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Feedback Controlled Bifurcation of Evolutionary Dynamics with Generalized Fitness

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***Abstract*— Coexistence and interaction of multiple strategies in a large population of individuals can be observed in a variety of natural and engineered settings. In this context, replicator-mutator dynamics provide an efficient tool to model and analyze the evolution of the fractions of the total population committed to different strategies. Although the literature addresses existence and stability of equilibrium points and limit cycles of these dynamics, linearity in fitness functions has typically been assumed. We generalize these dynamics by introducing a nonlinear fitness function, and we show that the replicator-mutator dynamics for two competing strategies exhibit a quintic pitchfork bifurcation. Then, by designing slow-time-scale feedback dynamics to control the bifurcation parameter (mutation rate), we show that the closed-loop dynamics can exhibit oscillations in the evolution of population fractions. Finally, we introduce an ultraslow-time-scale dynamics to control the associated unfolding parameter (asymmetry in the payoff structure), and demonstrate an even richer class of behaviors.**

***Index Terms*— Nonlinear Systems, Evolutionary Dynamics, Bifurcation**

I. Introduction

Evolutionary dynamics provide a set of powerful tools to model and analyze how the fractions of a population committed to different strategies evolve over time. These dynamics have been used to study evolution of language grammars [1]–[3], population genetics in biology [4], opinion formation in social-networks [5], [6], decision-making in multi-agent systems [7], [8], signaling systems [9], evolutionary graph theory [10], [11] and multi-agent systems [12]. The replicator-mutator dynamics model this evolution as a function of replication (commitment to strategies with higher rewards/pay-offs) and mutation (tendency for spontaneous switch among strategies). Depending on the rate of mutation and strategy interaction network, steady-state system-level behavior can be classified into three types: (i) dominance of a single strategy, (ii) coexistence of a few strategies, and (iii) the mixed equilibrium corresponding to equal fractions across strategies, also known as collapse of dominance. High mutation rate leads to collapse of dominance, whereas low mutation rate results in a single dominant strategy.

This model has been extensively studied in the literature for various numbersof strategies, distinct pay-off structures and strategy interaction networks [13]–[16]. Most of the studies in the literature have focused on the analysis of the stable equilibria of the replicator-mutator dynamics when there is assumed to be a symmetry in fitness functions associated with the strategies. However, [15], [16] explored asymmetry in fitness functions, and proved that the dynamics exhibit Hopf bifurcations and limit cycles for a circulant interaction network of $N \geq 3$ strategies.

In our previous work [17], we used replicator-mutator dynamics to investigate the mechanisms of social decision-making for a structured improvisational dance (with choreographer R. Lazier and composer D. Trueman from Princeton University) in which a group of dancers make a sequence of compositional choices among pre-defined dance motion primitives. The replicator-mutator dynamics were used to model the evolution of different fractions within a group of dancers committed to the different strategies (dance motion primitives) as a function of replication (commitment to motion primitives with higher fitness) and mutation (spontaneous switch between the primitives). We argued that the replicator-mutator dynamics provide a framework for analyzing dancers' decision-making strategies, and we incorporated a feedback mechanism for tuning the rate of mutation driven by the dancers' observations of the evolving subpopulation fractions. This yielded a feedback controlled bifurcation in the model dynamics which predict the persistence of strategies in the behavior of the group of dancers.

Further collaborations with Lazier, Trueman, and the dancers revealed that, during the structured improvisational dance, subgroup of dancers make decisions based on an artistic explore-exploit tradeoff, which is described by oscillations between dominance of a single strategy/coexistence of multiple strategies (exploitation) and collapse of dominance (exploration) in the steady-state behavior of the replicator-mutator model [18]. To capture this in the model in a systematic way, we introduced a nonlinear fitness function that enriches the steady-state behavior of the dynamics. We showed through simulations that our framework can emulate the oscillatory behavior in dance even in the case of only two interacting strategies at a time. The feedback of the bifurcation parameter was then used to study the duration of explore-exploit phases in the dance improvisation.

In the present study, we extend the model and analysis of feedback controlled bifurcations [19] of replicator-mutator dynamics with a generalized[1] nonlinear payoff. In Section II we first introduce our notation, the model and the tools we use from singularity theory [20]. In Section III

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[1]The term *generalized* has been used to imply that an appropriate choice of parameters for the nonlinear fitness will retrieve the linear fitness.

we incorporate the proposed nonlinear payoff structure in the well known replicator-mutator model. In Section IV, we prove conditions under which there exists a symmetric quintic bifurcation (a pitchfork bifurcation with a quintic, instead of cubic, stabilizing term) and a symmetric subcritical pitchfork bifurcation in its unfolding. The latter provides a hysteresis that we leverage in Section V by designing feedback dynamics for the bifurcation parameter to produce oscillations in the dynamics. We explore further ways to construct dynamics using bifurcation theory by explore the non-symmetric unfolding of the subcritical pitchfork in Section VI and introducing feedback dynnamics in an unfolding parameter in Section VII. We conclude in Section VIII.

