In Proceedings of the 54th Annual Allerton Conference on Communication, Control and Computing, IL, 2016

Synchronization Bound for Networks of Nonlinear Oscillators

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***Abstract*— Investigation of synchronization phenomena in networks of coupled nonlinear oscillators plays a pivotal role in understanding the behavior of biological and mechanical systems with oscillatory properties. We derive a general sufficient condition for synchronization of a network of nonlinear oscillators using a nonsmooth Lyapunov function, and we obtain conditions under which synchronization is guaranteed for a network of Fitzhugh-Nagumo (FN) oscillators in biologically relevant model parameter regimes. We incorporate two types of heterogeneity into our study of FN oscillators: 1) the network structure is arbitrary and 2) the oscillators have non-identical external inputs. Understanding the effects of heterogeneities on synchronization of oscillators with inputs provides a promising step toward control of key aspects of networked oscillatory systems.**

***Index Terms*— Complex Networked Systems, Nonlinear Oscillators, Synchronization, Lyapunov Analysis**

I. Introduction

Synchronization phenomena in networks of nonlinear oscillators have critical implications in biology, communications, computer science, power networks, and diverse other disciplines. In biological neuronal networks, synchronization can be beneficial, allowing for production of complex behavior, or detrimental, causing disorders such as Parkinson's disease [12] and epilepsy [4]. Understanding the principles underlying synchronization and related behavior in complex interconnected oscillatory systems is a necessary first step toward effective control for enhancement of desired dynamics and suppression of undesired dynamics.

Among multiple existing methods for finding necessary and sufficient conditions to determine stability of synchronization in nonlinear systems, the master stability function (MSF) approach establishes a necessary condition for synchronization in systems of oscillators with linear coupling [16]. Complementary sufficient conditions can be found by leveraging passivity properties of the oscillators [17] or by employing approaches based on contraction theory [1], [21]. However, the majority of synchronization conditions expressed in terms of a lower bound on network coupling strength are too loose to accurately describe the emergence of synchronization. Our approach is to build on the semi-passivity method described in [17], [23] to provide a tighter bound on the required coupling strength for synchronization in biologically relevant model parameter regimes.

*This research was supported in part by the Office of Naval Research under ONR grant N00014-14-1-0635 and by the National Science Foundation under Grant No. DGE-1656466.

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In this paper, we present a new sufficient condition for synchronization in a network of nonlinear oscillators whose dynamics can be represented by ordinary differential equations composed of polynomial functions of the state. This class of models generalizes well-known oscillator models including the Van der Pol oscillator, the FitzHugh-Nagumo (FN) neuronal model [6], [14], and the Hindmarsh-Rose neuronal model [8]. We consider dynamics that are strictly semi-passive and use a nonsmooth Lyapunov function [5] to find a sufficient condition for full synchronization in terms of a lower bound on coupling strength in an arbitrary network of oscillators with identical parameters. We apply this result to compute the bound for a network FN oscillators with identical external inputs to fully synchronize, and we show it is a tighter bound than bounds derived from related methods for relevant parameter regimes.

We then introduce the concepts of input-equivalence [20] and cluster synchronization [2], [22] to extend the nonsmooth Lyapunov analysis to networks of FN oscillators with non-identical external inputs. We calculate the sufficient condition for synchronization in clusters in a representative system to illustrate the utility of the nonsmooth Lyapunov method.

An understanding of how and when synchronization occurs promises to be an invaluable tool for informing experimental studies of oscillator ensembles and a basis for examining mechanisms for the emergence of abnormal synchronization.

II. Network Model

In this paper, we consider a network of n nonlinear oscillators with identical internal dynamics, and assume they interact over a connected, undirected graph $\mathcal{G}$. We let $\mathbf{x}^i \in \mathbb{R}^N$ denote the state of the i-th node, and we define the underlying dynamics as

$$\dot{\mathbf{x}}^i = \mathbf{f}(\mathbf{x}^i) + Bu^i \tag{1}$$

for $i = 1, \dots, n$. Each component of $\mathbf{f} : \mathbb{R}^N \to \mathbb{R}^N$ is a polynomial function of the state of the oscillator. $B \in \mathbb{R}^{N \times 1}$ captures how the social input u^i (due to influence from neighbors) affects the individual states of the i-th node. We assume B to be a vector of zeros with a one in its first row, thereby implying that the social input has a direct impact on only the first variable[1] of the state $\mathbf{x}^i$. The dynamics of an oscillator may also depend on an external input I^i. We examine the influence of identical and non-identical external inputs in Sections IV and V, respectively, in the case of Fitzhugh-Nagumo oscillators.

