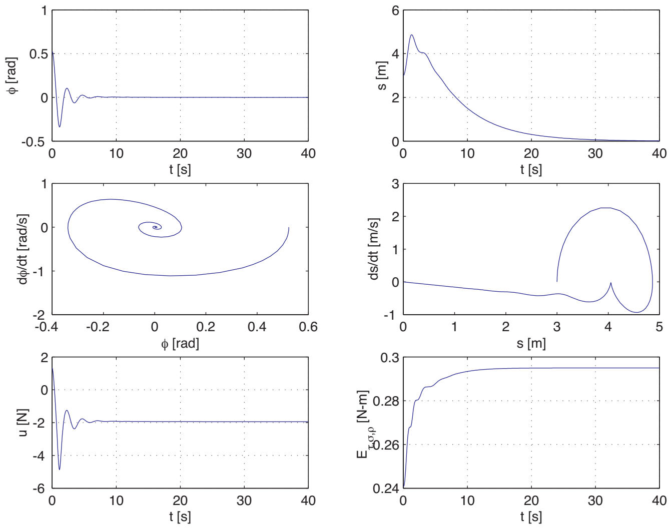

Fig. 3. Simulation of the controlled pendulum on an inclined plane.

At the bottom of Fig. 3 we have included a plot of the control law u and the Lyapunov function, i.e., the controlled energy $E_{\tau,\sigma,\rho,\epsilon}$. To keep the pendulum from falling past 90°, a large initial force is needed. But as the response reaches its steady state, the control law converges to $-(M+m)g\sin\psi = -1.9440$ N which is the force needed to keep the system statically from going down the incline. The controlled energy $E_{\tau,\sigma,\rho,\epsilon}$ converges to the value of $mgl = 0.2950$ N-m which is a maximum of $E_{\tau,\sigma,\rho,\epsilon}$ in (47) and corresponds to the value of $E_{\tau,\sigma,\rho,\epsilon}$ at the equilibrium.

Large Region of Attraction. We consider the same system with the inclination angle zero using the notation of §VI. Our goal is to get the control parameters to handle a large initial angle of the pendulum. To get a large W, choose $\kappa = 300$. Then, $W = (-1.4532, 1.4532) = (-83.26°, 83.26°)$. Choose $\rho = -0.02$, $\epsilon = 0.00001$, and $c = 0.015$. Since it is hard to visualize the level sets of the controlled energy $E_{\tau,\sigma,\rho,\epsilon}$, we consider the level sets with velocity zero. Let $a = 0.05$ and $\Sigma_a = \{(\phi, s) \in U \times \mathbb{R} \mid E_{\tau,\sigma,\rho,\epsilon}(\phi, s, 0, 0) \geq a\}$. The level sets of $E_{\tau,\sigma,\rho,\epsilon}(\phi, s, 0, 0)$ are shown in Fig. 4 and Σ_a is the shaded region. From the figure one can see that $\Sigma_a \subset W \times \mathbb{R}$. Let $\Omega_a = \{(\phi, s, \dot{\phi}, \dot{s}) \in \Sigma_a \times \mathbb{R}^2 \mid E_{\tau,\sigma,\rho,\epsilon}(\phi, s, \dot{\phi}, \dot{s}) \geq a\}$. Since $a_3(\phi)$ and a_4 in (50) are negative for $\phi \in W$, one can show that Ω_a is positively invariant and thus a region of attraction. Note that $\Sigma_a \times (0, 0)$ is contained in Ω_a.

Hence, we can see that the trajectories originating, for example, from $(\phi(0), s(0), \dot{\phi}(0), \dot{s}(0)) = (0.8, 0, 0, 0)$ or $(\phi(0), s(0), \dot{\phi}(0), \dot{s}(0)) = (1.2, -20, 0, 0)$ will converge to the origin. But we know that this estimation of the region of attraction from the level set of $E_{\tau,\sigma,\rho,\epsilon}$ with zero velocity could be conservative. To show this we present three different simulations. The first one originates from $(\phi(0), s(0), \dot{\phi}(0), \dot{s}(0)) = (0.9, 0, 0, 0)$ which is taken from the region of attraction given by Σ_a. The second one originates from $(\phi(0), s(0), \dot{\phi}(0), \dot{s}(0)) = (\pi/3, 8, 0, 0)$, and the third one originates from $(\phi(0), s(0), \dot{\phi}(0), \dot{s}(0)) = (4\pi/9, 5, 0, 0)$. The latter two initial conditions do not lie in the estimated region of attraction shown in Fig. 4. Fig. 5 shows the responses for the three different initial condition. Each row of plots corresponds to a different case. They all converge to the origin demonstrating a large region of attraction for the initial angle of the pendulum. Although we did not plot the force here, we note that we needed a large initial force in the third case, which is as discussed in § VI. This also explains that the large initial translational motion is unavoidable.

