Proceedings of the European Control Conference, Aalborg, Denmark, 2016

Optimal Leader Selection for Controllability and Robustness in Multi-agent Networks

Katherine Fitch and Naomi Ehrich Leonard

***Abstract*— Two optimal leader selection problems are examined for multi-agent networks. The optimal leader set is the set of $m > 0$ leaders that maximizes performance of a linear dynamic network. In the problem for controllability, each leader is identified with a control input, and performance is measured by average controllability and reachable subspace volume. In the problem for robustness, each leader responds to an external signal, the linear dynamics are noisy, and the performance is measured by the steady-state system error. Previously, we showed that the optimal leader set for robustness maximizes a joint centrality in the network graph. In this paper, we show how the optimal leader set for controllability depends also on measures of the graph, including information centrality of leaders and eigenvectors of the graph Laplacian. We explore a fundamental trade-off between optimal leader selection for controllability and for robustness, and we outline a distributed algorithm for the selection of a pair of leaders in trees.**

I. Introduction

The problems of optimally selecting leaders for controllability [1]–[5] and for robustness [6]–[9] in multi-agent networks have both received significant attention in the control literature. This is largely due to the broad applications of the problems, e.g., to robotic formation control [10], consensus [11], and collective animal behavior [12].

The objective in these problems is to choose a set of leaders that maximizes a system performance metric, namely a measure of controllability or robustness of the linear network dynamics. In this paper we focus on solving for optimal leader sets as a function of properties of the network graph. In previous work [6] we proved that the optimal leader set for robustness maximizes the *joint centrality* of the leader set in the network graph, and we showed that joint centrality depends on the information centrality of each of the leaders and a coverage term associated with the leader set. In this paper we find the leader sets that maximize controllability, as defined by two different metrics, and we show how these sets depend on information centrality of individual leaders and the eigenvectors of the graph Laplacian.

Structural controllability in multi-agent systems is the primary focus of [1]–[3], [13]. The authors of [1], and later the authors of [2], examined the graph theoretic characteristics required for a network to be controlled by a single leader. Means for extending these characteristics to the multiple leader setting were introduced in [2] using equitable partitions. Equitable partitions were also studied in [3]. In [14] it was argued that the more important question was whether or not a system is almost uncontrollable. Arbitrarily large graphs were considered in [15], and the fraction and locations of leaders needed for controllability were found for some canonical network topologies.

A related problem is to select leaders to maximize standard measures of controllability, which are defined in terms of the controllability Gramian. In [4] a strategy was developed for selecting leader nodes with performance guarantees relative to the smallest eigenvalues of the controllability Gramian, a measure inversely related to worst case input energy. The largest and smallest eigenvalues of the controllability Gramian were used in [16], where the relationship between energy cost bound and control time was demonstrated.

The trace of the controllability Gramian can be applied as a measure for average controllability as in [17]. In [5] the authors established that the trace of the controllability Gramian is modular while two other energy related control measures are submodular: trace of the inverse Gramian and log product of non-zero eigenvalues of the controllability Gramian, a measure of reachable volume. Control energy centralities were defined for each node in a network based on these measures of controllability.

The robustness of noisy, linear multi-agent dynamics has been studied using the H_2 norm as a measure of the steady-state system error, see, e.g., [18]. [7] introduced the optimal leader selection problem, in which leaders that respond to an external environmental signal are selected to maximize robustness to noise by minimizing the H_2 norm. This problem was studied further in [6], [8], [9]. In [6] we defined and interpreted the joint centrality of a set of nodes and proved its relationship to the optimal leader set for robustness. With the notable exception of [19], the problems of leader selection for controllability and leader selection for robustness have largely been treated as separate problems. However, the combined consideration of controllability and robustness is important for the design of versatile multi-agent systems.

Distributed selection of a single leader has been studied in [20], [21], and distributed algorithms for the calculation of node centrality metrics such as betweeness centrality [22] and harmonic influence centrality have been derived in [23]. However, distributed leader selection algorithms for multiple leaders that take into consideration both controllability and robustness have yet to be developed.

In this paper we first prove the optimal leader set to maximize average controllability and to maximize the reachable

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under grant DGE 1148900, NSF grant ECCS-1135724, ONR grant N00014-14-1-0635, and ARO grant W911NF-14-1-0431.