[1]In a neuronal oscillator context, the first variable x_1^i is typically interpreted as the underlying membrane potential.

We assume the social input u^i provides a linear diffusive coupling between neighbors in the graph $\mathcal{G}$. Let $A = [a_{ij}]$ with $a_{ij} \in [0, 1]$ represent the weighted adjacency matrix of $\mathcal{G}$. We represent the linear diffusive coupling term u^i as

$$u^i = \sum_{j=1}^{n} \gamma a_{ij}(x_1^j - x_1^i), \tag{2}$$

where the parameter $\gamma > 0$ is the coupling strength. Next, we define $\mathbf{x}_1 = [x_1^1, x_1^2, \ldots, x_1^n]^T$ and $\mathbf{u} = [u^1, u^2, \ldots, u^n]^T$ to represent the vectors of first variables of the system states and social inputs, respectively. The diffusive coupling between individual oscillators becomes

$$\mathbf{u} = -\gamma(D - A)\mathbf{x}_1 = -\gamma L \mathbf{x}_1, \tag{3}$$

where $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, $d_i = \sum_{j=1}^{n} a_{ij}$, and $L = D - A$ denotes the Laplacian of the underlying graph.

We restrict our analysis to systems where the dynamics are strictly semi-passive, which allows us to bound the dynamics of each variable for each oscillator.

Definition 2.1 (Strictly Semi-passive): A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$ is *strictly semi-passive* in a region $\mathcal{D} \subset \mathbb{R}^N$ if there exists a nonnegative function $V : \mathcal{D} \to \mathbb{R}_+$ such that $\mathcal{D}$ is open, connected and invariant under the dynamics, $V(\mathbf{x}) > 0$ for $\mathbf{x} \in \mathcal{D} \setminus \{0\}$, $V(0) = 0$, and $\dot{V} \leq \mathbf{y}^T\mathbf{u} - H(\mathbf{x})$, where $H(\mathbf{x}) > 0$ when $\|\mathbf{x}\| \geq r$ with the radius r being dependent on the system parameters.

A strictly semi-passive system behaves like a passive system whenever the system state is sufficiently away from the origin. As the trajectories of a semi-passive system eventually return to the ball of radius r around the origin, the trajectories of the system are ultimately bounded. Furthermore, when a group of n such semi-passive systems are interconnected by a linear diffusive coupling, the closed-loop system has ultimately bounded solutions [17], [18]. We let $\{\beta_1, \beta_2, \ldots, \beta_N\}$ represent the bounds on the state variables for individual oscillators.

III. Nonsmooth Lyapunov Analysis

In this section, we derive a sufficient condition for synchronization in the class of systems described in Section II. To do so, we first define the manifold of synchronized states, and then perform a stability analysis using a nonsmooth Lyapunov approach. By exploiting the properties of Dini derivatives of the associated nonsmooth Lyapunov function, our analysis yields the sufficient condition in terms of coupling strength and network connectivity.

Definition 3.1 (Complete synchronization manifold): The complete synchronization manifold $\mathcal{S}$ is an algebraic manifold in the state space of the full system wherein the states of individual systems are identical:

$$\mathcal{S} = \{\mathbf{x}^1, \ldots, \mathbf{x}^n \in \mathbb{R}^N | x^i = x^j, \ \forall\, i, j = 1, \ldots, n\}.$$

Definition 3.2 (Upper Dini derivative [11]): The upper Dini derivative, also called the upper right hand derivative, of a real valued function $v : \mathbb{R} \to \mathbb{R}$ is defined as

$$D^+ v(t) = \limsup_{h \to 0^+} \frac{v(t + h) - v(t)}{h}. \tag{4}$$

It provides an upper bound for right hand derivatives of v.