II. Notation and Background

Singularity theory provides a useful mathematical tool for investigating bifurcation in nonlinear dynamical systems [21]. In this section we provide a brief review of relevant results from singularity theory. Interested readers can refer to [20] for further details.

A scalar bifurcation problem

$$\psi(x,\lambda) = 0 \tag{1}$$

is defined by the set of scalar solutions x of (1) as the bifurcation parameter $\lambda \in \mathbb{R}$ is varied. In this setting, λ usually represents some relevant control parameter, whereas the variable x denotes the state of the underlying system. The zero set $\{(x,\lambda)|\psi(x,\lambda) = 0\}$ is called the *bifurcation diagram* of (1), and a point (x^*,λ^*) on the bifurcation diagram is called a *bifurcation point* if any neighborhood of λ^* contains a parameter value $\tilde{\lambda}$ such that the dynamics at $\tilde{\lambda}$ are topologically inequivalent from that at λ^*. For instance, the number or stability of equilibria, or periodic orbits of ψ might change with perturbations of λ from λ^*.

It directly follows from the implicit function theorem that a necessary condition for (x^*,λ^*) to be a bifurcation point is that $\psi_x(x^*,\lambda^*) = 0$, where ψ_x denotes the partial derivative of ψ with respect to x. In addition, if ψ satisfies

$$\begin{aligned} \psi_{xx}(x^*,\lambda^*) = \psi_\lambda(x^*,\lambda^*) = 0 \\ \text{and,} \quad \psi_{xxx}(x^*,\lambda^*) > 0, \quad \psi_{\lambda x}(x^*,\lambda^*) < 0, \end{aligned} \tag{2}$$

then (1) undergoes a *pitchfork bifurcation* at this point. At any pitchfork bifurcation point a single zero divides into three zeros, and it has a normal form of $\psi(x,\lambda) = x^3 - \lambda x$. As we will see later in Section IV, we can write similar conditions (9)-(13) for existence of a quintic pitchfork bifurcation which has a normal form of $\psi(x,\lambda) = x^5 - \lambda x$.

Now, by introducing $\alpha \in \mathbb{R}^l$, we consider an l-parameter family of bifurcation problems represented by $\Psi(x,\lambda,\alpha) = 0$. Moreover, the restriction that $\Psi(x,\lambda,0) = \psi(x,\lambda)$ allows us to recover the original scalar bifurcation problem as a special case. In addition, for any smooth perturbation term $\epsilon\tilde{\psi}(x,\lambda)$ with sufficiently small ϵ, there exists some parameter value $\alpha \in \mathbb{R}^l$ such that $\psi(x,\lambda) + \epsilon\tilde{\psi}(x,\lambda)$ and $\Psi(x,\lambda,\alpha)$ are *strongly equivalent* (i.e. Ψ is obtained from $\psi + \epsilon\tilde{\psi}$ via a local diffeomorphism of the form $(x,\lambda) \mapsto (X(x,\lambda),\Lambda(\lambda))$, and a nonzero function $S(x,\lambda)$). Such a Ψ

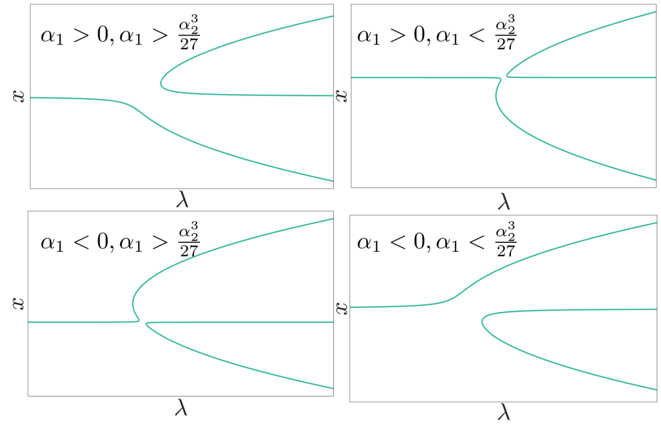


Fig. 1: Universal unfolding of a pitchfork ($-\epsilon$) bifurcation.

is called a l-parameter *unfolding* of ψ. However, unfolding of a bifurcation is not unique, and it is possible to obtain another unfolding $\tilde{\Psi}(x,\lambda,\beta) = 0, \beta \in \mathbb{R}^{\tilde{l}}$ for the same scalar bifurcation problem. Ψ will be called a *universal unfolding* of ψ if for any other $\tilde{l}$-parameter unfolding $\tilde{\Psi}$ we have $\tilde{l} \geq l$ and Ψ is strongly equivalent to $\tilde{\Psi}$. Then, l is called the *codimension* of ψ. The family of bifurcation diagrams associated with a universal unfolding is given by the set $\{(x,\lambda)|\Psi(x,\lambda,\alpha) = 0, \alpha \in \mathbb{R}^l\}$.

