Joint Centrality Distinguishes Optimal Leaders in Noisy Networks

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***Abstract*—We study the performance of a network of agents tasked with tracking an external unknown signal in the presence of stochastic disturbances and under the condition that only a limited subset of agents, known as leaders, can measure the signal directly. We investigate the optimal leader selection problem for a prescribed maximum number of leaders, where the optimal leader set minimizes total system error defined as steady-state variance about the external signal. In contrast to previously established greedy algorithms for optimal leader selection, our results rely on an expression of total system error in terms of properties of the underlying network graph. We demonstrate that the performance of any given set of noise-free leaders depends on their influence as determined by a new graph measure of the centrality of a set. We define the *joint centrality* of a set of nodes in a network graph such that a noise-free leader set with maximal joint centrality is an optimal leader set. In the case of a single leader, we prove that the optimal leader is the node with maximal information centrality for the noise-corrupted and noise-free leader cases. In the case of multiple leaders, we show that the nodes in the optimal noise-free leader set balance high information centrality with a coverage of the graph. For special cases of graphs, we solve explicitly for optimal leader sets. Examples are used to illustrate.**

***Index Terms*—Leader-follower dynamics, network analysis and control, networked control systems, network theory (graphs), optimization stochastic/uncertain systems.**

I. Introduction

Analysis of networked multiagent system dynamics has generated substantial research interest in recent years [1]–[3]. This is largely due to the broad range of applications for which the theory can be applied, including, for example, the design of vehicle networks [4], analysis of social networks [5], investigation of collective animal behavior [6], and more. Often in these applications, the network must learn an external signal, for example, in the case of a sensor network using a consensus to estimate an environmental signal [7]. However, when the signal is costly to sample, for example, because of energy consumption costs or costs associated with acquiring the necessary sensory or processing capability, it may become impractical for all agents in the network to measure the signal directly. If interagent sensing or communication is relatively inexpensive, then a more efficient solution involves a limited subset of agents, called leaders, measuring the signal directly, with the remaining agents, called followers, learning the signal through network connections. In this paper, we address the problem of selecting leaders, as a function of the network graph, to maximize network accuracy in tracking the external signal.

The problem is motivated by the quest for the design of high-performing engineered networks, such as sensor networks as well as by the search for conditions under which biological networks, such as animal groups, perform highly. For example, in migratory herds, the animals must learn, agree on, and move together along a single migration route. It is likely that only a subset of animals invests in a direct measurement of the route, particularly when it is easier to rely on observations of neighbors [8]. Reference [8] shows that the emergence of leaders and followers within a large, mobile population is an evolutionarily stable solution for a sufficiently high investment cost of sampling the migration route. In [9], the authors used a mathematical model to analyze this evolutionary dynamic and to compute the location of emergent leaders as a function of the network graph and the investment cost. The model yields a distributed adaptive dynamic for taking on leadership in this context; however, the evolutionary dynamics do not guarantee a steady-state solution that is optimal for the herd.

In this paper, we study the leader-follower network dynamics subject to stochastic perturbations [10], [11], examining cases in which there are one and two noise-corrupted leaders and in which there are any number of noise-free leaders. Our objective is to make rigorous how a leader set, as a function of properties of the network graph, affects the total system error of the group defined as the steady-state variance of the system about the external signal. Total system error can also be viewed as a measure of coherence, equivalently, the H_2 norm of the system dynamics [11], [12].

To this end, we develop a means of quantifying the combined influence of a set of leader nodes in a network on the total system error in the leader-follower dynamic. Intuitively, this influence should correspond to some notion of centrality of a set of nodes since a leader set that gives low system error must be well connected to other nodes in the network. Different types of centrality of a set of nodes were defined in [13], where the authors quantified degree, closeness, betweenness, and flow centralities of sets of nodes by extensions of the definitions for individuals. Illustrative examples were used in [13] to explore the relationship between those measures and network properties. In contrast to the literature, we derive a

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measure of centrality of a set of nodes, called *joint centrality*, by examining the performance measure, that is, total system error, and expressing performance in terms of graph measures. We note that applications of measures of centrality of a set of nodes include a broad range of research areas from emergency-response management [14] to a network connectivity analysis of the quality of innovative ideas [15].

Much of the recent research related to leader-follower multiagent systems with stochastic dynamics has focused on the development of offline leader selection algorithms that seek to find the leader set that minimizes total system error [11], [16]–[19]. These algorithms have been designed to be computationally efficient in approximating optimal solutions with proven bounds on the total system error relative to the optimal value of error. Many of the algorithms are iterative, adding to the leader set one agent at a time. This approach may preclude finding the optimal solution since the optimal set of l leaders does not necessarily include the optimal set of m leaders, $l > m$. The authors in [19] address this issue by considering a "swap" step within their iterative algorithm.

Our contributions are fourfold. First, we provide a new approach to solve the optimal leader selection problem in terms of network graph measures. In general, our approach reduces computational complexity significantly compared to the brute force computation. Second, we define a new notion of centrality of a set of nodes in an undirected, connected graph, that we call joint centrality. For the tracking dynamics of the leader-follower network, we show that the total system error is inversely proportional to the joint centrality of the leader set when the leaders are noise free. Thus, the solution to the optimal leader selection problem is the set of nodes that maximizes joint centrality.

The joint centrality of a set of nodes depends on the information centrality of each of the nodes and the resistance and biharmonics distances between pairs of nodes in the set. We show how to calculate joint centrality using entries from submatrices of the pseudoinverses of the Laplacian and squared Laplacian. We show that joint centrality of a set of nodes is a generalization of information centrality for a single node, and that the optimal leader set is composed of nodes that trade off high nodal information centrality with good coverage of the graph, that is, a well-distributed set with respect to resistance and biharmonic distances among nodes in the set.

Third, we consider the case of noise-corrupted leaders and we derive a modified notion of joint centrality, showing, in the cases of one and two noise-corrupted leaders, that total system error is inversely proportional to the modified joint centrality of the leaders. Fourth, we prove the explicit solution to the optimal leader selection problem in the case of cycle graphs and path graphs. A preliminary version of results in this paper, for the cases of one and two leaders, appears in [20].

This paper is organized as follows. In Section II, we introduce the network model dynamics and define the optimal leader selection problem. We review information centrality, resistance distance, biharmonic distance and other properties of the Laplacian in Section III. In Section IV we derive total system error for the general case of m noise-free leaders, define joint centrality of m nodes, and prove our main result. In Section V, we interpret joint centrality, we prove explicit solutions to the optimal leader selection problem in a few cases, and we extend our results to noise-corrupted leaders in the case of one and two leaders. We also discuss the connection to the problem of controllability of networks. We show an example application of joint centrality in Section VI. We conclude with a discussion in Section VII.

II. Model and Problem Statement

We consider a network of n agents tasked with tracking an external signal from the environment. We denote the external signal by $\mu \in \mathbb{R}$ and assume it to be a constant. Generalizations to vector-valued environmental signals are expected to be relatively straightforward and extensions to time-varying environmental signals are the topic of future work.

The state of agent i for $i = 1, \ldots, n$ is $x_i \in \mathbb{R}$, and it represents agent i's estimate of the signal μ. The state of the network is given by $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$. Agent i can measure the evolving relative state $x_j - x_i$ for each agent j in its set of neighbors $\mathcal{N}_i$. The availability of these measurements to agent i is the result of agent i directly sensing the relative state of its neighbors, for example, in the case that the state refers to position, or of neighbors communicating the value of their state to agent i.

The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ encodes the network topology. Each agent corresponds to a node in the set $\mathcal{V} = \{1, 2, \ldots n\}$, and we will use the terms "agents" and "nodes" interchangeably. $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, where the edge $(i, j) \in \mathcal{E}$ if $j \in \mathcal{N}_i$. The adjacency matrix is given by $A \in \mathbb{R}^{n \times n}$ where matrix element $a_{i,j}$ corresponds to the weight on edge (i, j).

We consider undirected, connected graphs. Recent results on effective resistance in directed graphs [21], [22] suggest the means to extend our theory in future work to the case of directed graphs. If the undirected graph contains edge (i, j), then $a_{i,j} = a_{j,i} > 0$; otherwise, $a_{i,j} = 0$. The degree matrix D is a diagonal matrix with entries $d_{i,i} = \sum_{j=1}^{n} a_{i,j}$. The associated Laplacian matrix is defined as $L = D - A$.

An agent $l \in \mathcal{V}$ is called a *leader* if it directly measures the external signal. Let $k_l > 0$ be the weight that agent l puts on its signal measurement. Any agent that is not a leader is called a *follower*. Let the set of leaders be denoted as S with cardinality m and the set of follower nodes, denoted by F, be the complement of S with cardinality $n - m$. The summation over s denotes summation over the leader set, while summation over i denotes summation over the entire set of leaders and followers. We use the index l_1 when it is necessary to identify one leader apart from the rest of the leader set.