Theorem 3.3: Consider the system described in (1) with a linear diffusive coupling on the first variable (2). Assume that (1) is strictly semi-passive. Then, whenever the coupling strength γ and the second smallest eigenvalue of the graph Laplacian $\lambda_2(L)$ (representing network connectivity) satisfy

$$\gamma\lambda_2(L) > \sum_{k=1}^{N} F_{1k} + h_1,$$

$$\text{and} \quad \sum_{k=1}^{N} F_{jk} + h_j < 0 \quad \forall j = 2, \ldots, N,$$

the complete synchronization manifold $\mathcal{S}$ is globally asymptotically stable, where F_{ij}'s and h_i's are functions of system parameters.

Proof: Earlier studies [5] have shown the effectiveness of nonsmooth Lyapunov functions in deriving the critical coupling strength for a complete graph of Kuramoto oscillators. Due to our interest in deriving a sufficient condition for synchronization in terms of a tight lower bound on the coupling strength we follow a similar philosophy, and introduce the following Lyapunov function:

$$V_0(\mathbf{x}) = \sum_{k=1}^{N} \max_{i,j=1,\ldots,n} (x_k^i - x_k^j). \tag{5}$$

The Dini derivative of this nonsmooth Lyapunov function can be expressed as

$$D^+ V_0(\mathbf{x}) = \sum_{k=1}^{N} \dot{x}_k^{m_k} - \dot{x}_k^{l_k}, \tag{6}$$

where m_k and l_k are defined as

$$m_k = \arg\max_{i=1,\ldots,n} \left(x_k^i\right),$$
$$l_k = \arg\min_{i=1,\ldots,n} \left(x_k^i\right).$$

As the dynamics of individual systems are identical, we can rewrite the Dini derivative as

$$D^+ V_0(\mathbf{x}) = \left(u^{m_1} - u^{l_1}\right) + \sum_{k=1}^{N} \left(f_k(\mathbf{x}^{m_k}) - f_k(\mathbf{x}^{l_k})\right),$$

where $f_k : \mathbb{R}^N \to \mathbb{R}$ represents the k-th component of the vector-valued function $\mathbf{f}$.

Let $L^i \in \mathbb{R}^{1\times n}$ denote the i-th row of the graph Laplacian L. Then, we have

$$u^{m_1} - u^{l_1} = \gamma(-L^{m_1} + L^{l_1})\mathbf{x}_1 = (\mathbf{e}_{l_1} - \mathbf{e}_{m_1})^\top \gamma L \mathbf{x}_1$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ constitutes the natural basis for $\mathbb{R}^n$. We can further simplify this expression in terms of the second smallest eigenvalue of the graph Laplacian by bounding the product $L\mathbf{x}_1$ as

$$\begin{aligned}(\mathbf{e}_l-\mathbf{e}_m)^\top \gamma L\mathbf{x}_1 &\leq \gamma\lambda_2(L)(\mathbf{e}_l-\mathbf{e}_m)^\top \mathbf{x}_1\\ &= \gamma\lambda_2(L)(x_1^{l_1}-x_1^{m_1}).\end{aligned}$$

This gives an expression for the derivative in terms of the internal dynamics and second smallest eigenvalue of the graph Laplacian:

$$D^+V_0(\mathbf{x}) = \sum_{k=1}^{N}\Big(f_k(\mathbf{x}^{m_k})-f_k(\mathbf{x}^{l_k})\Big)+\gamma\lambda_2(L)(x_1^{l_1}-x_1^{m_1}).$$

Now we perform a change of coordinates, where $w_k = x_k^{m_k}-x_k^{l_k}>0$ for all k. Then by separating each function into a linear term and a higher order term as

$$f_k(\mathbf{x}^{m_k}) = \mathbf{a}_k\cdot\mathbf{x}^{m_k}+g_k(\mathbf{x}^{m_k}),$$

we have

$$\Big(f_k(\mathbf{x}^{m_k})-f_k(\mathbf{x}^{l_k})\Big) = \mathbf{a}_k\cdot\mathbf{w}+g_k(\mathbf{x}^{m_k})-g_k(\mathbf{x}^{l_k}).$$

This allows us to capture the effect of nonlinearities by putting a bound on $g_k(\mathbf{x}^{m_k})-g_k(\mathbf{x}^{l_k})$, and bound the Dini derivative as

$$D^+V_0(\mathbf{w}) \leq \mathbf{1}\cdot F\mathbf{w}+\mathbf{1}\cdot\tilde{h}(\beta_1,\beta_2,\ldots,\beta_N)\mathbf{w}-\gamma\lambda_2(L)w_1.$$

Here, $\mathbf{1}$ is the vector of all ones, $F\in\mathbb{R}^{N\times N}$ has rows equal to the $\mathbf{a}_k$. The nonlinear behavior is captured by an $N\times N$ matrix $\tilde{h}\triangleq \mathrm{diag}\{h_1,h_2,\ldots,h_N\}$, where the functions $h_1,\ldots,h_N$ depend on the bounds $\beta_1,\ldots,\beta_N$ introduced in Section II.