Remark 2.1: For a supercritical pitchfork bifurcation problem ψ, $l = 2$, and the associated universal unfolding Ψ is strongly equivalent to the 2-parameter family of bifurcation problems given by $x^3 + \alpha_2 x^2 - \lambda x + \alpha_1 = 0,\ \alpha_1, \alpha_2 \in \mathbb{R}$. Based on the value of these two unfolding parameters α_1 and α_2, the associated bifurcation diagram is strongly equivalent to one of the persistent bifurcation diagrams of the pitchfork bifurcation (see Figure 1).

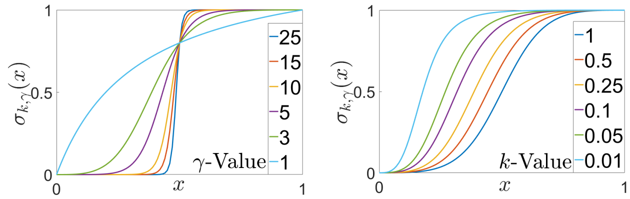
In this paper we use the recognition problem and unfolding theory for Z_2*-symmetric* quintic bifurcations ($-x^5 + \lambda x$). In the space of odd functions (Z_2 symmetry), the codimension of the quintic pitchfork is $l = 1$ [20, Table VI.5.1]. A normal form of its universal unfolding is $-x^5 + \lambda x + \alpha x^3$. We also recognize and unfold subcritical ($x^3 + \lambda x$) pitchfork bifurcations.

III. Replicator-Mutator Dynamics with Generalized Fitness

We begin by considering a large population of agents committed to N different strategies, wherein $x_i \in [0,1]$, $i \in \{1,2,\ldots,N\}$ represents the fraction of population committed to the i-th strategy. Furthermore, we define the fitness f_i associated with the strategy-i as

$$f_i = \sum_{j=1}^{N} b_{ij}\sigma_{k,\gamma}(x_j) = \sum_{j=1}^{N} b_{ij}\left(\frac{x_j^\gamma}{k(1-x_j)^\gamma + x_j^\gamma}\right), \tag{3}$$

where b_{ij} represents the payoff to an agent using strategy-i while interacting with agents committed to strategy-j and $B = [b_{ij}] \in \mathbb{R}^{N\times N}$ is the payoff matrix. In the subsequent analysis, we assume the payoffs to be non-negative (i.e. $b_{ij} \geq 0$), and an agent gets the maximum payoff (normalized to 1) during interaction with those using the same strategy.



(a) γ Dependence (b) k Dependence

Fig. 2: The influence of γ and k on the fitness function. Left panel: k is fixed at 0.25. Right panel: γ is fixed at 3.

Hence, $b_{ii} = 1$, and $b_{ij} \in [0, 1)$ for the off-diagonal entries of B. The Hill-function (i.e. $\sigma_{k,\gamma}$) parameters γ and k affect the nonlinearity in the fitness function (see Figure 2). Note that the original (linear) nature of the fitness function can be retrieved by setting $k = 1$ and $\gamma = 1$. We define the average fitness of the population ϕ as

$$\phi = \sum_{i=1}^{N} f_i x_i. \tag{4}$$

Next, we define the mutation rate q_{ij} (i.e. the probability of spontaneous switch from strategy-i to strategy-j) as

$$q_{ij} = \frac{\mu b_{ij}}{\sum_{j \neq i} b_{ij}}, \quad \text{and,} \quad q_{ii} = 1 - \mu, \tag{5}$$

where $\mu \in [0, 1]$ is the overall mutation strength which represents the probability of error in replication. As $\sum_{j=1}^{N} q_{ij} = 1$, it directly follows that the mutation matrix $Q = [q_{ij}]$ is row stochastic. In this framework, spontaneous switches (i.e. mutations) are always favored toward strategies with higher payoffs.

