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# Nonuniform Coverage and Cartograms

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***Abstract*— In this paper, we summarize our investigation of nonuniform coverage of a planar region by a network of autonomous, mobile agents. We derive centralized nonuniform coverage control laws from uniform coverage algorithms using cartograms, transformations that map nonuniform metrics to a near Euclidean metric. We also investigate time-varying coverage metrics and the design of control algorithms to cover regions with slowly varying, nonuniform metrics. Our results are applicable to the design of mobile sensor networks, notably when the coverage metric varies as data is collected such as in the case of an information metric. The results apply also to the study of animal groups foraging for food that is nonuniformly distributed and possibly changing.**

## I. INTRODUCTION

Sensor networks in space, in the air, on land, and in the ocean provide the opportunity for unprecedented observational capability. An important problem in this context is to determine how best to distribute sensors over a given area in which the observational field is distributed so that the likelihood of detecting an event of interest is maximized. If the probability distribution of the event is uniform over the area, then the optimal solution is uniform coverage, i.e., uniform distribution of sensors. On the other hand, if this probability distribution is nonuniform, then the sensors should be more (less) densely distributed in subregions with higher (lower) event probability. Further, if the probability distribution changes with time, then the nonuniform distribution should likewise change with time.

A related coverage problem derives from the classic objective analysis (OA) mapping error in problems of sampling (possibly time-varying) scalar fields [2]. OA is linear statistical estimation based on specified field statistics, and the mapping error provides a measure of statistical uncertainty of the model. Since reduced uncertainty, equivalent to increased entropic information, implies better measurement coverage, OA mapping error can be used as a coverage metric [2], [3].

Coverage problems also appear in models of social foraging. Backed by observations of animal behavior across species, biologists model distribution of animals over patchy resource environments according to a measure of patch suitability that depends on factors such as resource richness or conditions for survival [4], [5]. Suitability decreases in time as animals consume (and animals will abandon patches where suitability has declined). Coverage studies of changing, nonuniform environments may prove useful in helping to explain how animal groups move and redistribute.

Coverage algorithms for a group of dynamic agents in uniform fields (or nonuniform but symmetric fields) include those described in [6]–[9]. Further results are presented in [9] for a coverage metric defined in terms of the Euclidean metric with a weighting factor that allows for nonuniformity. In [6], [9], the methodology makes use of Voronoi cells and Lloyd descent algorithms.

In this paper we concentrate on planar regions and propose an approach to coverage control that makes use of existing algorithms designed for uniform coverage and extends these to nonuniform metrics. We are particularly interested in metrics defined in terms of non-Euclidean distance functions that effectively stretch and shrink space in lower and higher density regions of a given space. This yields optimal configurations where regions with a high density of resource or information are patrolled by more agents. Non-Euclidean distance metrics present challenges to existing techniques. For example, in the case of [6], [9], computing Voronoi cells with non-Euclidean metrics is computationally complex. For each point on a dense grid, one needs to compute the (non-Euclidean) distance to each agent and find the minimum.

The first step in our method is to compute a nonuniform change of coordinates on the original compact set with a non-Euclidean metric that maps to a new compact set with a near Euclidean metric. Such a map is called a *cartogram*. Inspired by the work of Gastner and Newman [10], we compute the cartogram from a diffusion equation. Gastner and Newman used cartograms in several applications [11], [12] where it is sufficient to compute single cartograms. Since we are interested in computing a series of cartograms for feedback control, we propose a method to compute cartograms that vary smoothly as a function of the density distribution.

A uniform control law can be used in the cartogram space since the metric in this space is almost Euclidean. The preimage of the control law yields convergent dynamics in the original space. We prove under certain conditions that these convergent dynamics optimize the nonuniform coverage metric. We extend to the time-varying metric case.

In section II we review the uniform coverage control of [6]. We describe the nonuniform coverage problem in section III. Cartograms are defined in section IV. Gastner and Newman's method for computing cartograms is reviewed, and our new approach to computing smooth cartograms is presented. In section V we prove our approach to nonuniform coverage control that makes use of cartograms and extends to slowly time-varying metrics. We illustrate with an example in section VI. Final remarks are given in section VII.

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## II. UNIFORM COVERAGE

In the uniform coverage approach of Cortés and Bullo [6], $n$ vehicles move in a region $\mathcal{D}$, with a polygonal boundary $\partial\mathcal{D}$. The vehicles obey first-order dynamics:

$$\dot{\mathbf{x}}_i = \mathbf{u}_i(\mathbf{x}_1, \ldots, \mathbf{x}_n), \tag{1}$$

where $\mathbf{x}_i$ is the position of the $i$th vehicle and $\mathbf{u}_i$ is the control input to the $i$th vehicle.