By construction, each element of $\mathbf{w}$ is positive, so we have $D^+V_0(\mathbf{w})\leq 0$ whenever

$$\gamma\lambda_2(L) > \frac{1}{w_1}\left(\mathbf{1}\cdot F\mathbf{w}+\mathbf{1}\cdot\tilde{h}(\beta_1,\beta_2,\ldots,\beta_N)\mathbf{w}\right).$$

We can write this as N separate conditions:

$$\gamma\lambda_2(L) > \sum_{k=1}^{N}F_{1k}+h_1 \quad (7)$$

$$\sum_{k=1}^{N}F_{jk}+h_j<0 \quad \forall j=2,\ldots,N. \quad (8)$$

Thus, $D^+V_0(\mathbf{w})\leq 0$, and increasing λ_2 will not change this property. So we have found a sufficient condition for local Lyapunov stability of the equilibrium state $\mathbf{w}=0$, which is equivalent to the manifold $\mathcal{S}$. Further, there exists a real number $\phi>0$ such that $D^+V_0(\mathbf{w})\leq-\phi\|\mathbf{w}\|_1$.

To show that $\mathcal{S}$ is attractive, we consider the following integral [7]:

$$\begin{aligned}V_0(\mathbf{w}(t))-V_0(\mathbf{w}(0)) &\leq -\int_0^t \phi\|\mathbf{w}(t)\|_1 dt\\ \Rightarrow V_0(\mathbf{w}(0)) &\geq V_0(\mathbf{w}(t))+\int_0^t\phi\|\mathbf{w}(t)\|_1 dt.\end{aligned}$$

As $V_0(\mathbf{w}(t))\geq 0$ for all $t\neq 0$ by construction, taking the limit $t\to\infty$ we have

$$V_0(\mathbf{w}(0)) \geq \phi\int_0^\infty \|\mathbf{w}(t)\|_1 dt. \quad (9)$$

So the integral in (9) is less than or equal to $V_0(\mathbf{w}(0))/\phi$, which takes a finite value, and the integrand is uniformly continuous. By Barbalat's Lemma, $\mathbf{w}\to 0$ as $t\to\infty$. Since x_k^i are bounded for all i and k, this means that $\mathcal{S}$ is attractive if there are trajectories that originate outside the set. Thus, the complete synchronization manifold $\mathcal{S}$ is globally asymptotically stable. ■

IV. FitzHugh-Nagumo Network With Identical External Inputs

The general argument presented in Section III can be specialized to a particular oscillator model in order to better understand the bound and to compare it with the bounds from other methods. Here, we specialize the bound in Theorem 3.3 to find a sufficient condition for synchronization of a network of FitzHugh-Nagumo (FN) oscillators [6], [14].

The FN model is a two-dimensional reduction of the four-dimensional Hodgkin-Huxley model of the membrane potential dynamics of neurons [9]. It is a comparatively simple model, but captures the distinct quiescent, firing, and saturated states of the system, which depend on the input into the model. We choose to analyze the FN model due to this combination of simplicity and range of possible dynamics.

We consider a network of n FN neuronal oscillators. Each FN oscillator $i=1,\ldots,n$ has two states ($N=2$) with dynamics modeled as

$$\begin{aligned}\dot{x}_1^i &= x_1^i-\frac{{x_1^i}^3}{3}-x_2^i+I^i+u^i\\ \dot{x}_2^i &= \epsilon(x_1^i+a-bx_2^i).\end{aligned} \quad (10)$$

The model parameters $\epsilon\ll 1$, a and b are all positive and the same for every oscillator i. The variable x_1^i represents the membrane potential and operates at a faster timescale than x_2^i, which is the recovery variable. We consider constant external inputs I^i that can be independently assigned to each oscillator in the network.