K. Fitch and N. E. Leonard are with the Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08540, USA.{ kfitch, naomi}@princeton.edu

subspace volume. For average controllability the optimal leader set depends only on information centrality of each leader in the network graph. For the reachable volume the optimal leader set depends on the eigenvectors of the graph Laplacian. We express two of the controllability centrality measures defined in [5] in terms of network graph measures. Second, we show how the optimal leader set for controllability is in tension with the optimal leader set for robustness.We outline a distributed algorithm for the selection of two leaders in tree graphs that trades off the two objectives. The algorithm results in a leader pair with the largest magnitude lower bound on robustness among candidate pairs which meet a minimum threshold on average controllability.

The paper is organized as follows. In Section II we define the optimal leader selection problems for controllability and robustness. We review modularity of controllability metrics and the control energy centralities proposed by [5] in Section III. In Section IV-A we prove the dependence of the trace of the controllability Gramian, and thus the average controllability centrality, on information centrality of the leader nodes. In Section IV-B we prove the dependence of the reachable volume on the leader nodes' entries in the eigenvectors of the graph Laplacian and show that volumetric control energy centrality for node i is dependent only on the ith terms of the Laplacian eigenvectors. In Section IV-C we review optimal leader selection for robustness. In Section V, we discuss the tension between optimal sets for controllability and for robustness, and outline a distributed algorithm for two leader selection in tree networks. We conclude in Section VI.

II. Model and problem definitions

A. Generalized model

We consider a network of n agents for which the state of agent i is $x_i \in \mathbb{R}$ and the state of the network is $\mathbf{x} = [x_1, x_2, ..., x_n] \in \mathbb{R}^n$. The network topology is represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ where each agent is represented by a node in the set $\mathcal{V} = \{1, 2, ..., n\}$. $\mathcal{N}_i$ is the set of neighbors of agent i. $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges in which edge $(i,j) \in \mathcal{E}$ if $j \in \mathcal{N}_i$. $A \in \mathbb{R}^{n \times n}$ is the adjacency matrix where element $a_{i,j}$ is the weight on edge (i,j). If $(i,j) \in \mathcal{E}$ then $a_{i,j} > 0$; otherwise $a_{i,j} = 0$. Here, we focus on connected, undirected graphs and therefore A is symmetric and $a_{i,j} = a_{j,i}$. The degree of node i is $d_i = \sum_{j=1}^{n} a_{i,j}$. The degree matrix, D, is the diagonal matrix of node degrees. The Laplacian matrix is defined as $L = D - A$ and the Moore-Penrose pseudoinverse of L is written as L^+.

We let c_i be the *information centrality* [24] of node i. c_i is related to L^+ by

$$\frac{1}{c_i} = L^+_{i,i} + \frac{K_f}{n^2},$$

where K_f is the Kirchhoff index of $\mathcal{G}$. In contrast with other centrality measures such as closeness and betweeness, information centrality takes into account all paths in the network. Information centrality of node i is closely related to resistance distance, $r_{i,j}$ between nodes i and j in an undirected graph, where

$$r_{i,j} = L^+_{i,i} + L^+_{j,j} - 2L^+_{i,j},$$

and it can be shown that

$$\sum_{i=1}^{n} r_{i,j} = \frac{n}{c_j}.$$

An additional distance measure between two nodes i and j in a graph is biharmonic distance, $\gamma_{i,j}$, where

$$\gamma_{i,j} = L^{2+}_{i,i} + L^{2+}_{j,j} - 2L^{2+}_{i,j} = \sum_{l=1}^{n} (L^+_{l,i} - L^+_{l,j})^2.$$

The dynamics considered in the following are modifications of linear consensus dynamics in which each agent i updates according to its relative state $x_j - x_i$ for $j \in \mathcal{N}_i$,

$$\dot{\mathbf{x}} = -L\mathbf{x}.$$

In the controllability problem, *leaders* are nodes that provide control input. In the robustness problem, leaders are nodes that respond to an external environmental signal. The remaining nodes, *followers*, update their state using only measurements of neighbor states.

B. Optimal leader selection problem for controllability

To investigate optimal leader selection for controllability, we start by assuming that a set S of $m > 0$ nodes are leaders, which act as control inputs to the network system. The network dynamics evolve according to

$$\dot{\mathbf{x}} = -L\mathbf{x} + B\mathbf{u} \tag{1}$$

where $\mathbf{u} \in \mathbb{R}^m$ is the control vector and B has m columns $b\mathbf{e}_i$ for $i \in S$, $\mathbf{e}_i$ are standard basis vectors and $b \in \mathbb{R}$.