The goal is to bring the vehicles, from their initial positions, to a (static) configuration that maximizes coverage of the domain. To define *maximum coverage*, Cortés and Bullo consider multicenter metric functions such as

$$\Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \max_{\mathbf{x}\in\mathcal{D}} \Big\{ \min_{i=1\cdots n} d(\mathbf{x}, \mathbf{x}_i) \Big\}, \tag{2}$$

where $d(\mathbf{x}, \mathbf{x}_i) = \|\mathbf{x} - \mathbf{x}_i\|$ is the Euclidean distance. Given the position of the $n$ vehicles, computing the metric requires computing the distance from any point $\mathbf{x} \in \mathcal{D}$ to the closest vehicle. The metric $\Phi$ is equal to the largest of these distances. As a result, the maximum distance between any point of the domain and the closest vehicle is always smaller than or equal to $\Phi$. Intuitively, a smaller $\Phi$ implies that the corresponding array of vehicles $\mathbf{x}_i$ achieves a better coverage of the domain $\mathcal{D}$.

Assuming that all of the vehicles have the same constant speed, $\Phi$ is proportional to the maximum time it takes for a vehicle to reach an arbitrary point of the domain. For this reason, Cortés and Bullo define optimal coverage as the minimum of the cost function $\Phi$.

One of the main results of [6] is the development of a stable procedure to bring the vehicles into a configuration that minimizes the metric $\Phi$. To this end, the Voronoi cell of each vehicle is computed repeatedly. The Voronoi cell for the $i$th vehicle is a polygonal subset of the domain $\mathcal{D}$ that contains all of the points that are closer to the $i$th vehicle than any other vehicle. Each vehicle is then directed to move toward the circumcenter of its Voronoi cell. Once all the vehicles reach the circumcenter of their Voronoi cell, the coverage metric $\Phi$ is minimum. Cortés and Bullo show that, from any initial position where the vehicles are not exactly on top of each other, their algorithm converges toward the optimal configuration.

## III. NONUNIFORM COVERAGE

We develop an approach that extends optimal coverage strategies to more general metrics, notably to nonuniform and time-varying metrics. We are particularly interested in metrics defined in terms of a (possibly time-varying) distance function that is non-Euclidean. In the approach of [6], the region of dominance of an agent might include points that can be reached more easily by other agents. In this paper, we consider the dominance region of an agent $\mathbf{x}_i$ as a defined region containing all of the points that are closer to $\mathbf{x}_i$ than any other agent in the sense of the non-Euclidean metric, i.e., that can be reached more easily by agent $\mathbf{x}_i$ than by any other agent, where ease in reaching a point depends on the density of resource or information. The dominance region is still a Voronoi cell, and the nonuniform density is introduced through the distance function used to compute the Voronoi cells. The nonuniform distance shrinks along paths where resources are sparse and increases along paths where resources are plentiful.

If the density of information $\rho : \mathcal{D} \to \mathbb{R}_0^+$ is not uniform, then we can define a non-Euclidean distance:

$$d_\rho(\mathbf{x}, \mathbf{x}_i) = \min_{\mathcal{C}_{\mathbf{x}}^{\mathbf{x}_i}} \left\{ \int_{\mathcal{C}_{\mathbf{x}}^{\mathbf{x}_i}} \sqrt{\rho}\, \mathrm{dl} \right\}.$$

We use $\sqrt{\rho}$ to weight the distance integral since, in two dimensions, multiplying the distances in each direction by $\sqrt{\rho}$ implies a net volume (or density) change of $\rho$.

In this paper we assume that the coverage metric $\Phi$ is a functional of a distance function $d_\rho$, which depends on the positions of the agents $\mathbf{x}_i$ and the domain $\mathcal{D}$, denoted

$$\Phi = (\Phi[d_\rho])\,(\mathbf{x}_i, \ldots, \mathbf{x}_n; \mathcal{D}). \tag{3}$$

Clearly, one can use any metric $\Phi$ that involves only the Euclidean distance, such as the multicenter function (2), and make it inhomogeneous by replacing the Euclidean distance $d$ with the weighted distance $d_\rho$. If $\rho$ represents the density distribution for information or resources, then optimal coverage solutions correspond to evenly distributed information or resources to each agent's dominance region.