A. FN Network as a Strictly Semi-passive System

In [17] it was shown that a single FN neuronal oscillator model with dynamics (10) is strictly semi-passive, and thus a network of FN oscillators is ultimately bounded. Following [23], a network of FN oscillators with linear diffusive coupling (2) was shown also to be strictly semi-passive. This can be done using a non-negative function

$$V_P = \sum_{i=1}^{n}\left(\frac{1}{2}{x_1^i}^2+\frac{1}{2\epsilon}{x_2^i}^2\right),$$

which has derivative satisfying

$$\begin{aligned}\dot{V}_P \leq &\underbrace{\sum_{i=1}^{n}x_1^iu^i}_{\mathbf{x}_1^T\mathbf{u}}\\ &\underbrace{-\sum_{i=1}^{n}\left(\frac{x_1^i}{3}\left({x_1^i}^3-3x_1^i-3I^i\right)+bx_2^i\left(x_2^i-\frac{a}{b}\right)\right)}_{H(\mathbf{x}_1,\mathbf{x}_2)}.\end{aligned}$$

It follows that if a single FN neuronal oscillator model is strictly semi-passive, then any network of FN oscillators connected by the linear diffusive coupling is also a strictly semi-passive system with ultimately bounded dynamics [23].

B. Synchronization Bound: Nonsmooth Lyapunov Function

In this section we apply the constructive proof of Theorem 3.3 to compute the corresponding sufficient condition for global asymptotic stability of the complete synchronization manifold $\mathcal{S}$ for a network of FN oscillators with dynamics (10), identical constant external inputs I^i, and linear diffusive coupling (2). From Section IV-A, the system is strictly semi-passive. Following (5), the nonsmooth Lyapunov function is

$$V(\mathbf{x}_1, \mathbf{x}_2) = \max_{i,j=1,\ldots,n} (x_1^i - x_1^j) + \max_{i,j=1,\ldots,n} (x_2^i - x_2^j).$$

The Dini derivative of this Lyapunov function is $D^+V(\mathbf{x}_1, \mathbf{x}_2) = \dot{x}_1^m - \dot{x}_1^l + \dot{x}_2^m - \dot{x}_2^l$. When the external inputs are identical, we can follow the general procedure of the proof and bound $D^+V(\mathbf{x}_1, \mathbf{x}_2)$ as follows:

$$\begin{aligned} D^+V(\mathbf{x}_1, \mathbf{x}_2) \leq & \left(1 + \epsilon + \frac{\beta_1^2}{3}\right)(x_1^m - x_1^l) \\ & + \gamma\lambda_2(L)(x_1^l - x_1^m) - (1 + b\epsilon)(x_2^m - x_2^l). \end{aligned}$$

Here, β_1 is the ultimate bound for the x_1 variable. Since each oscillator model has the same parameters, this bound is the same for each oscillator, independent of its position in the graph. Since the parameters b and ϵ are always positive, $x_2^m > x_2^l$. Thus, $-(1 + b\epsilon)(x_2^m - x_2^l) < 0$ and the condition (8) for synchronization is always satisfied.

To satisfy condition (7) we must have

$$\gamma\lambda_2(L) \geq 1 + \epsilon + \frac{\beta_1^2}{3} = \gamma\lambda_m^*. \tag{11}$$

This provides a sufficient condition for full synchronization of a network of FN oscillators with linear diffusive coupling and identical constant external inputs as a lower bound on the product of the coupling strength γ and the second smallest eigenvalue of the graph Laplacian $\lambda_2(L)$.

C. Synchronization Bound: Quadratic Lyapunov Function

In this section we use a quadratic Lyapunov function to compute a bound on $\gamma\lambda_2(L)$ that is sufficient for synchronization of a network of FN oscillators with linear diffusive coupling and identical inputs. This approach is an application of the procedure outlined in [23]. Earlier studies have evaluated similar bounds with quadratic Lyapunov functions for networks of Hindmarsh-Rose neurons [15].