Controllability of a consensus network can be defined by restricting dynamics to the disagreement subspace (orthogonal to the agreement subspace defined by the vector of all ones $\mathbf{1}_n \in \mathbb{R}^n$) [4]. Thus, we consider the reduced Laplacian $\bar{L} = QLQ^T$, and controllability Gramian

$$W_C = \int_{t_0}^{\infty} e^{-\bar{L}\tau} QBB^T Q^T e^{-\bar{L}\tau} d\tau \tag{2}$$

where the rows of $Q \in \mathbb{R}^{(n-1)\times n}$ form an orthonormal basis for $\mathbf{1}_n^\perp$. W_C is also the solution to the Lyapunov equation

$$\bar{L}W_C + W_C\bar{L}^T = QBB^TQ^T. \tag{3}$$

We let W_{C_i} be the controllability Gramian associated with one leader node, i. W_{C_i} satisfies (3) when $B = b\mathbf{e}_i$. The following four functions of the controllability Gramian provide four measures of controllability performance.

(a) **Average controllability**: $\text{tr}(W_C)$ provides a measure of average controllability over the controllable subspace and is equivalent to the H_2 norm of the dynamics (1).
(b) **Reachable volume**: $\text{ld}(W_C) = \log\left(\prod_{j=1}^{\text{rank}W_C} \lambda_j(W_C)\right)$ provides a measure of the volume of the controllable subspace reachable with one unit of input. When W_C is full rank $\text{ld}(W_C)$ is equal to the log determinant of W_C.

(c) **Average control energy**: tr$({W_C}^{-1})$ and tr$({W_C}^{+})$ provide measures of average control energy required to reach a random state in the controllable subspace.
(d) **Worst case input energy**: $\lambda_{min}(W_C)$ is inversely proportional to the input energy required to move in the least controllable direction in the controllable subpace.

We define four cases of the *optimal leader selection problem for controllability* as follows.

Problem 1 (Optimal leader selection problem for controllability)**.** *Given $m > 0$ and undirected, connected graph $\mathcal{G}$, find a set of m leaders S_C^* over all possible sets S of m leaders that optimizes a controllability metric $\alpha(W_C)$ for the leader-follower network dynamics (1), where $\alpha(W_C)$ is determined by one of the four performance measures:*

(a) Average controllability: $\alpha(W_C) = \text{tr}(W_C)$
(b) Reachable volume: $\alpha(W_C) = \text{ld}(W_C)$
(c) Average control energy: $\alpha(W_C) = \text{tr}(W_C^+)$
(d) Worst case input energy: $\alpha(W_C) = \lambda_{min}(W_C)$.

C. Optimal leader selection problem for robustness

To study leader selection for robustness, we let the system be subject to stochastic disturbances and the control input be state feedback with $B\mathbf{u} = -K\mathbf{x}$, and $K \in \mathbb{R}^{n\times n}$ a diagonal matrix where $K_{i,i} = k > 0$ if $i \in S$ and zero otherwise. S is the set of m leaders that provide state feedback in response to the environmental signal (taken without loss of generality to have reference value 0). The dynamics are

$$\dot{\mathbf{x}} = -(L + K)\mathbf{x} + \sigma d\mathbf{W} \tag{4}$$

where $\sigma d\mathbf{W}$ is a vector of increments drawn from a standard Wiener process with standard deviation of σ.

We construct the infinite-horizon controllability Gramian W_R for (4). In contrast to the system (1) studied for robustness, the system (4) has state matrix $-(L + K)$ and input defined by independent random perturbations to the state of each node. The Gramian is

$$W_R = \int_{t_0}^{\infty} e^{-(L+K)\tau}\ \sigma^2 I\ e^{-(L+K)\tau} d\tau,$$

which is the solution to the Lyapunov equation

$$(L + K)W_R + W_R(L + K)^T = \sigma^2 I.$$

The trace of W_R is the sum of the steady state variance of each node about the reference value. Equivalently, tr(W_R) is the H_2 norm of the system dynamics (4). Systems with lower tr(W_R) will have states that remain closer to the reference value despite the presence of noise. Thus, tr(W_R) measures robustness of the system (4); it is inversely related to how well the system rejects additive noise.