In [9], Cortés et al. design coverage control algorithms for a density-dependent metric defined, as a function of a given array of agents $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, by

$$\Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \int_{\mathcal{D}} \min_i \Big\{ f(d(\mathbf{x}, \mathbf{x}_i))\, \rho(\mathbf{x}) \Big\}\, \mathrm{d}\mathbf{x},$$

where $d$ is the Euclidean distance function, $f$ is a nondecreasing function and $\rho$ is the distribution density function. Because the metric depends on $d$, the cost function can be rewritten as

$$\Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} \int_{V_i} f(d(\mathbf{x}, \mathbf{x}_i))\, \rho(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

where the Voronoi cells $V_i$ are defined also by the Euclidean distance function as

$$V_i = \Big\{ \mathbf{x} \in \mathcal{D} \;\Big|\; d(\mathbf{x}, \mathbf{x}_i) \leq d(\mathbf{x}, \mathbf{x}_j)\ \forall j \neq i \Big\}.$$

As shown in [9], this means that the cost function can be seen as the contribution of $n$ dominance regions $V_i$, each of which is the Voronoi cell of an agent. Although this metric yields coverage solutions that are nonuniform, the information or resource will nonetheless *not* be equally distributed among corresponding dominance regions.

In this paper, we are interested in cost functions of the form (3). Indeed, a metric based on a nonuniform distance $d_\rho$ is more closely related to information gathering and sensing array optimization. One such problem is the detection of acoustic signals. In this case, $\sqrt{\rho}$ represents the nonuniform refractive index of the environment. The objective is to place the sensors in such a way that they can detect sources anywhere. In other words, one needs to minimize the weighted distance $d_\rho$ between any point in the domain and the agents.

Another typical problem consists in increasing the ability of the array on an uneven terrain. This situation is typical for mine hunting arrays in a standby mode; the optimal configuration minimizes the time that it would take to send one of the agents to a newly detected mine. In this case, the square root of $\rho(\mathbf{x})$ represents the roughness of the terrain, the infinitesimal time it takes to cross an infinitesimal path located in $\mathbf{x}$. The goal is to position the agents in such a way that any point of the domain can be reached by one of the agents in minimum time. The optimal solution corresponds to the minimum of the cost function in (3), where $d_\rho(\mathbf{x}, \mathbf{y})$ is the minimum travel time between points $\mathbf{x}$ and $\mathbf{y}$.

A similar practical situation can arise in the deployment of salesmen across a city with various speed zones. To respond to a call in as little time as possible, the salesmen make detours and avoid areas with low speed limits. In this example, the density $\rho$ is the square of the speed limit and the minimum of the cost function in (3) corresponds to an initial distribution of salesmen that minimizes the maximum response time to an arbitrary point in the city. The residual value of the cost function at the minimum gives the maximum time that the first customer has to wait before a salesman arrives on site.

In this paper, we assume that a particular cost function of the form (3) has been selected and that there exists a stable algorithm that brings a group of vehicles to the minimum of $\Phi$ for the Euclidean distance. We provide a methodology to modify this algorithm when the non-Euclidean distance (e.g., terrain roughness, acoustic refraction) is used.

## IV. Cartograms

Our approach to deriving coverage control strategies for nonuniform and time-varying metrics is to find a standard method to modify a control law defined for the Euclidean metric in such a way that it remains stable and converges to the minimum of the nonuniform metric. The method that we develop is based on a nonuniform change of coordinates that transforms the domain $\mathcal{D}$ with the non-Euclidean distance into another compact set $\mathcal{D}'$ where the distance is Euclidean or near Euclidean. Such transformations are commonly referred to as "cartograms" in computer graphics.

To motivate the notion of cartogram, consider how poorly census and election results are represented using standard geographical projections; such data are better plotted on maps in which the sizes of geographic regions such as countries or provinces appear in proportion to their population (as opposed to the geographical area). Such maps, which are cartograms, transform the physical space $\mathcal{D}$ into a fictitious space $\mathcal{D}'$ where the area element $A$ is proportional to a nonuniform density $\rho : \mathcal{D} \rightarrow \mathbb{R}_0^+$.

*Definition 1 (cartogram):* Given a compact domain $\mathcal{D} \subset \mathbb{R}^2$ and a density function $\rho : \mathcal{D} \rightarrow \mathbb{R}_0^+$, a cartogram is a $C^1$ (continuous everywhere and with continuous derivatives almost everywhere) mapping $\phi : \mathcal{D} \rightarrow \mathcal{D}' : \mathbf{x} \rightarrow \phi(\mathbf{x})$ such that

$$\det\left(\frac{\partial \phi}{\partial \mathbf{x}}\right) = \rho.$$

Our method for computing cartograms, inspired by the approach of Gastner and Newman [10], is presented in this section. Given a domain $\mathcal{D}$ and a density function $\rho$, there are infinitely many possible cartograms. As stated in [10], the objective is to minimize the distortion of the original figure. A perfect cartogram would not introduce any deformation and would satisfy

$$\frac{\partial \phi}{\partial \mathbf{x}} = \sqrt{\rho}\, \mathbb{I},$$

where $\mathbb{I}$ is the identity matrix. Clearly, such a cartogram does not exist for most density functions $\rho$. Nevertheless, we seek to reduce the distortion and to minimize $\max_{\mathbf{x}} \left\| \frac{\partial \phi}{\partial \mathbf{x}} - \sqrt{\rho}\, \mathbb{I} \right\|$, where $\| \cdot \|$ is any norm on the space of $2 \times 2$ matrices.