Throughout this paper, when a set S of m nodes is identified, we will assume they are the first m nodes in an ordering of the n nodes. Accordingly, we will denote the partition of an $n \times n$ matrix B as

$$B = \begin{bmatrix} B_S & B_{SF} \\ B_{FS} & B_F \end{bmatrix} \tag{1}$$

where B_S is an $m \times m$ matrix corresponding to nodes in set S, and B_F is an $(n - m) \times (n - m)$ matrix corresponding to the remaining nodes. We will further let l_1 be the first node in the ordered set S. We will denote the Moore Penrose pseudoinverse of a matrix B by B^+ and the conjugate transpose of B by B^*. We let $\mathbf{1_n}$ be the vector of n ones and $\mathbf{e}_j$ be the standard basis vectors for $\mathbb{R}^n$.

We assume that all leaders apply the same weight k to their measurement of the external signal, that is, $k_i = k > 0$ for $i \in S$ and $k_i = 0$ for $i \in F$. We assume that stochastic disturbances enter the dynamics as additive noise. We model the dynamics for each agent $i \in \mathcal{V}$ by the following stochastic process:

$$dx_i = -k_i(x_i - \mu)dt - L_i\mathbf{x}dt + \sigma dW_i \tag{2}$$

where L_i is the ith row of the Laplacian L, and σdW_i represents increments drawn from independent Wiener processes with standard deviation σ.

In the case that $k < \infty$, the dynamics of the leaders and followers are all noise corrupted. In [19], it was demonstrated that in the limit as $k \to \infty$, that is, in the case that leaders apply an arbitrarily large weight to tracking the external signal, the dynamics (2) describe the case of noise-free leaders. Thus, our model (2) describes both cases of noise-corrupted leaders ($k < \infty$) and noise-free leaders ($k \to \infty$).

To write (2) in vector form, let $K \in \mathbb{R}^n$ be the diagonal matrix with elements k_i, let $M = L + K$ and without loss of generality, let $\mu = 0$. Then, (2) becomes

$$d\mathbf{x} = -M\mathbf{x}dt + \sigma d\mathbf{W}. \tag{3}$$

Since we have assumed that $\mathcal{G}$ is connected, $-M$ is Hurwitz so long as $k_i = k > 0$ for some agent i, that is, S is nonempty.

Thus, for nonempty S, $\mathbf{x}$ will converge to a steady-state distribution about the value of the external signal, and the steady-state covariance matrix Σ of $\mathbf{x}$ is the solution to the Lyapunov equation

$$M\Sigma + \Sigma M^T = \sigma^2 I.$$

The steady-state variance of x_i is $\Sigma_{i,i}$, the corresponding diagonal element of Σ. Since the external signal is assumed to be constant, the system will converge to a steady-state distribution about the value of the external signal even if the nodes chosen as leaders do not guarantee system controllability.

Following [11] and [18], we define *total system error* as $\mathrm{Tr}(\Sigma) = \sum_{i=1}^n \Sigma_{i,i}$. We define group performance as the inverse of total system error, which measures *network tracking accuracy*.

By [23] we have that the covariance matrix of (3) is

$$\mathrm{Cov}\,(\mathbf{x}(t), \mathbf{x}(t)) = \sigma^2 \int_0^t e^{-M(t-\tau)} e^{-M^T(t-\tau)} d\tau. \tag{4}$$

Given that $\mathcal{G}$ is undirected, the Laplacian matrix L will be symmetric and it follows that M will be symmetric and normal. Let the eigenvalues of M be λ_i, $i \in \mathcal{V}$ with corresponding eigenvectors $\boldsymbol{\nu}_i$. Let Λ be the diagonal matrix with entries $\Lambda_{i,i} = \lambda_i$. Then there exists a unitary matrix U such that $U^*MU = \Lambda$ and (4) can be written as

$$\mathrm{Cov}\,(\mathbf{x}(t), \mathbf{x}(t)) = \sigma^2 \left(UR(t)U^*\right)$$

with

$$R(t) := \int_0^t e^{-(\Lambda+\Lambda^*)(t-\tau)} d\tau.$$

From [24], this gives

$$[\mathrm{Cov}\,(\mathbf{x}(t), \mathbf{x}(t))]_{i,j} = \sigma^2 \sum_{p=1}^n \frac{1 - e^{-2\mathrm{Re}(\lambda_p)t}}{2\mathrm{Re}(\lambda_p)} \nu_i^{(p)} \nu_j^{(p)*}.$$

Since M is symmetric, all eigenvalues of M will be real, and the steady-state variance of each node can be written as

$$\mathrm{Var}(x_i)_{ss} = \Sigma_{i,i} = \sigma^2 \sum_{p=1}^n \frac{1}{2\lambda_p} \left|\nu_i^{(p)}\right|^2. \tag{5}$$

Total system error follows from summing (5) over all i

$$\sum_{i=1}^n \Sigma_{i,i} = \sigma^2 \sum_{i=1}^n \frac{1}{2\lambda_i} = \frac{\sigma^2}{2} \sum_{i=1}^n M_{i,i}^{-1}. \tag{6}$$

Total system error defines the coherence of the network, and is equivalent to the H_2 norm of the system with output equation $\mathbf{y} = C\mathbf{x}$, where $C = I_n$ and I_n the $n \times n$ identity matrix [11], [12].

We define the *optimal leader selection problem* as follows.

Definition 1 (Optimal Leader Selection Problem): Given m and an undirected, connected graph $\mathcal{G}$, find a set of m leaders S^* over all possible sets S of m leaders that minimizes the total system error (6) for the leader-follower network tracking dynamics (3), that is, find

$$S^* = \arg\min_S \sigma^2 \sum_{i=1}^n \frac{1}{2\lambda_i} = \arg\min_S \frac{\sigma^2}{2} \sum_{i=1}^n M_{i,i}^{-1}.$$

III. Review of Properties of the Laplacian and Graph-Theoretic Measures

We briefly review relevant graph-theoretic measures and identities that will be applied in later sections. We start with the notion of information centrality, which was first introduced by Stephenson and Zelen in [25]. Information centrality can be understood by first defining the information in a path between any two nodes in $\mathcal{G}$ to be the inverse of the path length between those two nodes. Thus, the longer the path, the less information in that path. Total information between nodes i and j, denoted as $I_{i,j}^{\mathrm{tot}}$, is the sum of the information in all paths connecting nodes i and j. Reference [25] shows that total information can be calculated without path enumeration by using the group inverse of the Laplacian, which here is the pseudoinverse L^+

$$I_{i,j}^{\mathrm{tot}} = \left(L_{i,i}^+ + L_{j,j}^+ - 2L_{i,j}^+\right)^{-1}.$$

Information centrality for node i, denoted as c_i, is defined as the harmonic average of total information between node i and all other nodes in $\mathcal{G}$

$$c_i = \left(\frac{1}{n} \sum_{j=1}^n \frac{1}{I_{i,j}^{\mathrm{tot}}}\right)^{-1}. \tag{7}$$

In [26], Poulakakis *et al.* evaluated the certainty of each node i in a network of decision-makers accumulating stochastic evidence toward a decision. This certainty, denoted as μ_i, is defined as the inverse of the difference between the variance of the state x_i about the reference signal and the minimum achievable variance as $t \to \infty$. The authors applied the notion of information centrality to directly interpret μ_i in terms of structural properties of the underlying communication graph. It was proven that

$$\frac{1}{\mu_i} = \frac{\sigma^2}{2} L^+_{i,i} = \frac{\sigma^2}{2}\left(\frac{1}{c_i} - \frac{K_f}{n^2}\right) \tag{8}$$

where K_f is the Kirchhoff index of $\mathcal{G}$. The identity (8) implies that the ordering of nodes by certainty is equal to the ordering of nodes by information centrality [26]. We show in later sections that information centrality also plays a critical role in the solution to the optimal leader selection problem.