We define the *optimal leader selection problem for robustness* as follows.

Problem 2 (Optimal leader selection problem for robustness)**.** *Given $m > 0$ and undirected, connected graph $\mathcal{G}$, find a set of m leaders S_R^* over all possible sets S of m leaders that minimizes the H_2 norm, tr(W_R), for the leader-follower network tracking dynamics (4), i.e., find*

$$S_R^* = \arg\min_S \text{tr}(W_R).$$

D. Comparison of W_C and W_R

We observe that even though W_C and W_R are quite similar in formulation, the difference between choosing non-zero elements of B and non-zero diagonal elements of K is quite significant due to the fact that K influences the value of W_R as part of an element in an exponential function, whereas BB^T is simply multiplied by the state transition matrix. Thus, non-zero elements of B will have a different effect on the value of tr(W_C) than non-zero diagonal elements of K will have on the value of tr(W_R). Furthermore, to maximize average controllability one wants to *maximize* tr(W_C); however, to maximize robustness one wants to *minimize* tr(W_R). We demonstrate and discuss the implications of these observations on the resulting optimal leader sets for each problem in Sections IV and V.

III. Control energy centralities

The authors of [5] proved that the trace of the controllability Gramian, tr(W_C), is a modular set function. The implication of a modular set function is that each element of a subset independently contributes to the value of the function. Solving an optimization problem with a modular cost function is straightforward, as the total cost is the sum of each element's independent contribution to the cost function.

Summers, et al. proved that the trace of the (pseudo-) inverse of the controllability Gramian, tr$({W_C}^{+})$, and the log determinant of the controllability Gramian, ld(W_C), are both submodular functions of the leader set. In [8], it was shown that the trace of the robustness Gramian, tr(W_R), is also a submodular function of the leader set. A submodular set function has the property of diminishing returns, that is the addition of an element to a larger set has a smaller contribution than the addition of an element to a smaller set. Therefore, each element of a subset does not contribute independently as in modular set functions. Full solutions to opimization problems with nondecreasing submodular set functions are NP-hard, although greedy algorithms can provide a solution within a provable bound from the optimal solution [25]. Thus, a closed-form solution for maximizing tr(W_C) is obtainable, while optimizing tr$({W_C}^{+})$, ld(W_C), and tr(W_R) are each combinatorially difficult problems.

In [5], the authors defined three *control energy centralities* for each node i in a network based on the value of controllability measures (a)-(c) when i is selected as a single leader node. These control energy centralities are

- *Average controllability centrality*

$$C_{AC}(i) = \text{tr}(W_{c_i}) \quad i \in V$$

- *Average control energy centrality*

$$C_{ACE}(i) = -\text{tr}(W_{c_i}^+) \quad i \in V$$

- *Volumetric control energy centrality*

$$C_{VCE}(i) = \log\left(\prod_{j=1}^{\text{rank}W_{c_i}} \lambda_j(W_{c_i})\right) \quad i \in V.$$

The authors of [5] did not provide relationships between these centrality definitions and well defined measures of the network graph nor insight more generally on how a node's location in a network relates to the value of its three control energy centralities. For single leaders, the solutions to maximizing tr(W_C), and ld(W_C) and minimizing tr($W_C{}^+$) will align with the nodes that maximize the respective control energy centralities. In the following section, we make explicit the relationship between average controllability centrality C_{AC}, volumetric control energy centrality C_{VCE} and properties of the graph Laplacian L.

IV. Optimal leader selection results

A. Optimal leader selection for average controllability

The following theorem provides the optimal leader set S_C^* for Problem 1(a) in terms of properties of the network graph.

Theorem 1. *Consider the dynamics (1) with the undirected, connected graph $\mathcal{G}$ of order n. Let the set S be a set of m leaders. Then average controllability depends on the inverse of the information centrality of each node in S, and the optimal leader set S_C^* is composed of the m nodes with smallest information centrality.*

Proof. Consider the controllability Gramian W_C given by (2). We note that $\bar{L}$ has the same eigenvalues as L except for the zero eigenvalue, which we index by n. Let the diagonal matrix of eigenvalues and the matrix of right eigenvectors for $\bar{L}$ and L be $\bar{\Lambda}$, $\bar{V}$ and Λ, V, respectively. Then,

$$W_C = \bar{V}\Big(\int_{t_0}^{\infty} e^{-\bar{\Lambda}\tau}\bar{V}^T QBB^T Q^T \bar{V} e^{-\bar{\Lambda}\tau} d\tau\Big)\bar{V}^T d\tau. \quad (5)$$

Consider the case of a single controller node, indexed by l. Then, the vector B will have a single non-zero entry, b.