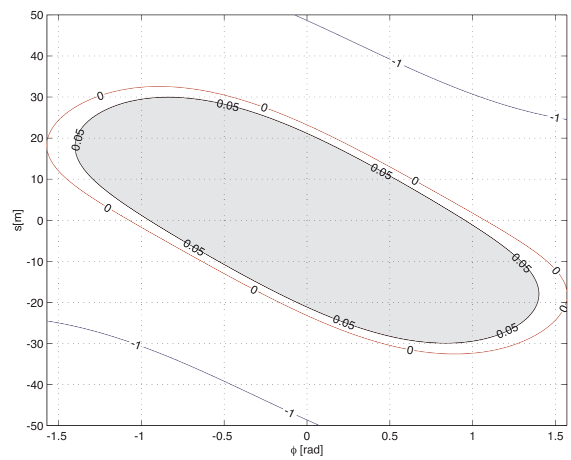

Fig. 4. The level sets of $E_{\tau,\sigma,\rho,\epsilon}(\phi, s, 0, 0)$. The shaded region is the set where $E_{\tau,\sigma,\rho,\epsilon}(\phi, s, 0, 0) \geq 0.05$.

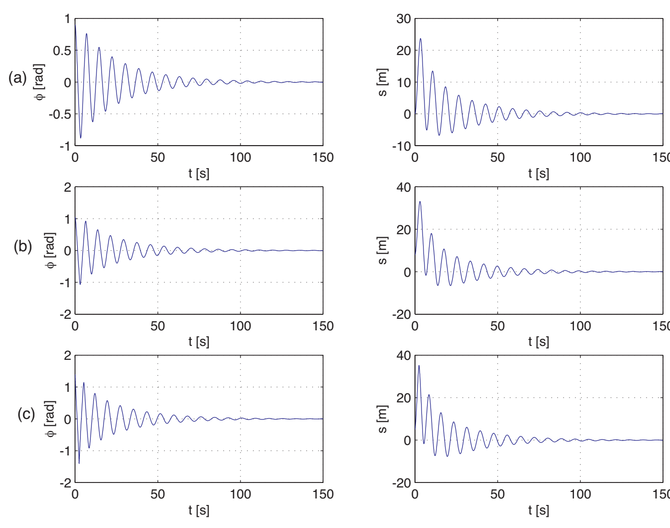

Fig. 5. Responses to various initial conditions: (a) $z(0) = (0.9, 0, 0, 0)$, (b) $z(0) = (\frac{\pi}{3}, 8, 0, 0)$, (c) $z(0) = (\frac{4\pi}{9}, 5, 0, 0)$.

Tracking. Next, we present tracking simulations. For simplicity, we consider the case where the inclination angle ψ is zero. Our goal is to make the cart track a given curve of constant acceleration a with the pendulum slanted by $\phi_a := \arctan(a/g)$. We can construct a controller combining the results from §V, §VI and §VIII. Let $r(t) = \frac{1}{2}at^2$ with $a = \frac{\pi}{6}g = 5.13\text{m/s}^2$ be the reference signal for the cart. Then $\phi_a = \pi/6(\text{rad}) = 30°$. First, we choose the following control gains: $\kappa = 30$, $\rho = -0.02$, $\epsilon = 0.0001$ and $c = 0.015$. Let e be the difference between the position of the cart s and the reference signal r. The first row and the second row of plots in Fig. 6 are the responses with this controller with the initial conditions $(\phi(0), s(0), \dot\phi(0), \dot s(0)) = (0, -2, 0, 0)$ and $(\phi(0), s(0), \dot\phi(0), \dot s(0)) = (\pi/3, 2, 0, 0)$, respectively. We can see that the angle of the pendulum converges to ϕ_a and the cart tracks the reference signal. However, this κ is not enough to handle a large initial angle difference roughly because it gives too small a W. So, we try another controller with $\kappa = 300$, $\rho = -0.02$, $\epsilon = 0.0001$, and $c = 0.015$ which was found earlier to get a large region of attraction in the regulation problem. The third row in Fig. 6 is the response with this controller with the initial condition $(\phi(0), s(0), \dot\phi(0), \dot s(0)) = (4\pi/9, 0, 0, 0)$. This controller achieves our objective.