Finally, we define the replicator-mutator dynamics for this large population committed to N different strategies by describing the evolution of population fractions x_i as

$$\dot{x}_i = \sum_{j=1}^{N} x_j f_j q_{ji} - \phi x_i = g_i(x_1, x_2, \ldots, x_N), \tag{6}$$

where $g_i : \mathbb{R}^N \to \mathbb{R}$ is a function on the $N - 1$ simplex $\Delta_{N-1} = \{(x_1, \ldots, x_N) \in \mathbb{R}^N | x_i \geq 0, \sum_{i=1}^{N} x_i = 1\}$. It can be shown that $\sum_{i=1}^{N} x_i$ remains constant along any trajectory of the replicator-mutator dynamics (6). In the sequel we focus on the interaction between two strategies (i.e. $N = 2$).

IV. Symmetric Quintic Pitchfork Bifurcation and Its Unfolding for a Two Strategy Case

It has been shown, e.g., [3], [15], that the replicator-mutator dynamics with $N = 2$ undergo a supercritical pitchfork bifurcation ($-x^3 + \lambda x$) when the fitness of strategies is governed by a linear function, which corresponds to setting $k = 1$ and $\gamma = 1$ in (3), and payoffs are symmetric (i.e. $b_{12} = b_{21}$). Here, we prove the existence of a symmetric quintic pitchfork ($-x^5 + \lambda x$) in the same dynamics for symmetric payoffs, small k and large γ.

Since the replicator-mutator dynamics (6) for a two strategy case ($N = 2$) is restricted to the 1-simplex Δ_1, the associated 2-dimensional dynamics can be represented by a scalar differential equation. In addition, if we assume the payoffs to be symmetric, i.e., $b_{11} = b_{22} = 1$ and $b_{12} = b_{21} = b \in (0, 1)$, the underlying 1-dimensional dynamics can be expressed as

$$\begin{aligned} \dot{x}_1 &= g(x_1, \mu, k, \gamma, b) \\ &= x_1(1 - \mu)\Big(\sigma_{\gamma,k}(x_1) + b\sigma_{\gamma,k}(1 - x_1)\Big) \\ &\quad + (1 - x_1)\mu\Big(b\sigma_{\gamma,k}(x_1) + \sigma_{\gamma,k}(1 - x_1)\Big) - \phi x_1, \end{aligned} \tag{7}$$

where the nonlinear Hill-function is given by (3) and ϕ is the average fitness (4).

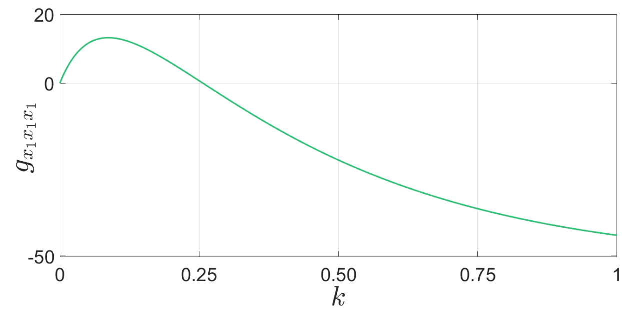


Fig. 3: Evolution of $g_{x_1x_1x_1}$ as a function of k.

A. Existence of a Symmetric Quintic Pitchfork Bifurcation

To show the existence of a symmetric quintic pitchfork in the scalar bifurcation problem

$$g(x_1, \mu, k, \gamma, b) = 0, \tag{8}$$

we first identify that the 1-dimensional reduced dynamics (7) is Z_2 symmetric with respect to $x_1 = 0.5$, i.e. (7) remains unchanged under the change of variable $x_1 \mapsto 1 - x_1$. Then by leveraging the results from [20, Proposition VI.3.4], we can conclude that this singularity in (8) has codimension 1, and hence we can use a scalar parameter for unfolding this bifurcation problem.

Proposition 4.1: Consider the dynamics (7) with fixed $b = 0.04$ and $\gamma = 3.0$. There exists a k^* such that the bifurcation problem at $k = k^*$, i.e., $g(x_1, \mu, k^*, 3.0, 0.04) = 0$, is strongly equivalent (in the sense of [20, Definition VI.2.5]) to the quintic bifurcation problem: $(x_1 - 0.5)^5 + (\mu - \mu^*)(x_1 - 0.5) = 0$, at the mixed equilibrium $x_1 = 0.5$ for some suitable μ^*.