Since $Q^T\bar{V}$ is equivalent to the first $n-1$ columns of V, we can represent the product in the integral of (5) as a function of the eigenvalues and eigenvectors of L. Then $W_C = b^2\bar{V}G\bar{V}^T$, where $G \in \mathbb{R}^{(n-1)\times(n-1)}$ has entries

$$g_{i,j} = \int_0^{\infty} e^{-\lambda_i\tau-\lambda_j\tau} v_{l,i}v_{l,j} d\tau = \frac{1}{\lambda_i+\lambda_j} v_{l,i}v_{l,j}. \quad (6)$$

Recall that we are interested in maximizing the trace of W_C and that trace is invariant under cyclic permutations. Thus

$$\begin{aligned} \text{tr}(W_C) &= b^2\text{tr}(\bar{V}G\bar{V}^T) = b^2\text{tr}(G\bar{V}^T\bar{V}) = b^2\text{tr}(G) \\ &= \sum_{i}^{n-1} \frac{b^2}{2\lambda_i} v_{l,i}^2 = \frac{b^2}{2} L_{l,l}^+ = \frac{b^2}{2}\left(\frac{1}{c_l} - \frac{K_f}{n^2}\right) \end{aligned} \quad (7)$$

where c_l is the information centrality of node l. From (7), for a single controller node, tr(W_C) is maximized by the node with the smallest information centrality.

Due to the modularity property, tr(W_C) with m leader nodes will be minimized when the set of leaders S_C^* consists of the m nodes with smallest information centrality. □

Corollary 1. *Consider the dynamics (1) with the undirected, connected graph $\mathcal{G}$ of order n. Then*

$$C_{AC}(i) = \text{tr}(W_{c_i}) = \frac{1}{2}\Big(\frac{1}{c_i} - \frac{K_f}{n^2}\Big).$$

We have thus shown that average controllability centrality defined by [5] is in fact inversely related to a well defined graph measure: information centrality. Corollary 1 implies that the more information central is a leader node the lower will be the average controllability.

B. Optimal leader selection for reachable subspace volume

The following theorem provides the optimal leader set S_C^* for Problem 1(b) in terms of properties of the network graph.

Theorem 2. *Consider the dynamics (1) with the undirected, connected graph $\mathcal{G}$ of order n. Let the set S be a set of m leaders. Then reachable volume can be written as*

$$\text{ld}(W_C) = \log\Big(\prod_{j=1}^{n-1}\Big(\sum_{i\in S} v_{i,j}^2\Big)\Big) + h$$

where $v_{j,i}$ is the ith entry in the jth right eigenvector of L and h is a constant that does not depend on leader set S. The optimal leader set is $S_C^* = \arg\max_S(\prod_{j=1}^{n-1}\Big(\sum_{i\in S} v_{i,j}^2\Big)$.

Proof. Using (5), the determinant of W_C is

$$\det(W_C) = b^2\det(\bar{V}G\bar{V}) = b^2\det(\bar{V})\det(G)\det(\bar{V}). \quad (8)$$

From (6), $G = \tilde{V}\Gamma\tilde{V}$ where $\tilde{V}, \Gamma \in \mathbb{R}^{(n-1)\times(n-1)}$. $\tilde{V}$ is a diagonal matrix with $\tilde{V}_{i,i} = v_{l,i}$ and the entries of Γ are $\Gamma_{i,j} = \frac{1}{\lambda_i+\lambda_j}$. Plugging in to (8) gives

$$\det(W_C) = b^2\det(\bar{V})\det(\tilde{V})\det(\Gamma)\det(\tilde{V})\det(\bar{V}). \quad (9)$$

The only term in (9) that depends on the choice of leader node is $\det(\tilde{V})^2$. Since $\tilde{V}$ is diagonal, its determinant is the product of its diagonal entries. Thus for a single leader l

$$\text{ld}(W_C) = \log\prod_{j=1}^{n-1} v_{l,j}^2 + h.$$

In the case of m leaders, $\tilde{V}_{j,j} = \sum_{i\in S_c} v_{i,j}$ and

$$\text{ld}(W_C) = \log\prod_{j=1}^{n-1}\Big(\sum_{i\in S_c} v_{i,j}^2\Big) + h.$$