The total information between any two nodes i and j is closely related to the *resistance distance* between them, denoted as $r_{i,j}$. Resistance distance between nodes in the undirected graph $\mathcal{G}$ is defined as the resistance distance between the corresponding two nodes in the electrical network analog to the graph $\mathcal{G}$. With [27] for an undirected graph $\mathcal{G}$

$$r_{i,j} = L^+_{i,i} + L^+_{j,j} - 2L^+_{i,j} = I^{\mathrm{tot}\,-1}_{i,j}. \tag{9}$$

It follows that

$$\sum_{i=1}^{n} r_{i,j} = \frac{n}{c_j}.$$

An additional measure with a similar form to that of resistance distance is the recently derived notion of biharmonic distance d_B [28]. This measure has been used to quantify distance between two points v_i, v_j on the surface of a discrete 3-D mesh

$$d_B(v_i, v_j)^2 = g_d(i,i) + g_d(j,j) - 2g_d(i,j)$$

where g_d is the discrete Green's function of the discretized, normalized bilaplacian $\tilde{L}^2$, equivalent to the pseudoinverse of $\tilde{L}^2$, and $\tilde{L}$ is the normalization of Laplacian L. In the context of 3-D meshes, the biharmonic distance has the advantage, over diffusion and geodesic distances, of providing a balance between local and global properties of a surface, reflecting overall connectivity for faraway points [28]. We define the *biharmonic distance between two nodes i and j in the graph $\mathcal{G}$*, which we denote $\gamma_{i,j}$, analogously without normalizing L

$$\begin{aligned}\gamma_{i,j} &= L^{2+}_{i,i} + L^{2+}_{j,j} - 2L^{2+}_{i,j} = \sum_{l=1}^{n}\left(L^+_{l,i} - L^+_{l,j}\right)^2 \\ &= (\mathbf{e}_i - \mathbf{e}_j)^T L^{2+}(\mathbf{e}_i - \mathbf{e}_j).\end{aligned} \tag{10}$$

We observe that the definition of biharmonic distance $\gamma_{i,j}$ of (10) is very similar to the definition of resistance distance $r_{i,j}$ of (9) with the difference being the use of the pseudoinverse of L^2 in the definition of $\gamma_{i,j}$ compared to the pseudoinverse of L in the definition of $r_{i,j}$. Since L^2 is symmetric and positive semidefinite, we immediately have that $\gamma^{1/2}$ is a metric. In fact, it can be viewed as a Manahalobis distance which, in this case, describes a dissimilarity measure between two vectors from a single distribution with covariance matrix L^2. Let Γ be the matrix with elements $\gamma_{i,j}$.

For completion, we note that resistance distance and biharmonic distance between nodes can be written in terms of the eigenvalues λ_i and eigenvectors $\boldsymbol{\nu}_i$ of the Laplacian L

$$r_{i,j} = \sum_{l=2}^{n} \frac{1}{\lambda_l}\left(\nu^i_l - \nu^j_l\right)^2$$

$$\gamma_{i,j} = \sum_{l=2}^{n} \frac{1}{\lambda_l^2}\left(\nu^i_l - \nu^j_l\right)^2.$$

Finally, the following properties of L^+ will be applied to the proofs. (See [26] for details.)

$$LL^+ = L^+L = I_n - \frac{1}{n}\mathbf{1_n}\mathbf{1_n}^T \tag{11}$$

$$\mathbf{1_n}^T L^+ = L^+\mathbf{1_n} = 0 \tag{12}$$

$$\mathrm{Tr}(L^+) = \frac{K_f}{n}. \tag{13}$$

IV. Joint Centrality and the Optimal m Noise-Free Leaders

In this section, we prove our main result on the general solution of the optimal leader selection problem by deriving an explicit expression for total system error with m noise-free leaders in terms of properties of the underlying graph. Before stating the theorem, we first define the *joint centrality of a set of m nodes* in a network graph.

Definition 2 (Joint Centrality): Let $\mathcal{G}$ be an undirected, connected graph of order n. Given integer $m < n$, let S be the set of any m nodes in $\mathcal{G}$. Choose an arbitrary element $l_1 \in S$. Let N be an $n \times n$ matrix with elements of N^{-1} given by

$$N^{-1}_{i,j} = L^+_{i,j} - L^+_{i,l_1} - L^+_{j,l_1} + L^+_{l_1,l_1}. \tag{14}$$

Following (1), $N^{-1}_{S\setminus l_1}$ is the $(m-1)\times(m-1)$ submatrix of N^{-1} corresponding to the elements of S less the first element l_1. Let $G = (N^{-1}_{S\setminus l_1})^{-1}$ and $\bar{G} = \begin{bmatrix} 0 & 0 \\ 0 & G\end{bmatrix} \in \mathbb{R}^{m\times m}$. Let $Q = \bar{G}\Gamma_S$, where Γ is given by (10). The *joint centrality of set S in $\mathcal{G}$* is defined as

$$\rho_S = n\left(\frac{K_f}{n} + n\det(G)\det\left(L^+_S\right) + \frac{1}{2}\mathrm{Tr}(Q) - \mathbf{1}_n^T Q\mathbf{e}_{l_1}\right)^{-1}. \tag{15}$$

Theorem 1 (Optimal Noise-Free Leader Set): Let $\mathcal{G}$ be an undirected, connected graph of order n. Let S be a set of m noise-free leaders. Then, the total system error (6) for the system dynamics (3) is

$$\sum_{i=1}^{n}\Sigma_{i,i} = \frac{\sigma^2}{2}\left(\frac{n}{\rho_S}\right) \tag{16}$$

where ρ_S is the joint centrality of leader set S given by (15). The optimal leader set is $S^* = \arg\max_S \rho_S$, the set of leader nodes with the maximal joint centrality.

We recall three lemmas that will be used in the proof of Theorem 1.

Lemma 1 [29]: Let $\mathbf{z}, \mathbf{y} \in \mathbb{R}^n$. A rank-1 update $\mathbf{z}\mathbf{y}^T$ for the Moore-Penrose pseudoinverse of a real valued matrix, $F \in \mathbb{R}^{n\times n}$, is given by

$$(F + \mathbf{z}\mathbf{y}^T)^+ = F^+ + H$$

where

$$H = -\frac{1}{\|\mathbf{w}\|^2}\mathbf{v}\mathbf{w}^T - \frac{1}{\|\mathbf{m}\|^2}\mathbf{m}\mathbf{h}^T + \frac{\beta}{\|\mathbf{m}\|^2\|\mathbf{w}\|^2}\mathbf{m}\mathbf{w}^T \quad (17)$$

and $\beta = 1 + \mathbf{y}^T F^+ \mathbf{z}$, $\mathbf{v} = F^+\mathbf{z}$, $\mathbf{h} = (F^+)^T\mathbf{y}$, $\mathbf{w} = (I - FF^+)\mathbf{z}$, and $\mathbf{m} = (I - F^+F)^T\mathbf{y}$.

Lemma 2 [30]: Let $X \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{m\times m}$, $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{m\times n}$ such that X, Z and $X + UZV$ are nonsingular. Then, $(X + UZV)^{-1}$ can be written as

$$(X + UVZ)^{-1} = X^{-1} - X^{-1}U(Z^{-1} + VX^{-1}U)^{-1}VX^{-1}.$$

Lemma 3 [31]: The determinant of a bordered matrix can be computed as follows:

$$\begin{vmatrix} X & \mathbf{u} \\ \mathbf{v}^T & d \end{vmatrix} = d|X| - \mathbf{v}^T(\mathrm{adj}X)\mathbf{u}$$

where $X \in \mathbb{R}^{p\times p}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$, and $d \in \mathbb{R}$.

Proof: (Theorem 1). We begin by studying terms in the total system error for finite $k > 0$ and then evaluate in the limit as $k \to \infty$. From (6), the total system error is proportional to $\mathrm{Tr}(M^{-1})$ where $M = L + K$. Let K_1 be the diagonal matrix with k in the first diagonal element and zeros elsewhere and let $K_{m-1} = K - K_1$. We derive an expression for $\mathrm{Tr}(M^{-1})$ by calculating two successive updates to L^+. We first show that if we define $N = L + K_1$, and thus $M = N + K_{m-1}$, then N^{-1} satisfies (14) for $k \to \infty$.

Let $\mathbf{e} = \mathbf{d}$ be vectors of length n with $\sqrt{k}$ in the l_1 (first) entry and zeros elsewhere where l_1 is a member of the leader set. Note that the choice of l_1 will not affect the value of joint centrality for a given leader set. Then $N^{-1} = (L + K_1)^{-1} = (L + \mathbf{e}\mathbf{d}^T)^{-1}$. Applying Lemma 1 we get that $(L + \mathbf{e}\mathbf{d}^T)^{-1} = L^+ + H$, with H given by (17) such that

$$N^{-1} = L^+ - L^+_{l_1}\mathbf{1_n}^T - \mathbf{1_n}L^{+T}_{l_1} + \frac{\left(1 + kL^+_{l_1,l_1}\right)}{k}\mathbf{1_n}^T\mathbf{1_n}. \quad (18)$$

Taking the limit as $k \to \infty$, the elements of N^{-1} can be written as (14).