Theorem 4.1: Consider a network of FN oscillators with dynamics (10), identical constant external inputs, and linear diffusive coupling (2). Suppose the coupling strength γ and second smallest eigenvalue of the graph Laplacian $\lambda_2(L)$ satisfy

$$\gamma\lambda_2(L) > \frac{(\epsilon - 1)^2}{4b\epsilon} + 1 + \frac{\beta_1^2}{3} = \gamma\lambda_s^*. \tag{12}$$

Then the complete synchronization manifold $\mathcal{S}$ is globally asymptotically stable.

Proof: Let $V_Q(\mathbf{w}_1, \mathbf{w}_2) = \frac{1}{2}(\|\mathbf{w}_1\|_2^2 + \|\mathbf{w}_2\|_2^2)$ be a positive-definite Lyapunov function, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are transformed coordinates that represent the differences between states in $\mathbf{x}_1$ and between states in $\mathbf{x}_2$, respectively.

The derivative of $V_Q(\mathbf{w}_1, \mathbf{w}_2)$ can be computed as

$$\dot{V}_Q(\mathbf{w}_1, \mathbf{w}_2) = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}_1\|_2^2 - b\epsilon\|\mathbf{w}_2\|_2^2 + \epsilon\mathbf{w}_1 \cdot \mathbf{w}_2. \tag{13}$$

Using, $u^i = -\gamma L\mathbf{x}_1$, $\mathbf{w}_1 \cdot L\mathbf{w}_1 \geq \lambda_2(L)\|\mathbf{w}_1\|_2^2$, and $|x_1^i| \leq \beta_1$, we can write

$$\begin{aligned} \dot{V}_Q \leq & \left(1 - \gamma\lambda_2 + \frac{\beta_1^2}{3}\right)\|\mathbf{w}_1\|_2^2 \\ & + (1 - \epsilon)\mathbf{w}_1 \cdot \mathbf{w}_2 - b\epsilon\|\mathbf{w}_2\|_2^2. \end{aligned}$$

When $\gamma\lambda_2(L) = \gamma\lambda_s^* = \frac{(\epsilon - 1)^2}{4b\epsilon} + 1 + \frac{\beta_1^2}{3}$, we have

$$\dot{V}_Q \leq -\left(\sqrt{b\epsilon}\|\mathbf{w}_2\|_2 - \frac{|\epsilon - 1|}{2\sqrt{b\epsilon}}\|\mathbf{w}_1\|_2\right)^2.$$

Thus, $\dot{V}_Q \leq 0$, and increasing γ will not change this property. So we have found a sufficient condition for Lyapunov stability of the equilibrium state $\mathbf{w}_1 = \mathbf{w}_2 = 0$ (and thus the complete synchronization manifold $\mathcal{S}$). Further, there is some κ such that $\dot{V}_Q \leq -\kappa(\|\mathbf{w}_1\|_2^2 + \|\mathbf{w}_2\|_2^2)$.

To show that $\mathcal{S}$ is attractive, we can evaluate the integral of $\dot{V}_Q$ as we did for the integral of the Dini derivative in the proof of Theorem 3.3. This completes the proof. ∎

D. Synchronization Bound: Master Stability Function

The Master Stability Function (MSF) approach is commonly used to calculate necessary conditions on coupling for synchronization in oscillator networks [16]. Given a particular coupling scheme, the MSF approach carries out a local stability analysis of the linearized dynamics, and derives a necessary condition for synchronization in terms of a lower bound on the coupling strength. Following the steps presented in [16], it can be shown that for an undirected network of FN oscillators connected with linear diffusive coupling, this necessary condition can be expressed as

$$\gamma\lambda_2(L) \geq 1 - b\epsilon - \beta_1^2. \tag{14}$$

E. Comparison of Bounds

We first compare the different bounds on $\gamma\lambda_2(L)$ computed above for global asymptotic stability of $\mathcal{S}$ in the case of a complete network graph of FN oscillators, i.e., there is a connection between every pair of oscillators. In this case the graph Laplacian is

$$L = (n - 1)\mathbb{I}_n - \mathbf{1}_n\mathbf{1}_n^T,$$

and $\lambda_2(L) = n$.