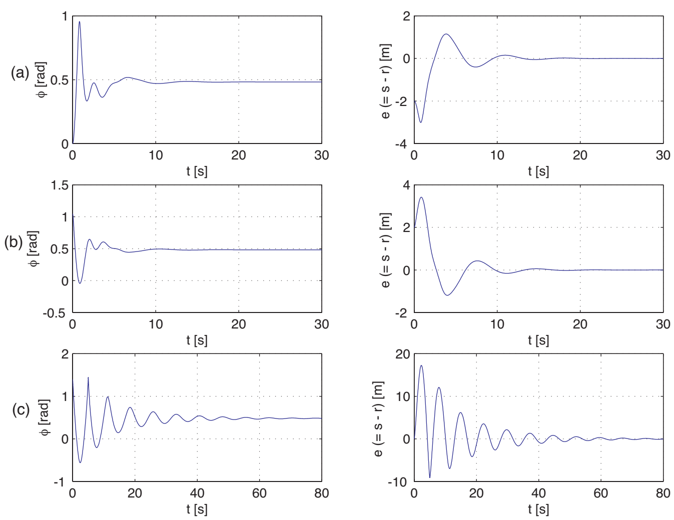

Fig. 6. Tracking responses with different initial conditions and different gains (Refer to the relevant part for more explanation): (a) $z(0) = (0, -2, 0, 0)$, (b) $z(0) = \left(\frac{\pi}{3}, 2, 0, 0\right)$, (c) $z(0) = \left(\frac{4\pi}{9}, 0, 0, 0\right)$.

X. Conclusions and Future Directions

In this paper we have described the method of controlled Lagrangians for a class of mechanical systems. We have shown how the combination of kinetic shaping and symmetry-breaking potential shaping leads to controllers which give asymptotic stability in the full state space and can handle certain types of tracking problems. The systems considered have symmetry in the kinetic energy but not necessarily in the potential energy.

In a forthcoming paper we shall describe the extensions of our results to a larger class of systems satisfying generalized matching conditions. A system satisfying the general matching condition is the pendulum on a rotor arm described in [11]. In recent papers we have addressed systems of the Euler-Poincaré type such as the rigid body with rotors, the heavy top with rotors and underwater vehicles (see [7], [14], [19], [44], [42] and references therein). Systems of Euler-Poincaré type were described briefly in [10].

We intend to make a number of other extensions of our work. For example, we intend to consider the swing-up problem for the pendulum and related problems which involve transfers between equilibria and/or periodic orbits. Use can be made in this setting of heteroclinic connections. This is related to the work of [15] and [27].

We plan to carry out the analysis of more general tracking problems perhaps using the techniques described in [17]. In addition we will carry out an analysis of various robustness issues in our nonlinear context. We have already made progress in understanding the robustness of our method to existing (physical) dissipation (see [42], [43] and [44]). In this work it is shown that friction contributes to stabilization in the unactuated directions and can be compensated for in the actuated directions. This was verified on an experimental inverted pendulum on a rotating rigid link. Some analysis of robustness to model parameter uncertainty in the energy shaping context has been carried out by [28], [35] and [47]. One situation that we have begun to investigate is stability in the presence of extra stable but unactuated degrees of freedom. Early work shows evidence that our approach provides some robustness in this regard. Finally, we intend to apply some of these ideas to the stabilization of nonholonomic systems using the energy-momentum results of [46]; see [45] and references therein for a start on this program.