Let $U = [-\sqrt{k}\mathbf{e}_2, \ldots, -\sqrt{k}\mathbf{e}_m] \in \mathbb{R}^{n\times(m-1)}$, let $V = U^T$ and let $\mathbb{I}_{m-1} \in \mathbb{R}^{(m-1)\times(m-1)}$ be the identity matrix. Then, $M^{-1} = (N + K_{m-1})^{-1} = (N + U\mathbb{I}V)^{-1}$. Applying Lemma 2, we obtain

$$(N + U\mathbb{I}V)^{-1} = N^{-1} - N^{-1}U(\mathbb{I} + VN^{-1}U)^{-1}VN^{-1}.$$

Let $G = (N^{-1}_{S\setminus l_1})^{-1}$ as in Definition 2. Then if we take the limit as $k \to \infty$, sum the diagonal elements of $M^{-1} = (N + UIV)^{-1}$, and apply the identities (12) and (13) we obtain

$$\begin{aligned}\sum_{j=1}^n M^{-1}_{j,j} = \frac{K_f}{n} + nL^+_{l_1,l_1} - \sum_{s_1,s_2\in S\setminus\{l_1\}} \sum_{i=1}^n G_{s_1,s_2} \\ \times\Big(L^+_{l_1,l_1}\Big(L^+_{l_1,l_1} - L^+_{l_1,s_1} - L^+_{l_1,s_2}\Big) + L^+_{l_1,s_1}L^+_{l_1,s_2} + \frac{1}{2} \\ \times\Big[\Big(L^+_{i,l_1} - L^+_{i,s_1}\Big)^2 + \Big(L^+_{i,l_1} - L^+_{i,s_2}\Big)^2 - \Big(L^+_{i,s_1} - L^+_{i,s_2}\Big)^2\Big]\Big).\end{aligned} \quad (19)$$

Consider the square bracketed terms of (19) in which we observe the emergence of biharmonic distance, γ. Substituting (10) and defining $\bar{G}$ as in Definition 2, we obtain

$$\begin{aligned}&\sum_{s_1,s_2\in S\setminus\{l_1\}} \sum_{i=1}^n G_{s_1,s_2} \\ &\times\frac{1}{2}\Big[\Big(L^+_{i,l_1} - L^+_{i,s_1}\Big)^2 + \Big(L^+_{i,l_1} - L^+_{i,s_2}\Big)^2 - \Big(L^+_{i,s_1} - L^+_{i,s_2}\Big)^2\Big]\Big) \\ &= -\frac{1}{2}\mathrm{Tr}(\bar{G}\Gamma_S) + \mathbf{1}_n^T[\bar{G}\Gamma_S]\mathbf{e}_{l_1}.\end{aligned}$$

Additional simplification is made by applying Lemma 3 to the middle terms on the right-hand side of (19). We obtain

$$\begin{aligned}&nL^+_{l_1,l_1} - n\sum_{s_1,s_2\in S\setminus\{l_1\}} G_{s_1,s_2} \\ &\times\left(L^+_{l_1,l_1}\left(L^+_{l_1,l_1} - L^+_{l_1,s_1} - L^+_{l_1,s_2}\right) + L^+_{l_1,s_1}L^+_{l_1,s_2}\right) \\ &= \frac{n}{\det(G^{-1})}\Bigg(L^+_{l_1,l_1}\det(G^{-1}) - \sum_{s_1,s_2\in S\setminus\{l_1\}} C_{N^{-1}s_1,s_2} \\ &\qquad\times\Big[L^+_{l_1,l_1}\Big(L^+_{l_1,l_1} - L^+_{l_1,s_1} - L^+_{l_1,s_2}\Big) + L^+_{l_1,s_1}L^+_{l_1,s_2}\Big]\Bigg)\end{aligned} \quad (20)$$

where $C_{N^{-1}}$ is the cofactor matrix of $N^{-1}_{S\setminus l_1} = G^{-1}$. We then let $\mathbf{L}^+_{l_1,s_i} = [L^+_{l_1,s_1}, \ldots, L^+_{l_1,s_{m-1}}]^T$ and $\mathbf{L}^+_{l_1,l_1} = [L^+_{l_1,l_1}, \ldots, L^+_{l_1,l_1}]^T$ be vectors in $\mathbb{R}^{m-1}$ and apply Lemma 3 to rewrite the expression in (20) as

$$\frac{n}{\det\left(N^{-1}{}_{S\setminus l_1}\right)}\begin{vmatrix} N^{-1}{}_{S\setminus l_1} & \mathbf{L}^+_{l_1,l_1} - \mathbf{L}^+_{l_1,s_i} \\ \mathbf{L}^+_{l_1,l_1} - \mathbf{L}^+_{l_1,s_i} & L^+_{l_1,l_1}\end{vmatrix}. \quad (21)$$

Using (14), we expand the determinant in (21) and perform algebraic manipulation to show that (21) simplifies to

$$n\det(G)\det\left(L^+_S\right).$$

Thus

$$\begin{aligned}\sum_{i=1}^n \Sigma_{i,i} &= \frac{\sigma^2}{2}\left(\frac{K_f}{n} + n\det(G)\det\left(L^+_S\right)\right. \\ &\qquad\left. + \frac{1}{2}\mathrm{Tr}(\bar{G}\Gamma_S) - \mathbf{1}_n^T[\bar{G}\Gamma_S]\mathbf{e}_{l_1}\right) \\ &= \frac{\sigma^2}{2}\left(\frac{n}{\rho_S}\right)\end{aligned}$$

where ρ_S is defined by (15). ■

V. Interpretation

In this section we provide interpretation of and intuition on the joint centrality measure, we prove explicit solutions to the optimal leader selection problem in a few cases, and we consider noise-corrupted leaders in the case of $m = 1$ and $m = 2$. Our central insight is that joint centrality of a set of nodes is a generalization of information centrality of an individual node: the joint centrality of a set of nodes is directly related to the information centrality of each individual node in the set and a coverage of the graph by the whole set, defined in terms of distribution of the set over the graph with respect to resistance and biharmonic distances. These components of joint centrality may be in tension, since the most information central nodes can be close to one another (e.g., in the path graph), in which case they may be insufficiently distributed over the graph to provide good coverage. The optimal leader set is composed of nodes that trade off high nodal information centrality (close to the center in the path graph example) with good coverage (close to the ends in the path graph example).

We begin in this section by examining the terms in the expression for joint centrality in the case of an arbitrary number of noise-free leaders m, and show the connection to information centralities and coverage. We solve the optimal leader selection problem in the case of a cycle graph and illustrate further with a more general example. We then specialize to the case of $m = 1$ leader, and show how joint centrality specializes to information centrality of the leader node, with or without noise corruption. Next we specialize to the case of $m = 2$ leaders, where the expression for joint centrality facilitates a close examination of the tradeoff between information centralities and coverage provided by the two leaders. We prove an explicit solution for the optimal set of two leaders in the case of the path graph. We also address the problem for $m = 2$ noise-corrupted leaders and provide intuition. We finish the section with a discussion of our results in light of greedy algorithms for finding optimal leader sets, and we make connections to controllability.

A. Joint Centrality and an Arbitrary Number of Leaders m

We interpret the results of Theorem 1 in the following two remarks. We then illustrate the notion of coverage by proving the explicit solution to the optimal leader set in the case of a cycle graph. We illustrate the tradeoff between centrality and coverage with an example network.

Remark 1: Using Theorem 1 to compute the total system error in terms of joint centrality of the m leader nodes provides a significant reduction in computation as compared to using the definition of total system error (6). Using joint centrality one only needs to compute the inverse of two $n \times n$ matrices L^+ and L^{2+} and then for each candidate set of leaders the inverse of an $(m-1) \times (m-1)$ matrix. This is in contrast to using the definition (6), which requires computing the inverse of the $n \times n$ matrix M for each candidate set of leaders.

Remark 2: Theorem 1 reveals how the solution to the optimal leader selection problem is an optimal tradeoff between high information centrality of the leader nodes and high resistance distances and biharmonic distances between leader nodes. To see this we examine the terms in (15) for joint centrality ρ_S.

First, the elements of N^{-1} given by (14) depend on resistance distances

$$N^{-1}_{i,j} = \frac{1}{2}(r_{i,l_1} + r_{j,l_1} - r_{i,j}).$$

Thus, $N^{-1}_{i,j}$ quantifies a joint resistance distance between a pair of nodes i, j and l_1, Then, $\det(G) = (\det(N^{-1}_{S\setminus l_1}))^{-1}$ depends on these joint resistance distances among leaders.

Second, by (8) each diagonal element of L^+_S corresponds to a leader node and depends directly on the inverse of its information centrality as follows:

$$L^+_{s,s} = \frac{1}{c_s} - \frac{K_f}{n^2}.$$

By (9) the off diagonal elements of L^+_S depend on information centralities and resistance distances between leaders

$$L^+_{s,t} = \frac{1}{2}\left(\frac{1}{c_s} + \frac{1}{c_t} - r_{s,t} - 2\frac{K_f}{n^2}\right).$$

Maximizing ρ_S requires a small $\det(G)\det(L^+_S)$, which suggests a key tradeoff between high information centrality of leaders and high resistance distances between leaders.