It follows that the set S_C^* of m leaders that maximizes $\prod_{j=1}^{n-1}\Big(\sum_{i\in S} v_{i,j}^2\Big)$ maximizes ld(W_C), the volume of the controllable subspace reachable with one unit of input. □

Corollary 2. *Consider the dynamics (1) with the undirected, connected graph $\mathcal{G}$ of order n. Let $Y = \det(\bar{V}\Gamma\bar{V})$. Then*

$$C_{VCE}(i) = \log\Big(Y\prod_{j=1}^{n-1} v_{i,j}^2\Big).$$

Theorem 2 and Corollary 2 show the dependence of ld(W_C) and volumetric control energy centrality on leader nodes' entries of the eigenvectors of the graph Laplacian.

We point out that ranking nodes by volumetric control energy centrality computed as $\prod_{j=1}^{n-1} v_{i,j}^2$ for each node i is significantly less computationally intensive than ranking through a calculation of the controllability Gramian and its determinant for each node i.

C. Optimal leader selection for robustness

We review leader selection for robustness by presenting a definition and theorem from [6].

Definition 1 (Joint centrality)**.** *Let $\mathcal{G}$ be an undirected, connected graph of order n. Given integer $m < n$, let S be the set of any m nodes in $\mathcal{G}$. The* joint centrality of set S *in $\mathcal{G}$ is defined as*

$$\rho_S = n\Big(\frac{K_f}{n} + nh_1(c_S, r_S) + \frac{1}{2}h_2(c_S, r_S, \gamma_S)\Big)^{-1}, \quad (10)$$

where h_1 is a function of the information centralities and resistance distances of the leader set and h_2 is a function of the information centralities, resistance distances and biharmonic distances of the leader set.

Theorem 3. *Consider the dynamics (4) with the undirected, connected graph $\mathcal{G}$ of order n. Let S be a set of m noise-free leaders. Then, tr(W_R) for the system dynamics (4) is*

$$tr(W_R) = \frac{1}{2}\Big(\frac{n}{\rho_S}\Big),$$

where ρ_S is the joint centrality of leader set S given by (10). The optimal leader set is $S_R^* = \arg\max_S \rho_S$, the set of leader nodes with the maximal joint centrality.

Proof. See [6]. □

From [6], it is known that h_1 will be optimized by a leader set with both high information centralities and high resistance distances among leaders in S. h_2 will be optimized by a leader set that trades off high biharmonic distances and information centralities. Therefore, node sets with high joint centrality will trade off information centralities and distribution over the graph. Joint centrality is a rigorous representation of the effects on robustness of the centrality and coverage over a network of a set of nodes.

V. Controllability versus robustness

Combining the results from Sections IV-A and IV-C we find a fundamental trade-off between selecting leaders for average controllability and selecting leaders for robustness. To maximize average controllability one simply selects the least information central nodes as leaders. Since the average controllability problem is modular, the solution does not depend on the relative positions of the leader nodes. Conversely, the problems of leader selection to maximize the volume reachable with one unit of energy and the problem of leader selection for robustness are both submodular; therefore the relative positions of nodes in the leader sets play a role in the optimal solutions. To maximize robustness the leader set must balance high information centrality of individual leader nodes with distribution of leader nodes over the network. Nodes with low information centralities lead to a leader set with high average controllability but often with low robustness. Therefore, robustness and average controllability cannot both be optimized by the same leader set in general graphs where all nodes do not have equivalent information centralities.

A. Optimal leader selection in a cycle graph

Consider a cycle graph with n nodes and uniform edge weights. The value of average controllability over the controllable subspace will be the same no matter which m nodes are leaders because all nodes have equivalent information centralities. For the robustness problem, in [6] it was proven that the optimal leader selection for robustness in a cycle graph corresponds to m nodes evenly distributed about the cycle. The optimal set of $m = 2$ leaders for reachable volume corresponds to any two nodes separated by a single node. Therefore the optimal leader set for reachable volume and the optimal leader set for robustness are in direct tension and cannot be simultaneously selected.