Appendix

I. General Discussion on controlled Lagrangians

We give a brief summary of a different perspective of the method of controlled Lagrangians taken by [2] and, [23], with a flavor of [29]. This will help us to understand the under-actuation structure and controlled Lagrangians. This appendix indicates how more general matching can be done. The advantage of our structured method in this paper is that it leads to explicit and relatively simple control laws that can be easily implemented in practice.

For simplicity we only consider Lagrangians of the kinetic minus potential energy form as follows:

$$L(q, \dot q) = \tfrac{1}{2}g(\dot q, \dot q) - V(q) \tag{56}$$

for $(q, \dot q) \in TQ$ with Q the configuration space of dimension n. The control u is a bundle map $u : TQ \to W_c \subset T^*Q$ where W_c is a subbundle of T^*Q. The subbundle W_c has the information on actuation structure. We call W_c the actuation cobundle and $W := g^\sharp W_c$ the actuation bundle. Hence, every underactuated mechanical system is denoted by a pair (L, W_c).

Suppose that we are given a system (L, W_c). Its Euler-Lagrange equations with control u are given by

$$\mathcal{E}_q(L) := \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u. \tag{57}$$

The equations in (57) can be written on the tangent space TQ as follows

$$\nabla_{\dot{q}}\dot{q} + g^\sharp \mathrm{d}V = v \tag{58}$$

where $v = g^\sharp u : TQ \to W \subset TQ$ and ∇ is the Levi-Civita connection of the metric g. The musical maps $g^\sharp : T^*Q \to TQ$ and $g^\flat : TQ \to T^*Q$ come from the isomorphism between TQ and T^*Q induced by a given Riemannian metric g. Suppose we have another Lagrangian system $(\tilde{L}, \tilde{W}_c)$ with Lagrangian $\tilde{L} = \frac{1}{2}\tilde{g} - \tilde{V}$ and actuation cobundle $\tilde{W}_c \subset T^*Q$. We want the two systems (L, W_c) and $(\tilde{L}, \tilde{W}_c)$ to be equivalent in the sense that for any choice of control $u : TQ \to W_c$ for the system (L, W_c), there is a control $\tilde{u} : TQ \to \tilde{W}_c$ such that both closed-loop systems produce the same ordinary differential equations and vice versa. First we transform the Euler-Lagrange equations for $(\tilde{L}, \tilde{W}_c)$ to the form (58) as follows

$$\tilde{\nabla}_{\dot{q}}\dot{q} + \tilde{g}^\sharp \mathrm{d}\tilde{V} = \tilde{v} \tag{59}$$

with $\tilde{v} : TQ \to \tilde{W} := \tilde{g}^\sharp \tilde{W}_c$ and $\tilde{\nabla}$ the Levi-Civita connection of the metric $\tilde{g}$. Comparison of (58) and (59) implies that the two systems are equivalent if and only if the following holds

$$\tilde{W} = W \tag{60}$$

$$\tilde{\nabla} - \nabla \in C^\infty(\mathrm{Sym}_2(T^*Q) \otimes W) \tag{61}$$

$$\mathrm{d}\tilde{V} \in \tilde{g}^\flat g^\sharp (W_c + \mathrm{d}V) \tag{62}$$

where $\mathrm{Sym}_2(T^*Q)$ is the $(0, 2)$ symmetric tensor field. The conditions (60)–(62) are the compact form of the matching conditions in [23].

We now give a procedure for finding systems equivalent to a given system (L, W_c). Choose a section $S \in C^\infty(\mathrm{Sym}_2(T^*Q) \otimes W)$ and define a torsion-free affine connection $\tilde{\nabla} := \nabla + S$ on TQ. The new connection $\tilde{\nabla}$ is the Levi-Civita connection of some Riemannian metric $\tilde{g}$ on Q if and only if there is a positive definite symmetric 2-form $\tilde{g}$ on Q such that

$$\tilde{\nabla}\tilde{g} = 0.$$

Assume that we found a Riemannian metric $\tilde{g}$ such that its unique Levi-Civita connection becomes $\tilde{\nabla}$. The Poincaré lemma implies that the existence of the function $\tilde{V}$ satisfying (62) is equivalent to the existence of a 1-form $\alpha \in W_c$ such that the 1-form $g^\flat g^\sharp(\alpha + \mathrm{d}V)$ is closed. Then the new Lagrangian system $\tilde{L} = \frac{1}{2}\tilde{g} - \tilde{V}$ with the control cobundle $\tilde{W}_c := \tilde{g}^\flat g^\sharp W_c$ is equivalent to the original system (L, W_c).