The term $\mathrm{Tr}(Q)$ in (15) is the sum of products of the biharmonic distances between pairs of leader nodes (from Γ_S), and terms in G. Since $\mathrm{Tr}(Q)$ is negative, maximizing joint centrality requires high biharmonic distances between pairs of leader nodes. Biharmonic distance between a pair of nodes depends strongly on global connectivity of the graph and together with resistance distances provides a measure of coverage of the graph by a node set. Thus, the joint centrality measure makes rigorous how the optimal leader set trades off high information centrality of each of the nodes in the set with a good coverage of the graph by the set of nodes.

To better understand the coverage term, we first consider the case of a cycle graph. Because each node in the cycle graph has the same information centrality, it is only the coverage term that matters in the optimization of joint centrality. We can use the cyclic structure of the graph Laplacian to explicitly solve for the optimal locations of m noise-free leaders. In the following theorem, we show that the optimal leader set is a set of nodes uniformly distributed about the cycle, which corresponds to a set that maximizes coverage of the graph.

Theorem 2 (Optimal Noise-Free Leader Set on a Cycle Graph): Let $\mathcal{G}$ be an undirected, unweighted cycle graph of order n. Let $m < n$ such that $p = n/m$ is an integer. Let S be a set of m noise-free leaders. Then, an optimal leader set S^* is any set S where the leaders are uniformly distributed around the cycle, that is, the geodesic distance between any leader and each of the other two closest leaders is $d_{s_a,s_b} = p$.

Proof: See Appendix A. ■

Next, to illustrate the tradeoff between nodal information centrality and coverage, we consider the unweighted, undirected, connected graph shown in Fig. 1. The optimal sets of one, two and three leaders are shown in yellow, green, and blue, respectively. Visually, it is clear that the optimal choice for a single leader (node 9, in yellow) has a central position in the network. In fact, node 9 has the highest information centrality c_i

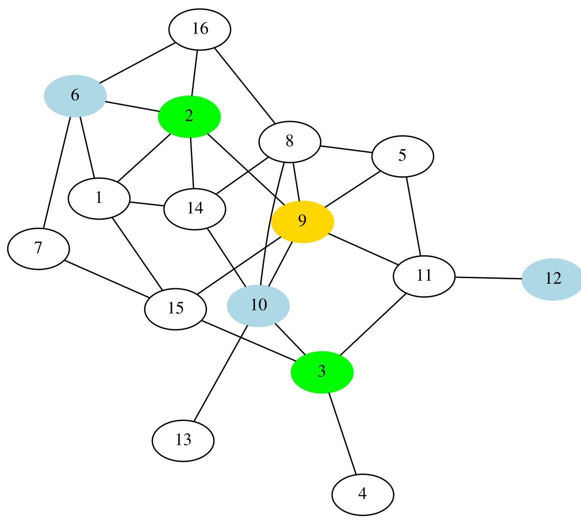

Fig. 1. Solutions to the optimal leader set for an example graph with 16 nodes. For $m = 1$ leader, the optimal solution is node 9, shown in yellow. For $m = 2$ leaders, the optimal solution is the set of nodes 2 and 3, shown in green. For $m = 3$ leaders, the optimal solution is the set of nodes 6, 10, and 12, shown in blue.

(7), consistent with Corollary 1 of Section V, where it is proved that the optimal single leader is the most information central node.

Interestingly, it is observed that the optimal single leader is not a member of the optimal set of two leaders (nodes 2 and 3, in green). This is due to the fact that the optimal two leaders need to trade off high information centrality as individuals with a joint coverage of the graph (see also Corollary 2 in Section V-C). For this reason, the optimal two leaders are well connected within the graph and distanced from each other.

The optimal three leaders (nodes 6, 10, 12, in blue) further illustrate the key tradeoff between leaders that are central and leaders that cover the graph. Although node 12 is not so well connected, its large resistance and biharmonic distances from nodes 6 and 10 make it part of the optimal three-leader set. That is, the three-node leader set has optimal joint influence on the graph, as encoded by the joint centrality of the set.

The three solutions illustrate how a leader selection algorithm that first selects a leader and then iteratively adds to the set would result in a suboptimal leader set for this example and likely in general. (See also the example in [11].)

B. Optimal Selection of a Single Noise-Corrupted or Noise-Free Leader

Joint centrality reduces to information centrality in the case of a single leader ($m = 1$), with or without noise corruption. Thus, the optimal single leader is the node with the highest information centrality.

Corollary 1 (Optimal Leader Set, $m = 1$): Let $\mathcal{G}$ be an undirected, connected graph of order n. Let $S = \{s\}$ be a set of one noise-corrupted leader ($k < \infty$) with information centrality c_s. Then, the total system error (6) for the system dynamics (3) is

$$\sum_{i=1}^{n} \Sigma_{i,i} = \frac{n\sigma^2}{2}\left(\frac{1}{k} + \frac{1}{c_s}\right).$$

If instead the leader set S is noise free, then the total system error (6) for the system dynamics (3) is

$$\sum_{i=1}^{n} \Sigma_{i,i} = \frac{n\sigma^2}{2}\left(\frac{1}{c_s}\right).$$

In both the noise-corrupted and the noise-free cases, the optimal leader set $S^* = \{s^*\} = \arg\max_s c_s$, the node with maximal information centrality c_{s^*}.

Proof: For a single leader, we only need to consider a rank-one update to the pseudoinverse of L. From (18), where $l_1 = s$, this is

$$N^{-1} = L^{-1} - L_s^{+}\mathbf{1_n}^T - \mathbf{1_n}L_s^{+T} + \frac{(1 + kL_{s,s}^{+})}{k}\mathbf{1_n}^T\mathbf{1_n}. \quad (22)$$

Summing the diagonal elements of (22) and applying (8), (12), and (13) yields

$$\sum_{i=1}^{n} N_{i,i}^{-1} = \frac{K_f}{n} + \frac{n}{k} + n\left(\frac{1}{c_s} - \frac{K_f}{n^2}\right) = \frac{n}{k} + \frac{n}{c_s}.$$

Subsequently, substituting into (6) gives the total system error

$$\sum_{i=1}^{n} \Sigma_{i,i} = \frac{n\sigma^2}{2}\left(\frac{1}{k} + \frac{1}{c_s}\right). \quad (23)$$

To obtain the total system error in the case of one noise-free leader, we take the limit of (23) as $k \to \infty$, which gives

$$\lim_{k\to\infty} \sum_{i=1}^{n} \Sigma_{i,i} = \lim_{k\to\infty} \frac{n\sigma^2}{2}\left(\frac{1}{k} + \frac{1}{c_s}\right) = \frac{n\sigma^2}{2}\left(\frac{1}{c_s}\right). \quad (24)$$

The total system error in (23) and in (24) is minimized when the leader has the highest information centrality. ■

Remark 3: Our definition of joint centrality derives from the definition of the optimal leader selection problem in terms of minimizing total system error (6). However, we have shown in Corollary 1 that joint centrality can be interpreted as a generalization of information centrality of a single node. This suggests the possibility of using joint centrality for generalizing from individual nodes to sets of nodes in problems where information centrality is a critical measure. For example, it is proved in [32] that information centrality of a node in a network performing distributed hypothesis testing determines its speed-accuracy tradeoff. Joint centrality may be useful for investigating the decision-making performance of a set of nodes in this context.

C. Joint Centrality and Two Noise-Free Leaders

In order to provide further intuition, we specialize Theorem 1 to the case of two noise-free leaders. In this case the expression for joint centrality simplifies as compared to the case of arbitrary m, and we can more closely examine the terms that determine the centrality versus coverage tradeoff in the optimal leader set.

Corollary 2 (Optimal Noise-Free Leader Set, $m = 2$): Let $\mathcal{G}$ be an undirected, connected graph of order n. Let $S_2 = \{s_1, s_2\}$ be a set of two noise-free leaders. Then, the total system error (6) for the system dynamics (3) is

$$\sum_{i=1}^{n} \Sigma_{ii} = \frac{\sigma^2}{2}\left(\frac{n}{\rho_{S_2}}\right) \tag{25}$$

where ρ_{S_2} is the joint centrality of S_2 given by (15), which specializes to

$$\rho_{S_2} = n\left(\frac{K_f}{n} + \frac{nL^+_{s_1,s_1}L^+_{s_2,s_2} - nL^{+\,2}_{s_1,s_2} - \gamma_{s_1,s_2}}{r_{s_1,s_2}}\right)^{-1}. \tag{26}$$

The optimal leader set is $S_2^* = \{s_1^*, s_2^*\} = \arg\max_{s_1,s_2} \rho_{S_2}$, the two nodes with the maximal joint centrality.