*Definition 2 (perfect cartogram):* For a given density function $\rho$, a perfect cartogram, if it exists, is a cartogram such that $\left\| \frac{\partial \phi}{\partial \mathbf{x}} - \sqrt{\rho}\, \mathbb{I} \right\| = 0$.

*Definition 3 (ideal cartogram):* For a given density function $\rho$, an ideal cartogram is given by

$$\underset{\phi}{\operatorname{Argmin}} \left( \max_{\mathbf{x}} \left\| \frac{\partial \phi}{\partial \mathbf{x}} - \sqrt{\rho}\, \mathbb{I} \right\| \right).$$

### A. Cartograms with fixed boundaries

Gastner and Newman [10] showed how to construct a cartogram using a diffusion equation. Their work has shown that, among all known methods to compute cartograms, the diffusion method introduces very little distortion and produces maps that are the closest to the perfect diagonal form $\sqrt{\rho}\, \mathbb{I}$.

To describe the method of [10], we first address the case in which, for a given $\rho$, the normal component of $\nabla \rho$ along the boundary $\partial \mathcal{D}$ vanishes. In this case, there exists a cartogram $\phi : \mathcal{D} \rightarrow \mathcal{D}'$, where $\mathcal{D}' = \mathcal{D}$. To show the existence of the cartogram and to determine a method to compute it, we imagine that the domain $\mathcal{D}$ is filled with a fluid whose initial density is given by $\rho$. As time evolves, the gradient of the density creates motion and the density of the fluid tends to homogenize. Let us consider the density $c(\mathbf{x}, t)$ at point $\mathbf{x}$ and time $t$. It satisfies the diffusion equation

$$\frac{\partial c}{\partial t} = \nu \Delta c\ ,$$

where the initial condition is $c(\mathbf{x}, 0) = \rho(\mathbf{x})$, the boundary condition is $\frac{\partial c}{\partial n} = 0$, and $\nu > 0$ is arbitrary. For $t \rightarrow +\infty$, the density $c$ tends to a constant distribution $c_\infty$ and, at any time $t$ and at any position $\mathbf{x}$, we have $c(\mathbf{x}, t) > 0$. As a result, we can define a velocity field:

$$\mathbf{v}(\mathbf{x}, t) = -\nu \frac{\nabla c}{c}(\mathbf{x}, t).$$

Given the initial position $\mathbf{x}_0$ at which a particle is released at time $t = 0$, the velocity field above determines the position at any later time $t$. The flow (i.e., the trajectories) of the velocity field is a function of time and of the initial position. We denote by $\mathbf{x}(t; \mathbf{x}_0)$ the unique trajectory that satisfies

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v}\left(\mathbf{x}(t; \mathbf{x}_0), t\right), \\ \mathbf{x}(0; \mathbf{x}_0) = \mathbf{x}_0. \end{cases}$$

The domain $\mathcal{D}$ is compact; hence the trajectories $\mathbf{x}$ are at least $C^1$ on any *finite interval* of time $[0, t]$ (see, e.g., [13]). In this case, however, $c$ is the solution of the diffusion equation and the magnitude of its gradient decays exponentially with time while $c$ approaches its average, $c_\infty$. As a result, the velocity field $\mathbf{v}$ also decays exponentially in time. This is a sufficient condition for the trajectories $\mathbf{x}(t, \mathbf{x}_0)$ to be $C^1$ on the *infinite* interval $t \in [0, +\infty[$. We define

$$\phi(\mathbf{x}_0) = \lim_{t \to +\infty} \mathbf{x}(t, \mathbf{x}_0).$$

The limit exists, is unique, and is a $C^1$ function of its argument $\mathbf{x}_0$. To check that this transformation is a cartogram, we need to show that the area element starting in $x_0$ is indeed scaled by a factor $\rho(x_0)$. Recall that Liouville's theorem determines how area elements $A$ change along trajectories:

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln A \bigg|_{\mathbf{x}(t;\mathbf{x}_0),t} = \mathrm{div}\left(\mathbf{v}(\mathbf{x}, t)\right). \tag{4}$$

Direct computation shows that

$$\mathrm{div}\left(\mathbf{v}\right) = -\frac{\nu}{c}\Delta c + \frac{1}{\nu}\mathbf{v}^2.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln c = \frac{1}{c}\frac{\partial c}{\partial t} + \frac{\mathbf{v} \cdot \nabla c}{c} = -\mathrm{div}\left(\mathbf{v}\right).$$

As a result, Liouville's equation (4) simplifies to

$$A(t) = A(0)\, e^{-\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \ln c \,\mathrm{d}t} = A(0)\, \frac{c(\mathbf{x}_0, 0)}{c(\mathbf{x}, t)} = A(0)\, \frac{\rho(\mathbf{x}_0)}{c(\mathbf{x}, t)}.$$

For $t \to +\infty$, the density $c$ becomes constant in space and we have

$$\det\left(\frac{\partial \phi}{\partial \mathbf{x}_0}(\mathbf{x}_0)\right) = \lim_{t \to +\infty} \frac{A(t)}{A(0)} = \frac{\rho(\mathbf{x}_0)}{c_\infty}.$$

### *B. Cartograms with moving boundaries*

The conclusions reached for cartograms with constant boundaries do not translate immediately to cases where $\frac{\partial \rho}{\partial n} \neq 0$ on the boundary of the domain. In this case, we cannot apply the method described in the previous section, and theorems about existence and smoothness of the diffusion problem, as well as about the advection of the velocity field, are not applicable either. Gastner and Newman [10] suggest extending the density to a larger domain where Neumann boundary conditions are enforced. Given a function $\rho : \mathcal{D} \to \mathbb{R}_0^+$, one can select a larger domain $\mathcal{D}_0 \supset \mathcal{D}$ and pick an arbitrary function $\hat{\rho} : \mathcal{D}_0 \to \mathbb{R}_0^+$ such that $\hat{\rho}$ is identical to $\rho$ in $\mathcal{D}$. Typically, $\mathcal{D}_0$ has an area of 4 or 9 times the initial domain $\mathcal{D}$. The goal is to design $\hat{\rho}$ in such a way that $\frac{\partial \hat{\rho}}{\partial n} = 0$ at the edges of the larger domain $\mathcal{D}_0$ (see Figure 1). This permits the computation of the diffusion cartogram for the large domain $\mathcal{D}_0$ with fixed boundaries, followed by a restriction of the transformation to $\mathcal{D}$ to obtain the cartogram for the initial domain. This procedure is dependent on the choice of the embedding domain $\mathcal{D}_0$. Given $\mathcal{D}_0$, it also depends on how the extended density function $\hat{\rho}$ is constructed in $\mathcal{D}_0 \setminus \mathcal{D}$.

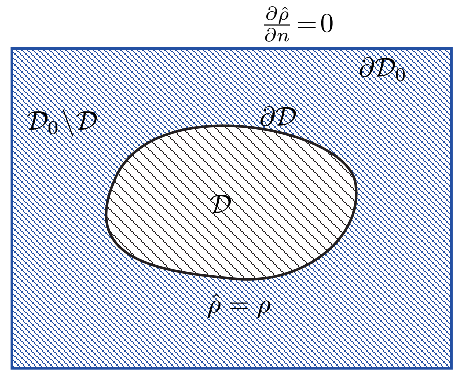


Fig. 1. Proposed approach: when computing a cartogram for a domain $\mathcal{D}$ that has an arbitrary shape or for which the normal derivative of the density function $\rho$ does not vanish at the boundary, a large rectangle $\mathcal{D}_0 \supset \mathcal{D}$ is selected. The density $\rho$ is extended outside $\mathcal{D}$ by enforcing Neumann boundary conditions at the boundary of the large rectangle, requiring continuity of $\hat{\rho}$ at the edge with $\mathcal{D}$, and setting the Laplacian of $\hat{\rho}$ to a constant value outside $\mathcal{D}$. This defines a unique extension $\hat{\rho}$ which is continuous and has continuous derivatives almost everywhere.

Gastner and Newman showed the importance of applying a "neutral buoyancy" condition, which keeps the total area under consideration constant. To construct $\hat{\rho}$, they first computed the average density in $\mathcal{D}$. In $\mathcal{D}_0 \setminus \mathcal{D}$, they filled $\hat{\rho}$ with a constant equal to the mean density in $\mathcal{D}$.