B. Optimal leader selection in a random graph

Consider a random unweighted network with $n = 100$ nodes as in Fig. 1, and the $m = 3$ nodes that optimize the leader selection Problems 1(a), 1(b), 1(c) and 2. The leader set that maximizes average controllability is colored in green, maximizes reachable volume in orange, minimizes average control energy in red, and maximizes robustness in blue. The direct tension between optimizing the leader set for controllability metrics and optimizing the leader set for robustness is visually apparent. The optimal leader sets for maximizing average controllability, maximizing reachable volume, and minimizing average input energy are on the periphery of the network, and it is interesting to note that one node in particular is a member of all three sets. The nodes in the optimal leader set for robustness are in central, but distributed, locations. Furthermore, nodes in the optimal leader set for robustness are less susceptible to becoming disconnected from the network through edge failures. A trade-off must be made if both controllability and robustness are important for the multi-agent system.

The controllability Gramian measures are defined over the controllable subspace and none guarantee full controllability. It is possible that a leader set that maximizes tr(W_C) corresponds to few highly controllable nodes or many weakly controllable nodes. It may therefore be of interest to account also for the rank of the controllability Gramian.

C. Design of a distributed leader selection algorithm

An important application of the results above is in the design of distributed leader selection algorithms that balance controllability and robustness. Using the results of Section IV we have developed a distributed algorithm for the selection of two leaders in a tree graph that converges in finite time to the pair that yields the highest lower bound on robustness among candidate pairs that satisfy a minimum threshold on average controllability.

The outline of the algorithm is as follows. Each leaf i in the tree (nodes with degree one) uses a message passing

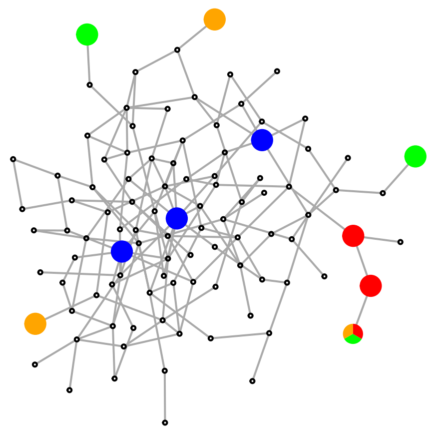

Fig. 1. Random undirected graph with $n = 100$ nodes highlighting optimal leader sets of $m = 3$ nodes for problems 1a (average controllability: green), 1b (reachable volume: orange), 1c (average control energy: red) and 2 (robustness: blue).

protocol to determine its own information centrality as well as the information centrality and resistance distance to the leaf j that is the farthest in terms of resistance distance. Then each leaf i calculates average controllability for leader pair (i, j) and passes information to its neighbor when certain criteria on average controllability are met.

The step repeats with neighbor nodes and then their neighbors. The process for leaf i is complete when information is passed to a *stopping node*. Let $a \in \mathcal{N}_i$, $p \in \mathcal{N}_a$ and q be a node along the path from i to j. Then node a will be a stopping node, and won't pass on information to node p, if its estimate of average controllability for leader pair (p, q) is lower than the acceptable limit for average controllability or when (p, q) does not improve robustness relative to (a, q). When the process is complete for each leaf node, each stopping node calculates and broadcasts its lower bound on robustness for its candidate leader pair. The candidate pair with the largest lower bound on robustness becomes the leader pair. Details will appear in a future publication.

VI. Final Remarks

We have examined and provided new insights on the optimal leader selection problem for leader-follower multi-agent systems. We proved that the optimal leader set for average controllability consists of the least information central nodes in the network. We proved the relationship between the optimal leader set for reachable volume and entries in the eigenvectors of the network graph Laplacian. From these we derived expressions for average controllability centrality and volumetric control energy centrality in terms of well defined graph measures. We showed how the optimal leader sets for controllability metrics are in tension with the optimal leader set for robustness, and thus require a trade-off if both features are desirable. Finally, we outlined a distributed algorithm for leader selection that takes into account this trade-off.

Future directions include characterizing $\text{tr}(W_C^+)$ and $\lambda_{min}(W_C)$ in terms of properties of the graph and expanding the distributed leader selection algorithm to accommodate sets of more than two leaders.

References

[1] H. G. Tanner, "On the controllability of nearest neighbor interconnections," in *Proc. IEEE CDC*, 2004, pp. 2467–2472.