Our new bound using the nonsmooth Lyapunov function can be compute from (11) as

$$\gamma n > 1 + \epsilon + \frac{\beta_1^2}{3},$$

whereas the bound computed using the quadratic Lyapunov function is given by

$$\gamma n > \frac{(\epsilon - 1)^2}{4b\epsilon} + 1 + \frac{\beta_1^2}{3}.$$

An earlier work [20], used a contraction analysis, and the corresponding sufficient condition was given as

$$\gamma n > \frac{1}{\epsilon}.$$

On the other hand, the master stability function based approach yields the following necessary condition:

$$\gamma n \geq 1 - b\epsilon - \beta_1^2.$$

Whenever $\epsilon/(1-\epsilon) < 1/(2\sqrt{b})$, our new bound from the nonsmooth analysis is tighter than the bound from the quadratic Lyapunov function. Additionally, when $\epsilon < 3/(3 + 3\epsilon + \beta_1^2)$, our new bound is tighter than the contraction theory based bound as well. For biologically plausible firing behavior of an FN oscillator, numerical simulations typically use $b \in [0, 1]$ and small values of ϵ ($\approx \frac{1}{12}$), which in turn tends to result in $\beta_1 \approx 2$. In this parameter regime, our nonsmooth analysis yields a tighter bound compared to the bounds obtained from earlier approaches based on the quadratic Lyapunov function and contraction theory.

We next compare the bounds for a general network graph. In Figure 1 we compare the bound from the nonsmooth Lyapunov approach with the bound from the quadratic Lyapunov function approach by ploting the ratio of λ_m^* to λ_s^*. The ratio is plotted for $\epsilon \in [0, 0.3]$ and $b \in [0, 1]$, which are parameter values commonly used to provide biologically relevant behavior with the FN model. For these conditions, $\lambda_m^* < \lambda_s^*$, and the ratio gets smaller with decreasing ϵ. This implies that in these parameter regimes, the bound from our new nonsmooth approach is tighter than the bound from the quadratic Lyapunov function approach.

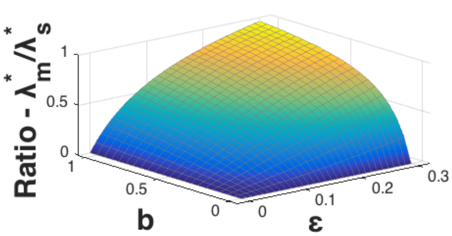

Fig. 1. **Bound comparison:** Ratio of synchronization conditions for the nonsmooth and quadratic Lyapunov approaches. For the biologically relevant parameter ranges plotted, the bound derived from the nonsmooth approach is always tighter.

Comparing the bound from the nonsmooth Lyapunov stability analysis, which is sufficient for synchronization, with the bound from the MSF approach, which is necessary for synchronization, provides insight into where the bounds perform well and how we can improve them in further work [19]. The necessary condition is $\gamma\lambda_2 \geq 1 - b\epsilon - \beta_1^2$, and the sufficient condition is $\gamma\lambda_2 \geq 1 + \epsilon + \frac{\beta_1^2}{3}$. The difference between these bounds is $\epsilon(1+b) + \frac{4}{3}\beta_1^2$. For models with a small ϵ parameter, as are typical, the accuracy of these bounds is limited by the bound on the dynamics, β_1^2. This suggests that to get closer to a condition that is both necessary and sufficient for synchronization, we should use a method that does not rely on the bound on the dynamics.

V. FitzHugh-Nagumo Network with Non-Identical External Inputs

When the external inputs I^i to individual FN oscillators in a network are not the same, the network separates into synchronized clusters, i.e. groups of oscillators with identical behavior, depending both on the distribution of external inputs and on the network structure [2], [22]. Oscillators must be input-equivalent in order for synchronization to occur [20]. Here, we use the notion of input-equivalence to extend our analysis to networks of nonlinear oscillators with non-identical constant external inputs I^i.

Definition 5.1 (Input-equivalence): Two FN oscillators i and j are input-equivalent if

$$I^i + u^i(t) = I^j + u^j(t) \quad \forall t.$$

A. Nonsmooth Lyapunov Analysis

We now extend our result from Section IV to a network of FN oscillators with non-identical inputs. We provide a sufficient condition under which each of a set of oscillators that are input-equivalent will synchronize as a cluster.