For our control design problem, the method above has an important flaw: $\hat{\rho}$, the initial condition for the diffusion problem, is not continuous at the boundary of $\mathcal{D}$. As a result, existence, uniqueness, and smoothness of the solution of the diffusion problem are not guaranteed. This is not necessarily a problem when producing only one cartogram. Our objective, however, is to produce continuous sequences of maps. Indeed, we will need the cartogram to vary smoothly when the density function is changed. For example, transferring Lyapunov functions from the cartogram space to the physical plane requires the existence of continuous derivatives.

We propose the following alternative to the method of [10]. Given $\rho$ in the domain of interest $\mathcal{D}$, we compute $\frac{\partial \rho}{\partial n}$ at the boundary of $\mathcal{D}$ and the total flux across $\partial \mathcal{D}$. We define the extended density $\hat{\rho}$ as follows:

- Inside $\mathcal{D}$, $\hat{\rho}(\mathbf{x}) = \rho(\mathbf{x})$.
- Outside $\mathcal{D}$, $\hat{\rho}$ is the solution of

$$\begin{cases} \Delta \hat{\rho} = \dfrac{-1}{\mathrm{Area}(\mathcal{D}_0 \setminus \mathcal{D})} \displaystyle\int_{\mathcal{D}} \Delta \rho(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\ \qquad = \dfrac{-1}{\mathrm{Area}(\mathcal{D}_0 \setminus \mathcal{D})} \displaystyle\oint_{\partial \mathcal{D}} \frac{\partial \rho}{\partial n} \,\mathrm{d}l, \\ \dfrac{\partial \hat{\rho}}{\partial n}\bigg|_{\partial \mathcal{D}_0} = 0, \quad \hat{\rho}|_{\partial \mathcal{D}} = \rho|_{\partial \mathcal{D}}. \end{cases} \tag{5}$$

The equations above define $\hat{\rho}$ inside $\mathcal{D}_0 \setminus \mathcal{D}$ as the solution of a linear problem with inhomogeneous Neumann boundary conditions. The Laplacian of $\hat{\rho}$ is constant in $\mathcal{D}_0 \setminus \mathcal{D}$, and its value is set so it compensates exactly the flux through the inside hole $\mathcal{D}$. Indeed, Green's equality requires

$$\int_{\mathcal{D}_0} \Delta \hat{\rho} \,\mathrm{d}\mathbf{x} = \oint_{\partial \mathcal{D}_0} \frac{\partial \hat{\rho}}{\partial n} \,\mathrm{d}l = 0.$$

Since this problem is compatible, standard results in functional analysis [14], [15] guarantee that the solution is unique and belongs to the Sobolev space $H_1$, which contains the functions on $\mathcal{D}_0$ that are continuous everywhere and for which the derivatives are continuous almost everywhere. This guarantees also that the resulting extended density, $\hat{\rho}$, can be used as the initial condition of the diffusion problem and provides a $C^1$ solution $c$. The resulting transformation $\phi(\mathbf{x})$ is unique and varies smoothly (i.e., in a $C^1$ fashion) when the input density $\rho$ is changed.

Gastner and Newman showed how the diffusion problem on a rectangle can be efficiently solved using the Fourier transform of $c(\mathbf{x}, t)$. This transforms the problem into an ordinary differential equation where the variables are the Fourier coefficients [10]. The only difference between our procedure and that of Gastner and Newman is how the density $\rho$ is extended from the domain of interest $\mathcal{D}$ to the larger square $\mathcal{D}_0$. To solve (5) and obtain $\hat{\rho}$ on $D_0 \backslash D$, we use the finite element method (see [14], [15]). Since the equations giving the extended density $\hat{\rho}$ are linear, the computational cost is negligible with respect to the time that it would take to compute the nonlinear boundaries of the Voronoi cells for the non-Euclidean metric.

## V. Nonuniform coverage control

### A. Method

Cartograms can be used to extend any algorithm that minimizes the uniform coverage metric, based on the Euclidean distance, to an algorithm that minimizes a nonuniform coverage metric dependent on an arbitrary "weighted" distance $d_\rho$. Indeed, starting from a non-Euclidean distance $d_\rho$, a perfect cartogram gives a transformation $\mathbf{y} = \phi(\mathbf{x})$ such that the distance function is Euclidean for $\mathbf{y}$. As a result, one can apply the uniform coverage algorithm to the $\mathbf{y}$ coordinates and prove convergence in the transformed space. In Theorem 1 below, we prove conditions under which convergence to the minimum of the uniform metric in the transformed space implies convergence to the minimum of the nonuniform metric in the original domain $\mathcal{D}$. The control law in the physical space for a system of agents with dynamics given by (1) can then be recovered as

$$\mathbf{u} = \dot{\mathbf{x}} = \left.\frac{\partial \phi^{-1}}{\partial \mathbf{y}}\right|_{\phi(\mathbf{x})} \dot{\mathbf{y}}.$$

### B. Convergence

Assume that a feedback control law has been designed and converges to the unique minimum of a cost function based on the Euclidean distance. We consider a nonuniform distance $d_\rho$ and investigate how the control law for the Euclidean distance behaves in a near perfect cartogram of $\rho$. We show that, for $C^1$, strictly positive $\rho$, the non-Euclidean cost function has a unique minimum. Furthermore, the cartogram inverse-mapped feedback control converges toward this minimum.