Proof: We use k as an unfolding parameter to verify the existence of a symmetric quintic pitchfork in the dynamics (7). Following the recognition problem in [20, Proposition VI.2.14], we seek μ^*, k^* that satisfy

$$g(0.5, \mu^*, k^*, 3.0, 0.04) = 0, \tag{9}$$
$$g_{x_1}(0.5, \mu^*, k^*, 3.0, 0.04) = 0 \tag{10}$$
$$g_{x_1x_1x_1}(0.5, \mu^*, k^*, 3.0, 0.04) = 0, \tag{11}$$
$$g_{x_1x_1x_1x_1x_1}(0.5, \mu^*, k^*, 3.0, 0.04) \neq 0, \tag{12}$$
$$\text{and,} \quad g_{x_1\mu}(0.5, \mu^*, k^*, 3.0, 0.04) \neq 0. \tag{13}$$

We first solve (9) in order to express μ^* as a function of k^*, and plug this solution into (11) so that $g_{x_1x_1x_1} = 0$ can be represented as a function of k^*. Note that (10) holds true whenever $x_1 = 0.5$, irrespective of the values of μ^* and k^*.

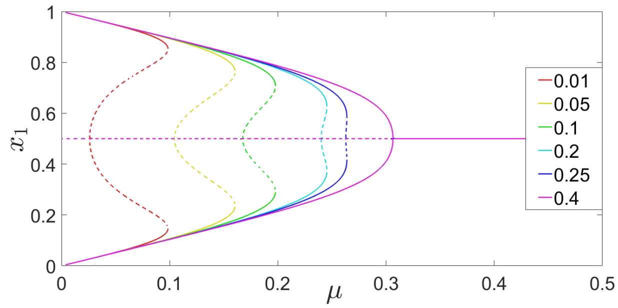


Fig. 4: Bifurcation diagrams of the replicator-mutator dynamics (6) corresponding to six different values of the k parameter in the nonlinear fitness function. We have used $\gamma = 3.0$ for each of these bifurcation diagrams. $k^* \simeq 0.257$. Solid lines show stable branches, whereas dashed lines represent unstable branches.

However, solving (11) to obtain a closed form solution for k^* is non-trivial. So, we examine the evolution of $g_{x_1x_1x_1}$ as a function of k (see Figure 3), and this reveals the presence of an isolated zero for (11) at $k = k^* \simeq 0.257$. We can now also easily verify that

$$g_{x_1x_1x_1x_1x_1}(1/2, \mu^*, k^*, 3.0, 0.04) < 0,$$
$$\text{and,} \quad g_{x_1\mu}(1/2, \mu^*, k^*, 3.0, 0.04) > 0,$$

which completes the recognition problem for the symmetric quintic bifurcation. ∎

Remark 4.2: For $\gamma \neq 3.0$ we expect a codimension 1 algebraic manifold (locally, a line) of symmetric quintic pitchfork singularity in the (k, γ) plane. The existence of this structure will be analyzed in a future work.

B. Symmetric subcritical pitchfork bifurcation in the unfolding of the symmetric quintic pitchfork bifurcation

Proposition 4.3: Consider the dynamics (7) with fixed $b = 0.04$ and $\gamma = 3.0$. For $k > k^*$ the quintic pitchfork unfolds into a supercritical pitchfork bifurcation. For $k < k*$ quintic pitchfork unfolds into a subcritical pitchfork bifurcation with a quintic stabilizing term.

Proof: Variations of k around the special value $k = k^*$ unfolds the Z_2-symmetric quintic pitchfork without breaking the Z_2 symmetry. That is, it follows from [20, Theorem VI.3.3] that $g(x_1, \mu, k, 3.0, 0.04) = 0$ provides a Z_2-symmetric unfolding of the quintic pitchfork around $(x, \mu, k) = (0.5, \mu^*, k^*)$. For $k > k^*$ the cubic term is negative, which unfolds the quintic pitchfork into a supercritical pitchfork bifurcation. For $k < k*$ the cubic term is positive, which unfolds the quintic pitchfork into a subcritical pitchfork bifurcation with a quintic stabilizing term. ∎

The unfolding of the quintic pitchfork for k above and below k^* is shown in Figure 4. As leveraged in the next section, the tristability observed in the case $k < k^*$ provides sufficient richness to capture oscillatory behaviors in the evolution of fractions of population committed to two strategies.

V. Two time-scale oscillatory behavior from symmetric subcritical pitchfork bifurcation

As illustrated in Figure 4, for $k < k^*$ the replicator-mutator dynamics (6) undergo a symmetric subcritical pitchfork bifurcation with a quintic stabilizing term. By providing a hysteresis between the mixed equilibrium and the dominant solutions, this gives rise to a novel kind of multi-stability . Whenever $k < k^*$ two stable dominant solutions co-exist with the stable mixed equilibrium. In our subsequent analysis we leverage this feature to enable a fast switching behavior in the underlying population fraction x_1.