[2] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 162–186, 2009.

[3] S. Martini, M. Egerstedt, and A. Bicchi, "Controllability analysis of multi-agent systems using relaxed equitable partitions," *Int. J. Systems, Contr and Comm.*, vol. 2, pp. 100–121, 2010.

[4] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," *IEEE TCNS*, vol. 1, no. 1, pp. 3287–3292, 2014.

[5] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On submodularity and controllability in complex dynamical networks," *arXiv:1404.7665v2*, 2015.

[6] K. Fitch and N. E. Leonard, "Joint centrality distinguishes optimal leaders in noisy networks," *IEEE Trans. Cont. Network Sys.*, 2016.

[7] S. Patterson and B. Bamieh, "Leader selection for optimal network coherence," in *Proc. IEEE CDC*, 2010, pp. 2693–2697.

[8] A. Clark, L. Bushnell, and R. Poovendran, "A supermodular optimization framework for leader selection under link noise in linear multi-agent systems," *IEEE Trans. Autom. Control*, vol. 59, pp. 283 – 296, 2014.

[9] F. Lin, M. Fardad, and M. R. Jovanovic, "Algorithms for leader selection in stochastically forced consensus networks," *IEEE Trans. Autom. Control*, vol. 59, pp. 1789–1802, 2014.

[10] W. Ren, "Multi-vehicle consensus with a time-varying reference state," *Syst. Control Lett.*, vol. 56, no. 7, pp. 474–483, 2007.

[11] R. Olfati-Saber and J. S. Shamma, "Consensus filters for sensor networks and distributed sensor fusion," in *Proc. IEEE CDC*, 2005, pp. 6698 – 6703.

[12] G. F. Young, L. Scardovi, A. Cavagna, I. Giardina, and N. E. Leonard, "Starling flock networks manage uncertainty in consensus at low cost," *PLoS Comput. Biol.*, vol. 9, pp. 1–7, 2013.

[13] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, pp. 167–173, 2011.

[14] N. J. Cowan, E. J. Chastain, D. A. Vilhena, J. S. Freudenberg, and C. T. Bergstrom, "Nodal dynamics, not degree distributions, determine the structural controllability of complex networks," *PLoS One*, vol. 10.1371/journal.pone.0038398, 2012.

[15] C. Enyioha, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, "Controllability and fraction of leaders in infinite networks," in *Proc. IEEE CDC*, 2014, pp. 1359–1364.

[16] G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and Baowen, "Controlling complex networks: How much energy is needed?" *PRL*, vol. 108, pp. 218 703–1 – 218 703–5, 2012.

[17] B. Marx, D. Koenig, and D. George, "Optimal sensor and actuator location for descriptor systems using generalized gramians and balanced realizations," in *Proc. ACC*, 2004, pp. 2729–2734.

[18] G. F. Young, L. Scardovi, and N. E. Leonard, "Rearranging trees for robust consensus," in *Proc. IEEE CDC*, 2011, pp. 1000–1005.

[19] A. Clark, L. Bushnell, and R. Poovendran, "On leader selection for performance and controllability in multi-agent systems," in *Proc. IEEE CDC*, 2012, pp. 86–93.

[20] S. Vasudevan, J. Kurose, and D. Towsley, "Design and analysis of a leader election algorithm for mobile ad hoc networks," in *Proc IEEE ICNP*, 2004, pp. 350 – 360.

[21] N. Malpani, J. L. Welch, and N. Vaidya, "Leader election algorithms for mobile ad hoc networks leader election algorithms for mobile ad hoc networks leader election algorithms for mobile as hoc networks," in *Proc. DIALM*, 2000, pp. 96–103.

[22] W. Wang and C. Y. Tang, "Distributed computation of node and edge betweeness on tree graphs," in *Proc. IEEE CDC*, 2013, pp. 43–48.

[23] L. Vassio, F. Fagnami, P. Frasca, and A. Ozdaglar, "Message passing optimization of harmonic influence centrality," *IEEE Trans. Cont. Network Sys.*, vol. 1, no. 1, pp. 109–120, 2014.

[24] K. Stephenson and M. Zelen, "Rethinking centrality: Methods and examples," *Social Networks*, vol. 11, pp. 1–37, 1989.

[25] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher, "An analysis of approximations for maximizing submodular set functions–I," *Math. Program.*, vol. 14, pp. 265–294, 1978.