Corollary 5.2: Consider a network of FN oscillators with dynamics (10), non-identical constant external inputs, and linear diffusive coupling (2). Suppose that the oscillators can be partitioned into C distinct sets $\mathcal{C}_k$, $k = 1, \ldots, C$ such that all pairs in each set are input-equivalent [20]. Let L_k be the Laplacian of the subgraph for the oscillators in $\mathcal{C}_k$. Define the cluster synchronization manifold as

$$\mathcal{S}_\mathcal{C} = \{\mathbf{x^1}, \ldots, \mathbf{x^n} \in \mathbb{R}^2 : x^i = x^j,\ \forall\, i,\, j \in \mathcal{C}_k,\ \forall k\}.$$

$\mathcal{S}_\mathcal{C}$ is globally asymptotically stable if for all k

$$\gamma\lambda_2(L_k) > 1 + \epsilon + \frac{\beta_{1,k}^2}{3}.$$

Proof: By input-equivalence, we treat each set of FN oscillators separately. Since the internal dynamics of each oscillator are identical, we can use the result from Theorem 3.3 for FN oscillators as in (11) for each set $\mathcal{C}_k$. ■

Example 5.3 (Cluster Synchronized Graph): We illustrate our result by considering a network of FN oscillators interacting over the undirected graph in Figure 2, which can be partitioned into three ($C = 3$) input-equivalent sets: (1) a cycle graph C_m, (2) a complete graph K_m, and (3) a single central node connected to every element in both K_m and C_m. We simulate such a system with $m = 50$, and external input 0 to elements in C_m, external input 0.1 to the central node, and external input 0.4 to elements in K_m. When $b = 0.8$, $\epsilon = 0.08$, and $\gamma = 0.1$ for all connections, we observe the dynamics represented in Figure 3. All oscillators in the complete graph synchronize, while those in the cycle graph do not. We calculate the second smallest eigenvalues of the graph Laplacians for each subgraph, and find that

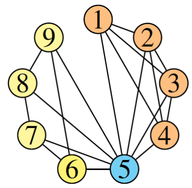

Fig. 2. Graph used in the example illustrated in the case $m = 4$.

for the complete graph $\lambda_2(L_K) = 50$, while for the cycle graph $\lambda_2(L_C) = 0.0158$. Since the oscillator parameters are homogeneous, $\beta_{1,k} \approx 2$ for both C_m and K_m. Thus, the sufficient condition for synchronization is $\lambda_m^* = 2.41$ for each subgraph: $\lambda_2(L_K)$ is above the synchronization bound, while $\lambda_2(L_C)$ is far below. In this example, we have used the synchronization condition as a guideline to design a system that exhibits different types of dynamic behavior.

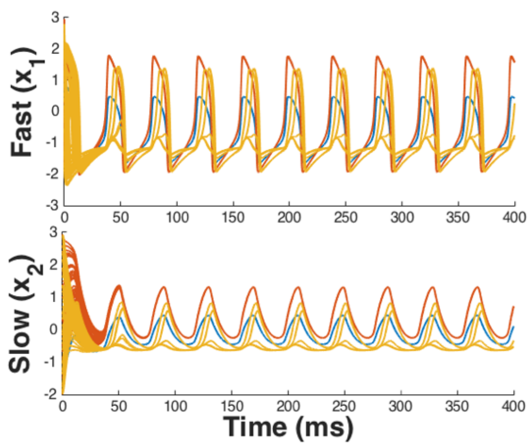

Fig. 3. **Dynamics of cluster synchronization:** The x_1 and x_2 dynamics for 101 FN oscillators arranged according to the graph structure depicted in Figure 2 with $m = 50$. Cluster synchronization is apparent in one cluster (the complete graph in orange-red), but the other input-equivalent cluster (the cycle graph in yellow-gold) does not synchronize. These results are consistent with the bounds from Corollary 5.2.

Conclusion

We have used a nonsmooth Lyapunov function to determine new sufficient conditions for synchronization in networks of nonlinear oscillators. This function was previously used to find tight bounds for synchronization in a complete graph of Kuramoto oscillators. We provide a general framework and a specialization to the FN model that illustrates its effectiveness. The bounds reported for the FN model improve on previously reported bounds as well as the bound we calculate in this work using an alternative method [20], [23]. Finding sufficient conditions for synchronization in systems with nonlinear coupling, time delays, and heterogeneous node dynamics has been explored using the quadratic Lyapunov and contraction analyses in [3], [10], [13]. We expect that these bounds may also be improved with nonsmooth Lyapunov functions.

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