We begin by introducing a slowly varying feedback to control the bifurcation parameter μ so that it can oscillate around the bifurcation point. In particular, for a two-strategy case with symmetric payoffs (i.e. $b_{12} = b_{21} = b$) this fast-slow closed-loop dynamics can be expressed as

$$\begin{aligned}\dot{x}_1 &= x_1(1-\mu)\Big(\sigma_{\gamma,k}(x_1) + b\sigma_{\gamma,k}(1-x_1)\Big) \\ &\quad + (1-x_1)\mu\Big(b\sigma_{\gamma,k}(x_1) + \sigma_{\gamma,k}(1-x_1)\Big) - \phi x_1 \\ \dot{\mu} &= -K_\mu(x_1-\alpha_1)(\alpha_2-x_1)\mu(1-\mu), \end{aligned} \tag{14}$$

where $\alpha_1 \in (0.5, 1)$ and $\alpha_2 \in (0, 0.5)$ represent two thresholds, and $0 < K_\mu \ll 1$ represents the extent of separation between the time-scales of the $\dot{x}_1$ and $\dot{\mu}$ dynamics. The feedback dynamics (14) imply that in absence of a strong dominance, the mutation rate decreases and drives the population fractions away from the mixed equilibrium. If either of the strategies is dominating (i.e. $x_1 \geq \alpha_1$ or $x_1 \leq \alpha_2$), the mutation rate increases and breaks the dominance.

The lines $x_1 = \alpha_1$ and $x_1 = \alpha_2$ belong to the nullcline of the $\dot{\mu}$-dynamics on the $x_1 - \mu$ plane. When these lines intersect the critical manifold of the replicator-mutator dynamics (where $\dot{x}_1 = 0$) along its unstable branches, the closed-loop system (14) exhibits two distinct oscillatory behaviors for sufficiently low value of K_μ (see Figure 5). Each of these oscillations corresponds to the periodic switches between the mixed equilibrium and either of the strongly dominant solutions. Also, these limit cycles are symmetric with respect to the mixed equilibrium ($x_1 = 0.5$).

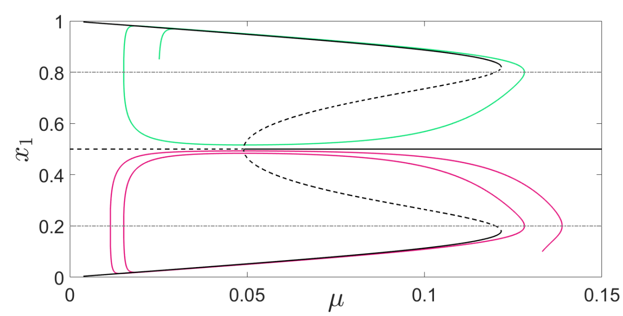


Fig. 5: Co-existence of two stable limit cycles (green and magenta), shown on the bifurcation diagram for (14). Stable manifolds of the fast dynamics are shown in solid lines, whereas the corresponding unstable manifolds are shown in dashed lines. Parameters are $\gamma = 3.0$, $k = 0.02$, $K_\mu = 0.3$, $b = 0.04$, $\alpha_1 = 0.2$, and $\alpha_2 = 0.8$.

By using techniques from geometric singular perturbations and blow-up theory [22], one can prove existence and stability of these limit cycles. Because of space constraints we leave the proof for our next publication and focus instead on showing how symmetry-breaking in the associated payoff structure can lead to oscillations between two dominant solutions. We provide geometric intuition on how these cycles can be constructed in the limit $K_\mu \to 0$ by showing the singular (i.e. $K_\mu \to 0$) phase portrait (see Figure 6), Because x_1 evolves at a time-scale which is much faster than that for μ, trajectories spend most of the time on the critical manifold. The singular limits of the two limit cycles are constructed as closed singular trajectories, which merge along the horizontal part of the critical manifold where $x_1 = 0.5$. Because μ is decreasing in that region, both singular cycles approach the vertical line $\mu = 0$. There, they split in upward and downward direction, respectively. At the intersection with the upper and lower branches of the critical manifold, where μ is increasing, they slide along the critical manifold until the fold singularity, where they jump back to the horizontal branch of the critical manifold, much in the same way as a standard relaxation oscillator.