*Theorem 1 (nonuniform coverage by cartograms):* Consider a $C^1$ cost function $\left(\Phi[d_\rho]\right)(\mathbf{x}_i, \ldots, \mathbf{x}_n; \mathcal{D})$ that depends only on the distance $d_\rho(\mathbf{a}, \mathbf{b}) = \min_{\mathcal{C}_\mathbf{a}^\mathbf{b}} \int_{\mathcal{C}_\mathbf{a}^\mathbf{b}} \sqrt{\rho}\ \mathrm{d}l$ between $n$ agent positions and points in the domain $\mathcal{D}$. We assume that $\Phi$ has a unique, nondegenerate minimum for the Euclidean distance $d_1(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$. We also assume that there exists a feedback control law $\dot{\mathbf{x}}_i = \mathbf{v}_i(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ that brings the vehicles to the minimum *for the Euclidean distance* $d_1$.

Given a density function $\rho : \mathcal{D} \to \mathbb{R}_0^+$, consider a cartogram $\phi : \mathcal{D} \to \phi(\mathcal{D})$. We consider applying the control law for the Euclidean distance in the cartogram space; hence

$$\dot{\mathbf{y}}_i = \mathbf{v}_i(\mathbf{y}_1, \ldots, \mathbf{y}_n),$$

where $\mathbf{y}_i = \phi(\mathbf{x}_i)$. The corresponding dynamics in the physical space $\mathcal{D}$

$$\dot{\mathbf{x}}_i = \mathbf{u}_i = \left.\frac{\partial \phi^{-1}}{\partial \mathbf{y}}\right|_{\phi(\mathbf{x})} \dot{\mathbf{y}}_i$$

yield a convergent sequence. In the neighborhood of a perfect cartogram, we have the following:

1. There is a unique minimum of $\left(\Phi[d_\rho]\right)(\mathbf{x}_i, \ldots, \mathbf{x}_n, \mathcal{D})$ on $\mathcal{D}$.
2. The agents converge to an equilibrium configuration that tends continuously to the unique minimum as $\max_\mathbf{x} \left\| \frac{\partial \phi}{\partial \mathbf{x}} - \sqrt{\rho}\ \mathbb{I} \right\| \to 0$.

The proof can be found in [1].

### C. Space-time optimal coverage

The method developed in this paper is well suited for time-varying metrics. For example, when the density $\rho$ is a physical quantity, such as the refractive index, it changes according to the fluctuations in the environment (e.g., sources, sinks, diffusion, advection). The objective analysis (OA) information map in [3] is an example of a consumable resource that varies as sensors move around the area to be covered. The numerical method presented above is aimed at producing cartograms that depend smoothly on the density function $\rho$. In other words, for a density function $\rho(\mathbf{x}, t)$ that is $C^1$ in time, we find a family of cartograms $\phi_t : \mathcal{D} \to \mathcal{D}'_t$ where both the transformation $\phi_t$ and the transformed space $\mathcal{D}'_t$ change with time in a $C^1$ fashion.

For autonomous, nonuniform metrics, we proved uniqueness of the optimal configuration and convergence to this position. The algorithm applies well to the case of time-varying metrics. If the density function changes slowly enough (in comparison to agent speed) and, at any time $t$, the maximum distortion $\frac{1}{\sqrt{\rho}} \left\| \frac{\partial \phi}{\partial \mathbf{x}} - \sqrt{\rho}\ \mathbb{I} \right\|$ is sufficiently small, then convergence can be inferred by our theorem. The requirement that $\rho(t)$ does not change too fast guarantees that the cartogram does not change too fast, and, hence, the boundary $\phi(\mathcal{D})$ does not change too fast with respect to vehicle speed. As a result, the motion of the cartogram boundary (slow dynamics) and the motion of the vehicles (fast dynamics) are almost decoupled, and we infer convergence from the fact that vehicles are converging to the equilibrium on timescales much shorter than the timescale at which $\phi(\mathcal{D})$ changes.