Proof: In the case of two leaders, $G = 1/r_{s_1,s_2}$. Equation (25) follows directly from simplification of (16) and (15) from Theorem 1. ■

Remark 4: Following Remark 2, we see that in the two-leader case the term $\det(G)\det(L_S^+) = (L^+_{s_1,s_1}L^+_{s_2,s_2} - L^{+\,2}_{s_1,s_2})/r_{s_1,s_2}$, which is small for large leader information centrality and large resistance distance between leaders. The term $\mathrm{Tr}(Q)$ is proportional to $-\gamma_{s_1,s_1}/r_{s_1,s_2}$. For this term to be small, the biharmonic distance should be large relative to the resistance distance between leaders.

We prove an explicit formula for the optimal two noise-free leader set in the case of a path graph of order n.

Corollary 3 (Optimal Noise-Free Leader Set on a Path Graph, $m = 2$): Let $\mathcal{G}$ be an undirected, unweighted path graph of order n, which is the cycle graph with one link removed. Let $S_2 = \{s_1, s_2\}$ be a set of two noise-free leaders. The optimal leader set S^* is $s_1^* = \mathrm{rnd}((n/5) + (1/2))$ and $s_2^* = \mathrm{rnd}((4n/5) + (1/2))$, where rnd is rounding to the closest integer.

Proof: See Appendix B. ■

We observe that for large n, the optimal two leader locations on the path approach 0.2 and 0.8 of the path length (starting from one end). This is in contrast with the cycle, where the optimal two leaders maintain a distance between each other equal to 0.5 of the number of nodes. Considering that the path is simply a cycle with one edge removed, it is interesting to observe that for large n, removing an edge from a cycle will cause the fraction of nodes between the optimal two leaders to increase from 0.5 to 0.6. That is, the optimal two leaders in the path are more spread out towards the two endpoints. The locations of the optimal two leaders in the path can be understood to be the optimal solution to the tradeoff between high information centrality of two symmetrically distributed leaders, which increases as the two leaders get closer to the midpoint and thus to each other, and good coverage, which requires the two leaders to be sufficiently distant from each other. The optimal two-leader set does not include the optimal single leader set, which is the node at the midpoint of the path, following Corollary 1 of Section V-B.

D. Joint Centrality and Two Noise-Corrupted Leaders

To address the case of two noise-corrupted leaders, where $k < \infty$, we define a k-dependent joint centrality of a set of two nodes. We then derive the solution to the optimal leader selection problem for two noise-corrupted leaders by calculating the total system error in terms of the k-dependent joint centrality of the two-leader set.

Theorem 3 (Optimal Noise-Corrupted Leader Set, $m = 2$): Let $\mathcal{G}$ be an undirected, connected graph of order n. Let $S_2 = \{s_1, s_2\}$ be a set of two noise-corrupted leaders ($k < \infty$). Define ρ_{kS_2}, the k-dependent joint centrality of S_2, as

$$\rho_{kS_2} = n\left(\frac{K_f}{n} + \frac{n\left[1 + k\left(L^+_{s_1,s_1} + L^+_{s_2,s_2}\right)\right]}{k\left(2 + kr_{s_1,s_2}\right)} + \frac{nk^2\left(L^+_{s_1,s_1}L^+_{s_2,s_2} - L^{+\,2}_{s_1,s_2}\right) - k^2\gamma_{s_1,s_2}}{k\left(2 + kr_{s_1,s_2}\right)}\right)^{-1}. \tag{27}$$

Then, the total system error (6) for the system dynamics (3) is

$$\sum_{i=1}^{N} \Sigma_{ii} = \frac{\sigma^2}{2}\left(\frac{n}{\rho_{kS_2}}\right). \tag{28}$$

The optimal leader set is $S_2^* = \{s_1^*, s_2^*\} = \arg\max_{s_1,s_2} \rho_{kS_2}$, the two nodes with the maximal k-dependent joint centrality.

Prior to proving Theorem 3, we state a lemma from [33] that provides a simplification of the Woodbury formula in the case of a rank one update to a matrix.

Lemma 4 [33]: For rank one square matrix H and nonsingular X and $X + H$, $(X + H)^{-1}$ can be written as

$$(X + H)^{-1} = X^{-1} - \frac{1}{1+g}X^{-1}HX^{-1}$$

where $g = \mathrm{Tr}(HX^{-1})$.

Proof: (Theorem 3). Let K_1, K_2 be rank one matrices with $K_{1_{s_1,s_1}} = k$, $K_{2_{s_2,s_2}} = k$ where $k > 0$ and all other elements of K_1, K_2 are zero. Let $K = K_1 + K_2$ and $N = L + K_1$. Then, $M = L + K = N + K_2$.

By applying Lemma 4, we compute

$$\begin{aligned} M^{-1} &= (N + K_2)^{-1} \\ &= N^{-1} - \frac{1}{1 + \mathrm{Tr}(K_2N^{-1})}N^{-1}K_2N^{-1}. \end{aligned} \tag{29}$$

By (18)

$$\begin{aligned} \mathrm{Tr}(K_2N^{-1}) &= 1 + kL^+_{s_2,s_2} - 2kL^+_{s_2,s_1} + kL^+_{s_1,s_1} \\ &= 1 + kr_{s_1,s_2}. \end{aligned} \tag{30}$$

Plugging (30) into (29) yields total system error (6)

$$\begin{aligned} \sum_{i=1}^{n} \Sigma_{i,i} &= \frac{\sigma^2}{2}\sum_{i=1}^{n} M^{-1}_{i,i} \\ &= \frac{\sigma^2}{2}\sum_{i=1}^{n}\left(N^{-1}_{i,i} - \frac{1}{2 + kr_{s_1,s_2}}(N^{-1}K_2N^{-1})_{i,i}\right). \end{aligned}$$

Expanding N^{-1} in terms of L^+ and applying (12) and (13) gives

$$\sum_{i=1}^{n} M_{i,i}^{-1} = \frac{n}{k} + \frac{K_f}{n} + nL_{s_1,s_1} - \frac{1}{2+kr_{s,p}} \times \left(k \sum_{i=1}^{n} \left(L_{i,s_1}^+ - L_{i,s_2}^+\right)^2 + nk\left(L_{s_1,s_2}^+\right)^2 - 2nL_{s_1,s_2}^+ - 2nkL_{s_1,s_1}^+ L_{s_1,s_2}^+ + 2nL_{s_1,s_1}^+ + nk\left(L_{s_1,s_1}^+\right)^2 + \frac{n}{k} \right).$$

Rearranging terms and substituting from (10) results in

$$\sum_{i=1}^{n} \Sigma_{i,i} = \frac{\sigma^2}{2} \left(\frac{K_f}{n} + \frac{n + nk\left(L_{s_1,s_1}^+ + L_{s_2,s_2}^+\right)}{k\left(2+kr_{s_1,s_2}\right)} + \frac{nk^2\left(L_{s_1,s_1}^+ L_{s_2,s_2}^+ - {L_{s_1,s_2}^+}^2\right) - k^2\gamma_{s_1,s_2}}{k\left(2+kr_{s_1,s_2}\right)} \right) = \frac{\sigma^2}{2}\left(\frac{n}{\rho_{kS_2}}\right).$$

■

We observe that the k-dependent joint centrality (ρ_{kS_2} from Theorem 3) plays the same role in determining total system error with two noise-corrupted leaders (28) as joint centrality (ρ_{S_2} from Corollary 2) plays in determining total system error with two noise-free leaders (25). Further, as expected, in the limit as $k \to \infty$ we see that ρ_{kS_2} approaches ρ_{S_2}. To better understand the results in the case of finite k, we compute the Taylor series expansion of ρ_{kS_2} (28) about $k = 0$:

$$\rho_{kS_2} = 2k + \left(r_{s_1,s_2} - \left(L_{s_1,s_1}^+ + L_{s_2,s_2}^+\right) - \frac{4K_f}{n^2} \right) k^2 + O(k^3). \tag{31}$$

Thus, for $k \ll 1$, ρ_{kS_2} can be approximated by $2k + (r_{s_1,s_2} - (L_{s_1,s_1}^+ + L_{s_2,s_2}^+) - (4K_f/n^2))k^2$.

Remark 5: It can be seen in (31) that $\rho_{kS_2} \to 0$ as $k \to 0$. This follows since at $k = 0$ there are no leaders and thus no centrality of leaders. For $k \ll 1$, the approximation $2k + (r_{s_1,s_2} - (L_{s_1,s_1}^+ + L_{s_2,s_2}^+) - (4K_f/n^2))k^2$ of ρ_{kS_2} reveals a tradeoff similar to the tradeoff in the noise-free case. The tradeoff implies that the k-dependent joint centrality is maximized for large resistance distance r_{s_1,s_2} between the two leaders and for large information centrality of each of the two leaders. In the case of a symmetric graph where each node has the same information centrality, the optimal leader set is the one in which the pair has maximum resistance distance.

We prove in the case of the cycle graph, where every node has the same information centrality, that the optimal two noise-corrupted leaders correspond to an antipodal pair of nodes, that is, a pair with maximal resistance distance. This is the same solution as in the case of noise-free leaders on the cycle.