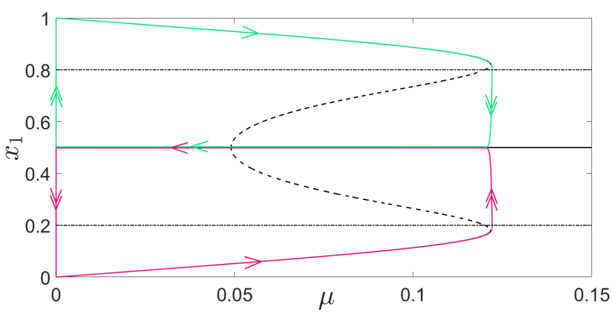


Fig. 6: Geometric construction of the two stable limit cycles (green and magenta) in the limit $K_\mu \to 0$, shown on the bifurcation diagram for (14). Parameters are $\gamma = 3.0$, $k = 0.02$, $b = 0.04$, $\alpha_1 = 0.2$, and $\alpha_2 = 0.8$.

For small values of K_μ these two singular cycles perturb into two exponentially stable limit cycles, corresponding to the two oscillatory behaviors (as shown in Figure 5). Then using Fenichel theory [23], one can show that these two cycles are $\mathcal{O}(e^{-1/K_\mu})$-close to each other in the region where they shadow the horizontal branch of the critical manifold. For K_μ sufficiently small, tiny perturbations (possibly due to noise) can lead the system to switch spontaneously between these two cycles.

VI. Non-symmetric unfolding of the subcritical pitchfork bifurcation

Although the subcritical pitchfork has codimension 0 under Z_2-symmetry [20, Proposition VI.3.4], we can unfold it into one of the persistent bifurcation diagrams of the pitchfork (as shown in Figure 1) by relaxing the Z_2-symmetry condition. A natural way to break the Z_2-symmetry in the replicator-mutator dynamics (6) with $N = 2$ is to let the payoffs become asymmetric, i.e. $b_{12} \neq b_{21}$. Note that in the absence of Z_2-symmetry, the subcritical pitchfork bifurcation has codimension 2.

We begin by introducing a new parameter $\tilde{b}$ such that $b_{12} = b + \tilde{b}$ and $b_{21} = b - \tilde{b}$. This new parameter $\tilde{b}$ provides the first unfolding parameter. Another alternative way of breaking Z_2-symmetry involves considering different γ's in the fitness functions associated to the different population fractions x_i, i.e. we now define the fitness functions as $\sigma_{\gamma_i,k}(x_i)$ for all $i = 1, \dots, N$. In particular, for the $N = 2$ case, we pick $\gamma_1 = \gamma + \tilde{\gamma}$ and $\gamma_2 = \gamma - \tilde{\gamma}$ where $\tilde{\gamma}$ provides the second unfolding parameter. The resulting (non-Z_2 symmetric) unfolding can be expressed as

$$\begin{aligned} & G(x_1, \mu, k, \gamma, b, \tilde{\gamma}, \tilde{b}) \\ &\quad = x_1(1-\mu)\Big(\sigma_{\gamma_1,k}(x_1) + b_{12}\sigma_{\gamma_2,k}(1-x_1)\Big) \\ &\qquad + (1-x_1)\mu\Big(b_{21}\sigma_{\gamma_1,k}(x_1) + \sigma_{\gamma_2,k}(1-x_1)\Big) - \phi x_1. \end{aligned}$$

Note that the two unfolding parameters have distinct effects in breaking the Z_2-symmetric bifurcation diagrams. For $\tilde{\gamma} = \tilde{b} = 0$, $x_1 = 0.5$ is a solution of $G(x_1, \mu, k, \gamma, b, 0, 0) = 0$ for all μ, k, γ. Although the Z_2-symmetry is lost for $\tilde{\gamma} \neq 0$, it still holds that $G(x_1, \mu, k, \gamma, b, \tilde{\gamma}, 0)|_{x_1=0.5} = 0$ for all $\mu, k, \gamma, b, \tilde{\gamma}$. On the contrary, if $\tilde{b} \neq 0$, then $x_1 = 0.5$ is not a generic solution of $G(x_1, \mu, k, \gamma, b, \tilde{\gamma}, \tilde{b}) = 0$. The two unfolding parameters in the normal form of the universal unfolding of the pitchfork are distinguished in a similar way.

We consider the unfolding G for $\gamma = 3.0$, $b = 0.04$, and $k < k^*$, which implies the presence of a subcritical pitchfork for $\mu = \mu^*_{SC}$ if $\tilde{\gamma} = \tilde{b} = 0$. For $\tilde{\gamma} \neq 0$, $\tilde{b} \neq 0$, we obtain a universal unfolding of the subcritical pitchfork, as illustrated in the top-left panel of Figure 1. Similar to the recognition of the symmetric quintic pitchfork in Section IV, this result can be easily (but lengthy) be verified by applying [20, Proposition III.4.4] in a semi-analytical way.