II. Moving Systems

This appendix summarizes the relationship between the Lagrangian system with forces in the fixed frame and that in the moving frame that was used in §VIII on tracking.

Consider a Riemannian manifold $\mathcal{S}$, a submanifold Q, and a space M of embeddings of Q into $\mathcal{S}$. Let $m_t \in M$ be a given curve. If a particle in Q is following a curve $q(t)$, and if Q moves by superposing the motion m_t, then the path of the particle in $\mathcal{S}$ is given by $m_t(q(t))$. Thus, its velocity in $\mathcal{S}$ is given by $T_{q(t)}m_t \cdot \dot{q}(t) + \mathcal{Z}_t(m_t(q(t)))$, where $\mathcal{Z}_t(m_t(q)) = \frac{d}{dt}m_t(q)$. The Lagrangian L on $T\mathcal{S}$ is the kinetic minus potential energy: $L(\tilde{q}, \tilde{v}) = \frac{1}{2}||\tilde{v}||^2 - U(\tilde{q})$. Consider a Lagrangian L_{m_t} in TQ of the usual form of kinetic minus potential:

$$\begin{aligned} L_{m_t}(q, v) &= \tfrac{1}{2}||T_q m_t \cdot \dot{q}(t) + \mathcal{Z}_t(m_t(q))||^2 - U(m_t(q)) \\ &= L(m_t(q), T_q m_t \cdot v + \mathcal{Z}_t(m_t(q))). \end{aligned} \tag{63}$$

Assume that $\tilde{q}(t) := m_t(q(t)) \in \mathcal{S}$ satisfies the following Euler-Lagrange equations with an exterior force:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{q}}} - \frac{\partial L}{\partial \tilde{q}} = F \tag{64}$$

where $F : T\mathcal{S} \to \mathcal{T}^*\mathcal{S}$ is a given exterior force. By the Lagrange–d'Alembert principle (see [31]), the following holds: any family of curves $\tilde{q}_\epsilon(t) \in \mathcal{S}$ with

$$\begin{aligned} &\tilde{q}_0(t) = \tilde{q}(t) = m_t(q(t)), \\ &\tilde{q}_\epsilon(a) = m_a(q(a)), \quad \tilde{q}_\epsilon(b) = m_b(q(b)) \end{aligned} \tag{65}$$

for all small ϵ, satisfies

$$\begin{aligned} &\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b L(\tilde{q}_\epsilon(t), \dot{\tilde{q}}_\epsilon(t))dt \\ &\qquad + \int_a^b F(\tilde{q}(t), \dot{\tilde{q}}(t)) \cdot \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{q}_\epsilon(t)dt = 0. \end{aligned} \tag{66}$$

Now pick an arbitrary family of curves $q_\epsilon(t) \in Q$ such that

$$q_0(t) = q(t),\ q_\epsilon(a) = q(a),\ q_\epsilon(b) = q(b) \tag{67}$$

for all small ϵ. Define $\tilde{q}_\epsilon(t) = m_t(q_\epsilon(t))$. Then we can readily check that $\tilde{q}_\epsilon(t)$ satisfies (65) and thus (66). The following equations immediately follow from the definitions and the arguments in the above:

$$\begin{aligned} &\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b L_{m_t}(q_\epsilon(t), \dot{q}_\epsilon(t))dt \\ &\qquad = -\int_a^b F_{m_t}(q(t), \dot{q}(t)) \cdot \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q_\epsilon(t)dt \end{aligned}$$

where $F_{m_t} : TQ \to T^*Q$ is defined by

$$F_{m_t}(q, v) = T^*_{m_t(q)}m_t \cdot F(m_t(q), T_q m_t \cdot v + \mathcal{Z}_t(m_t(q))).$$

By the Lagrange-d'Alembert principle, the above variational equations imply that $q(t) \in Q$ satisfies the following Euler-Lagrange equations with forces :

$$\frac{d}{dt}\frac{\partial L_{m_t}}{\partial \dot{q}} - \frac{\partial L_{m_t}}{\partial q} = F_{m_t}. \tag{68}$$

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