Corollary 4 (Optimal Noise-Corrupted Leader Set on a Cycle, $m = 2$): Let $\mathcal{G}$ be an undirected, unweighted cycle graph of order n where n is even. Let $S_2 = \{s_1, s_2\}$ be a set of two noise-corrupted leaders ($k < \infty$). The optimal leader set S^* is any two nodes with maximal resistance distance $r_{s_1,s_2} = n/4$, which corresponds to geodesic distance $d_{s_1,s_2} = n/2$ and antipodal nodes.

Proof: See Appendix C. ■

To further investigate the role of finite k we computed the optimal noise-corrupted leader set for the path graph of order $n = 51$. For $k = 2$ and higher values, the solution corresponds to the optimal solution in the noise-free case given by Corollary 3, that is, $S^* = \{11, 41\}$. In the case of $k = 0.0001$, the optimal solution is $S^* = \{13, 39\}$, that is, the optimal noise-corrupted leaders are a little closer to the center of the path. The trend persists for larger n. For example, for a path graph of order $n = 101$, for $k = 2$ and higher values, the solution corresponds to the optimal solution in the noise-free case given by Corollary 3, that is, $S^* = \{21, 81\}$, and in the case of $k = 0.0001$, the optimal solution is $S^* = \{25, 77\}$.

E. Optimization Algorithms for Leader Selection

A number of optimization algorithms have been derived in the literature to approximately solve the optimal leader selection problem. In [11] a greedy algorithm was proposed. The authors argued that the greedy algorithm may be too computationally intensive for very large networks, and they derived alternative algorithms that use a bound on the total system error to improve efficiency. These algorithms add a leader to the optimal set one at a time. In [19] convex optimization was used to quantify bounds on performance and an efficient greedy approach was proposed. This algorithm uses a swap procedure to reduce the error associated with choosing one leader at a time. In [18] the total system error was proved to be a supermodular function of the leader set, and this allowed for the development of algorithms that approximate the optimal solution up to a provable bound.

The dependence of the optimal leader set on joint centrality explains how S_m^*, the set of m optimal leaders, is not in general a subset of S_{m+1}^*, the set of $m+1$ optimal leaders. That is, while the total system error is a supermodular function of the leader set, it is not a modular function of the leader set. Thus, any approach that chooses optimal leaders one at a time will in general find only a suboptimal solution. This was illustrated in the example in Section V-A. Likewise, in the case of the cycle graph, since we have shown that the optimal leaders are uniformly distributed around the cycle, a greedy method will give the optimal solution for $m = 2^a$ where $a = 0, 1, 2, 3 \ldots$, but otherwise a suboptimal solution.

The results in the present paper complement the results on optimization algorithms by characterizing the optimal leader set in terms of graph centrality and coverage measures and making it possible in some cases to solve explicitly for the optimal leader set.

F. Connections to Controllability of Networks

The covariance matrix $W_m = \sigma^2 \int_0^\infty e^{-M\tau} e^{-M^T\tau} d\tau$, given by (4) for $t \to \infty$, is the infinite-horizon controllability

Gramian of the n-dimensional state-space system (3). The inputs to the state-space equation are the independent noise terms introduced at each node and the input matrix is σI_n. The total system error (6) that defines performance in this paper is equivalent to the trace of W_m. Thus, the optimal leader selection problem requires to choose the $m < n$ nonzero diagonal elements of K in $M = L + K$ that minimizes the trace of W_m. Although a controllability Gramian determines performance, choosing a set of leaders to minimize the impact of noise on the network coherence is not the same problem as choosing a set of leaders to optimize controllability.

Controllability of networks is studied in the literature, for example, in [34]–[36]. There the problem is to choose a set of $q < n$ input nodes that define the input to the network dynamics, that is, that determine the rank q input matrix B to guarantee or optimize controllability. In [34], the problem is considered with respect to optimization of performance metrics defined in terms of the infinite-horizon controllability Gramian $W_c = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau$, where A defines the zero-input network dynamics. For example, if the q input nodes guarantee controllability, then W_c is invertible and minimizing the trace of W_c^{-1} minimizes the input energy needed for control. Even without controllability, the trace of W_c can be maximized to minimize average input energy. It is shown in [34] that the trace of W_c is a modular function of the set of q input nodes. This implies that the set of q input nodes that maximize the trace of W_c is contained in the set of $q + 1$ input nodes that maximize the trace of W_c. This modularity result does *not* apply to the problem studied in the present paper. Indeed, as discussed in Section V-E the trace of W_m is not a modular function of the m leaders that minimize total system error but rather a supermodular function of these m leaders as proved in [18]. Likewise, as pointed out in [18], an approach that chooses leaders to guarantee controllability, does not address the impact of noise, and deviations in behavior can result even when noise is introduced at a single node.

VI. Joint Centrality and Synthetic Lethality in Saccharomyces Cerevisiae

To further investigate joint centrality of a set of nodes, we apply it in the analysis of synthetically lethal (SL) genes of the functional gene network of Saccharomyces Cerevisiae, also known as baker's yeast. A functional gene network is one in which nodes in the network represent genes and edges between pairs of nodes represent the function or process by which the pair of genes interact. S. Cerevisiae has served as a platform for studying genetics of human diseases and is therefore an important model for biological studies [37]. Here, we focus on instances of synthetic lethality, which occur when the deletion of two genes (A and B) is lethal to the organism and the deletion of A alone or B alone is not lethal.

Using the probabilistic functional gene network of S. Cerevisiae from [37] (5808 genes with 362,421 edges that represent functional couplings), we calculated the two-node joint centrality for every pair of genes in the network. Then we applied experimental interaction data from the BioGrid database to identify SL pairs of nodes [38]. Fig. 2

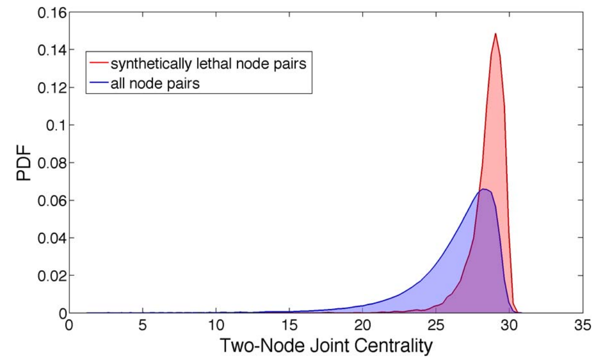

Fig. 2. Distribution of two-node joint centrality for every node pair (blue) in the functional gene network of S. Cerevisiae and distribution of two-node joint centrality of synthetically lethal node pairs (red).

shows the probability distribution function of two-node joint centrality for all pairs of genes (blue) against the probability distribution function of two-node joint centrality for SL pairs of genes (red). The distributions were constructed by fitting non-parametric distributions with a normal kernel function to normalized histograms of joint centrality calculations for all node pairs and for all SL node pairs.

A clear distinction between the two distributions in Fig. 2 is apparent. The distribution of two-node joint centralities for SL node pairs is more highly skewed towards high values of joint centrality than the distribution of two-node joint centralities for all node pairs.

We note that SL pairs of nodes are also distinguishable from all other pairs due to their having a higher average degree. This is expected, however, as there is likely a research bias towards testing high degree nodes for synthetic lethality (the set of SL pairs is not necessarily the complete set but rather the set that has been identified thus far). Accordingly, we do not suggest that joint centrality is the only way to predict possible SL pairs. Instead, we suggest that two-node joint centrality provides a natural measure for predicting SL pairs, because it takes into account the joint influence of a pair of nodes on the entire network. In contrast, a measure of pairwise average degree only considers independent, local interactions.

VII. Final Remarks

In this paper we examine the optimal leader selection problem in a leader-follower network dynamic subject to stochastic disturbances. The objective is for the network to track an external, unknown signal, where leaders can take measurements of the external signal but followers must rely only on their measurements of their neighbors. Performance is defined as the inverse of total steady-state error of the system about an external, unknown signal to be tracked, and the optimal set of m leaders maximizes performance over all possible sets of m leaders.

In contrast to approaches in the literature, which focus on derivation of greedy algorithms, our approach is to derive total system error as a function of a measure of the underlying network graph. To do so we define the joint centrality measure of a set of nodes, such that total system error is inversely proportional to joint centrality. We prove that the optimal leader set corresponds to the set of m leaders with maximal joint centrality. We show that joint centrality of a set of nodes depends directly on the information centrality of each node in the set and resistance distances and biharmonic distances between pairs of nodes in the set, which can be interpreted as a coverage of the graph by the set as a whole. We discuss how the optimal solution is the set of leader nodes that trades off high information centralities of individuals nodes with a good coverage of the graph by the set. We show that joint centrality specializes to information centrality in the case of a single node, and that the optimal leader, with or without noise corruption, is the most information central node.