VII. Three time-scale chaotic behavior via unfolding of subcritical pitchfork bifurcation

The universal unfolding of the subcritical pitchfork G provides a principled way of modeling richer dynamical behavior by adding a third variable on an ultraslow time-scale modulating one of the unfolding parameters. Similarly to [24], the hierarchy between state variable (x_1), bifurcation parameter (μ), and unfolding parameters ($\tilde{b}, \tilde{\gamma}$) is dynamically reflected in the hierarchy of time-scales. We augment our model (14) by introducing the following dynamics for $\tilde{b}$:

$$\dot{\tilde{b}} = K_b \left[-\tilde{b} + \frac{x_1 - 0.5}{(x_1 - 0.5)^2 + 0.5} \right] \tag{15}$$

where $0 < K_b \ll K_\mu \ll 1$ represent the time-scale separations. The rational behind dynamics (15) is as follows. If $\tilde{b} > 0$, the Z_2-symmetry of equally preferred modules is broken in favor of the one corresponding to the upper attractive limit cycle, and vice-versa for $\tilde{b} < 0$. Figure 7 shows that, for $\tilde{b} > 0$, trajectories indeed converge to the upper limit cycle.

Dynamics (15) lets $\tilde{b}$ decrease if $x_1 > 0.5$ and increase if $x_1 < 0.5$. At the behavioral level, the effect of introducing dynamics (15) is thus to switch the favored state of the (x_1, μ) subsystem. Similarly to the Lorentz chaotic system, there exists a parameter range in which the switch between the upper and lower cycle is chaotic, as illustrated in Figure 8.

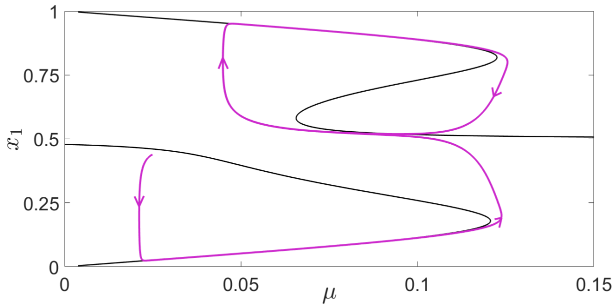


Fig. 7: Illustration of a solution trajectory of (14) converging to the upper limit cycle when $\tilde{b} = 0.0049$.

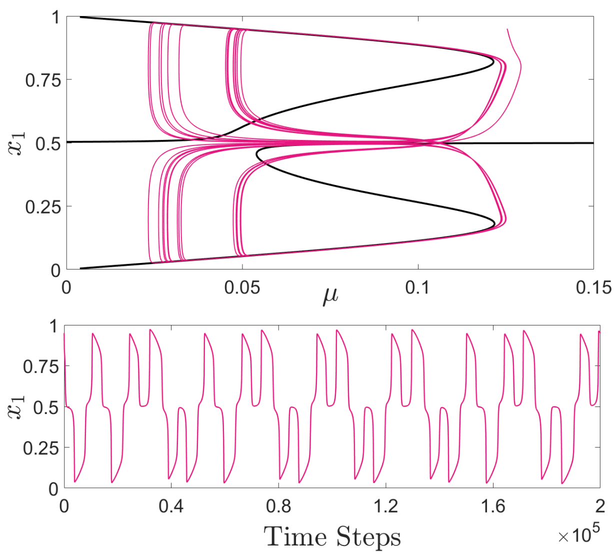


Fig. 8: Chaotic oscillation in population fractions for the three time-scale dynamics (14,15). Parameters are $\gamma = 3.0$, $k = 0.02$, $b = 0.04$, $\alpha_1 = 0.2$, $\alpha_2 = 0.8$, $K_\mu = 0.1$ and $K_b = 0.00005$.

VIII. Conclusion

In this paper we have introduced a nonlinear fitness function within the framework of replicator-mutator dynamics. This nonlinear fitness function, in addition to providing a generalization of the more commonly used linear fitness functions, also enables a richer class of behaviors. In particular, for a special case (i.e. $N = 2$) the replicator-mutator dynamics exhibit a quintic pitchfork bifurcation. Then, by introducing a slow dynamics to control the bifurcation parameter we synthesize an oscillatory behavior in associated population fractions. Moreover, we have shown that it is possible to obtain seemingly unpredictable oscillations with larger amplitude by controlling the asymmetry in the payoff structure at a much slower timescale. These results show a range of ways that bifurcation theory can be used as a constructive means to design closed-loop adaptive dynamics. Future work will investigate the bifurcation problems present in the competition of three or more strategies (i.e. $N = 3$ or more), and explore how introducing feedback at multiple time scales can influence the closed loop dynamics.

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