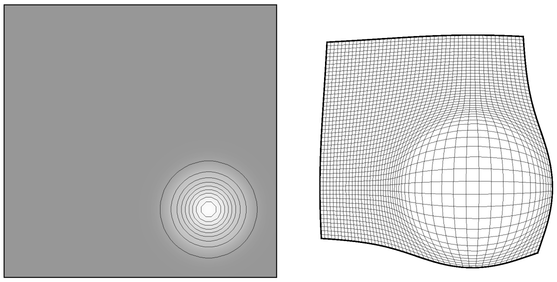

Fig. 2. Cartogram of the unit square. Left panel: Physical domain $\mathcal{D}$ with level sets of density function $\rho(x, y)$ given by (7). Right panel: Cartogram $\mathcal{D}'$ and image of a Cartesian mesh.

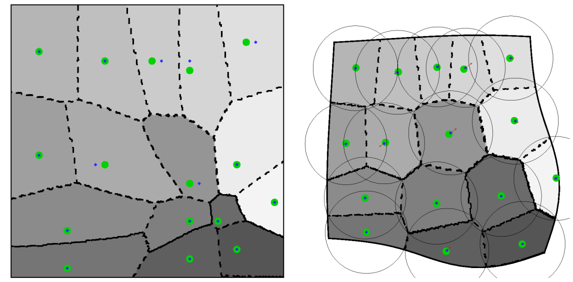

Fig. 3. Convergence of sixteen vehicles to static nonuniform coverage of density function $\rho(x, y)$ given by (7). Left panel: Resulting positions of the vehicles in physical space $\mathcal{D}$. Right panel: Solution in cartogram space $\mathcal{D}'$.

## VI. Example

As an example, we let $\mathcal{D}$ be the unit square and we consider the multicenter coverage metric (2), where we replace the Euclidean distance $d$ with a non-Euclidean distance $d_\rho$:

$$\Phi[d_\rho](\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \max_{\mathbf{x}\in\mathcal{D}} \left\{ \min_{i=1\cdots n} d_\rho(\mathbf{x}, \mathbf{x}_i) \right\}. \quad (6)$$

We set the density function $\rho : \mathcal{D} \to \mathbb{R}_0^+$ to

$$\rho(x, y) = \frac{3}{40} + e^{-\frac{\left(x-\frac{3}{4}\right)^2 + \left(y-\frac{1}{4}\right)^2}{\left(\frac{1}{10}\right)^2}}, \quad (7)$$

which represents our nonuniform interest in the features contained inside the unit square. The density $\rho$ is plotted in the left panel of Figure 2. The lower right quadrant of the square has a much higher density and must be covered more densely than the rest of the square. In the analogy with a group of animals, the peak at $\left(\frac{3}{4}, \frac{1}{4}\right)$ represents a region with larger food supply. In the analogy with a mine hunting array, the peak is a region where the agents move more slowly. To be able to respond anywhere in minimum time, the vehicles must be closer to each other in the lower right quadrant.

To derive coverage control laws for $n = 16$ vehicles with dynamics given by (1), we first perform a cartogram of the area (see the right panel of Figure 2). The area near the peak of the Gaussian source is stretched by the transformation and represents about 30% of the mapped domain $\mathcal{D}'$, while it accounts for less than 10% of the physical domain $\mathcal{D}$.

We apply the uniform coverage control law of [6] in $\mathcal{D}'$ which guarantees convergence to the optimal configuration in $\mathcal{D}'$ and, by Theorem 1, convergence to optimal nonuniform coverage in $\mathcal{D}$. Figure 3 shows the steady-state configuration of the vehicles in $\mathcal{D}$ (left panel) and $\mathcal{D}'$ (right panel). The optimal configuration segments $\mathcal{D}'$ into sixteen Voronoi cells of equal area, but, in the physical space, this corresponds to sixteen (nonpolygonal) regions of unequal area; i.e., coverage is increased in the lower right quadrant. Examples of nonuniform coverage using the cartogram approach in the case of a slowly time-varying density can be found in [1].

## VII. Final Remarks

We investigated the use of cartograms to achieve time-varying, nonuniform coverage of a spatial domain by a group of agents. Our method permits generalizing many existing uniform coverage algorithms to nonuniform metrics. It also provides a simple and fast control law; e.g., as compared to computing Voronoi cells for a nonuniform metric.

In ongoing work we are investigating extensions of the proposed approach to a possibly fast changing density such as statistical uncertainty in a model of a spatial field, which changes with the motion of the sampling agents. Additionally, a distributed version of the proposed approach may be possible, provided that each agent computes its own local diffusion equation. In this case information passed from neighbors would be used to determine boundary conditions.

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