We solve explicitly for the optimal leader set in the case of the cycle graph and the optimal two-leader set in the case of the path graph. Further, we extend the notion of joint centrality and the optimal leader set to the case of two noise-corrupted leaders. Finally, we provide additional illustration of joint centrality and its more general applicability by using it in the analysis of synthetically lethal gene pairs in a functional gene network. Because joint centrality can be interpreted as a generalization of information centrality, we expect it to prove useful in generalizing to an optimal set of nodes in problems where information centrality distinguishes individual nodes, such as in the case of optimizing the speed-accuracy tradeoff in a network performing distributed hypothesis testing as studied in [32].

Our optimal leader selection results are relevant both to control design, for example, enabling accuracy and efficiency in sensor networks, and to analysis, for example, finding conditions that yield the high performance observed in collective animal behavior. One future direction is to extend the optimal leader selection results of this paper to directed networks by applying the definition in [21], [22] of effective resistance in directed graphs towards a definition of joint centrality in directed graphs. Another compelling future direction is to derive distributed, on-line algorithms that solve the optimal leader selection problem, leveraging our solutions that depend on measures of the graph.

Appendix A
Proof of Theorem 2

Proof: We begin by assuming m nodes on the cycle have been selected as leaders and let $M = L + K$ where K is a matrix with a value of k in the entries along the main diagonal corresponding to the leader nodes and zeros elsewhere. We partition M in the usual way. Since we are assuming noise-free leaders, to compute total system error we need only to consider the sum of the diagonal elements of the inverse of the submatrix M_F. M_F can be written as a block diagonal matrix where each block corresponds to a set of connected follower nodes between two leader nodes. Each block M_{F_i} will itself be a tridiagonal matrix of the form

$$M_{F_i} = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & & \\ 0 & & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}.$$

In the case where there is one follower node in between two leader nodes the corresponding diagonal block in M_F will be one element with an entry of 2.

Similar to previous sections, total system error for noise-free leaders will be proportional to the trace of M_F^{-1}, which here is equivalent to the total sum of eigenvalues of each $M_{F_i}^{-1}$. By [39] we have that the eigenvalues of $M_{F_i}^{-1}$ are

$$\lambda_{z_{ij}} = \frac{1}{2 - 2\cos\left(j\frac{\pi}{w_i+1}\right)} \quad j = 1, \ldots, w_i$$

where w_i is dimension of M_{F_i}. The average value of the eigenvalues of a block is then

$$\bar{\lambda}_{z_i} = \sum_{j=1}^{w_i} \lambda_{z_{ij}} = \frac{1}{6}w_i + \frac{1}{3}.$$

Therefore, minimizing the total sum of eigenvalues is equivalent to minimizing the sum over i of w_i^2. It follows that the minimum is achieved when $w_1 = w_2 = w_3 = \cdots$, or in other words when the dimension of each block is the same. This corresponds to the leaders being evenly distributed around the cycle with shortest distances between leaders equal to $d_{s_1,s_2} = n/2$. ■

Appendix B
Proof of Corollary 3

Proof: Resistance distance in a path graph simplifies to $r_{i,j} = \|i - j\|$ and

$$\begin{aligned} L_{j,j} &= \frac{\sum_{i=1}^n r_{i,j}}{n} - \frac{K_f}{n^2} \\ &= \frac{(n-j)(1+n-j) - j + j^2}{2n} - \frac{K_f}{n^2}. \end{aligned} \tag{32}$$

The substitution of (32) into the expression (26) for ρ_{S_2}, where, without loss of generality, we take $s_2 > s_1$, which gives

$$\rho_{S_2}^{-1} = \frac{1}{n}\left(-\frac{1}{6} + \frac{n + n^2 - s_1 - s_2}{4} + \frac{\left(2s_1^2 + 2s_2^2 - s_2(3n + s_1)\right)}{3} \right). \tag{33}$$

We then take partial derivatives of (33) with respect to s_1 and s_2 to find the minimum of $\rho_{S_2}^{-1}$ to be $s_1 = \mathrm{rnd}((n/5) + (1/2))$ and $s_2 = \mathrm{rnd}((4n/5) + (1/2))$. The rounding of s_1 and s_2 can be checked by observing from (33) that the level sets of $\rho_{S_2}^{-1}$ are ellipses in s_1, s_2. Computing the semiaxis lengths of the ellipses shows that the nearest integer values of s_1 and s_2 that minimize $\rho_{S_2}^{-1}$ indeed determine the optimal leader set. ■

APPENDIX C
PROOF OF COROLLARY 4

Proof: For a circulant graph $L^+_{s_1,s_1} = L^+_{s_2,s_2} = L^+_{s,s}$ and, thus, $\gamma_{s_1,s_2} = (1/4)\sum_{i=1}^n (r_{i,s_1} - r_{i,s_2})^2$. The k-dependent joint centrality ρ_{kS_2} (27) simplifies to

$$\rho_{kS_2} = n\left(\frac{K_f}{n} + \frac{n{L^+_{s,s}}^2 - n{L^+_{s_1,s_2}}^2 - \sum_{i=1}^n (r_{i,s_1} - r_{i,s_2})^2}{4r_{s_1,s_2}}\right)^{-1}.$$

By applying (9) and rearranging terms, we have

$$\rho_{kS_2} = \frac{n^2}{4}\left(\frac{K_f}{n^2} + \frac{2}{k} + 4L^+_{s,s} - r_{s_1,s_2} - \frac{k}{4}\frac{\sum_{i=1}^n (r_{i,s_1} - r_{i,s_2})^2}{2 + kr_{s_1,s_2}}\right)^{-1}. \tag{34}$$

Using the electric circuit analog of resistance distance and applying Kirchhoff's laws, the resistance distance between any two nodes in a cycle can be written as

$$\frac{1}{r_{i,j}} = \frac{1}{d_{i,j}} + \frac{1}{n - d_{i,j}} \tag{35}$$

where $d_{i,j}$ is the geodesic distance between nodes i and j. The maximum resistance distance is $r_{i,j} = n/4$, which is obtained between two nodes with $d_{i,j} = n/2$.

Simplifying the $\sum_{i=1}^n (r^+_{i,s_1} - r^+_{i,s_2})^2$ term of (34) by inserting (35) gives

$$\begin{aligned}\sum_{i=1}^n (r_{i,s_1} - r_{i,s_2})^2 &= \sum_{i=1}^n \left(d_{i,s_1} - d_{i,s_2} + \frac{d^2_{i,s_2} - d^2_{i,s_1}}{n}\right)^2 \\ &= \frac{d_{s_1,s_2}(d_{s_1,s_2} - n)\left(d^2_{s_1,s_2} - nd_{s_1,s_2} - 2\right)}{3n}.\end{aligned} \tag{36}$$

Substituting (36) into (34) results in

$$\begin{aligned}\rho_{kS_2} = \frac{n^2}{4}&\left(\frac{K_f}{n^2} + \frac{2}{k} + 4L^+_{s,s} - \frac{d_{s_1,s_2}(n - d_{s_1,s_2})}{n}\right. \\ &\left. - \frac{kd_{s_1,s_2}(d_{s_1,s_2} - n)\left(d^2_{s_1,s_2} - nd_{s_1,s_2} - 2\right)}{6n\left(2n + kd_{s_1,s_2}(n - d_{s_1,s_2})\right)}\right)^{-1}.\end{aligned} \tag{37}$$

To determine how ρ_{kS_2} changes as a function of d_{s_1,s_2}, we take the partial derivative of (37) with respect to d_{s_1,s_2} to give

$$\begin{aligned}\frac{\partial \rho^{-1}_{kS_2}}{\partial d_{s_1,s_2}} = &-\frac{1}{4}(n - 2d_{s_1,s_2}) \\ &- \frac{nk\left[2\left(-d_{s_1,s_2} + d^3_{s_1,s_2}\right) + \left(1 - 3d^2_{s_1,s_2}\right)n + d_{s_1,s_2}n^2\right]}{3\left(2n + d_{s_1,s_2}k(-d_{s_1,s_2} + n)\right)^2} \\ &- \frac{k^2\left[-2d^5_{s_1,s_2} + 5d^4_{s_1,s_2}n - 4d^3_{s_1,s_2}n^2 + d^2_{s_1,s_2}n^3\right]}{12\left(2n + d_{s_1,s_2}k(-d_{s_1,s_2} + n)\right)^2}.\end{aligned} \tag{38}$$

Since $d_{s_1,s_2} \leq n/2$, the first term of (38) will always be nonpositive. In addition, it can be shown algebraically that for $n > 3$. the two bracketed expressions in the second and third terms will be greater than zero. Therefore. $\rho^{-1}_{kS_2}$ decreases as d_{s_1,s_2} increases, reaching its minimum at the maximal value of $d_{s_1,s_2} = n/2$, corresponding to $r_{s_1,s_2} = n/4$. ■

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