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Stability of a Bottom-heavy Underwater Vehicle*

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Conditions for stability of underwater vehicle dynamics are established as a function of vehicle shape, mass distribution and equilibrium velocity using methods from geometric mechanics.

Key Words—Stability; dynamic stability; underwater vehicle dynamics; marine systems; stabilization.

Abstract—We study stability of underwater vehicle dynamics for a six-degree-of-freedom vehicle modeled as a neutrally buoyant, submerged rigid body in an ideal fluid. We consider the case in which the center of gravity and the center of buoyancy of the vehicle are noncoincident such that gravity introduces an orientation-dependent moment. Noting that Kirchhoff's equations of motion for a submerged rigid body are Hamiltonian with respect to a Lie–Poisson structure, we derive the Lie–Poisson structure for the underwater vehicle dynamics with noncoincident centers of gravity and buoyancy. Using the energy–Casimir method, we then derive conditions for Lyapunov stability of relative equilibria, i.e. stability of motions corresponding to constant translations and rotations. The conditions reveal for the vehicle stability problem the relevant design parameters, which in some cases can be interpreted as control parameters. Further, the formulation provides a setting for exploring the stabilizing and destabilizing effects of dissipation and externally applied control forces and torques. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The study of submerged bodies has a long history, but is receiving renewed attention in light of challenging, new problems in underwater motion control that are emerging from a growing industry in underwater vehicles for deep sea exploration. This type of vehicle, often referred to as a submersible to distinguish it from a submarine, is typically small and versatile, with performance requirements, design capabilities and even physical configuration differing from those of a traditional submarine. Further, many of the currently active underwater vehicles are remotely operated, and there is a more recent trend to make them completely autonomous.

In order for unmanned underwater vehicles to succeed, they must be able to control their own motion reliably and efficiently. Because underwater vehicle models are nonlinear, it is promising to explore the use of modern control techniques that exploit system nonlinearities. For example, in the event of an actuator failure on board an unmanned underwater vehicle, nonlinear control methods can be used for motion control where techniques based on linearization would fail to be useful (Leonard, 1995a,b). Other nonlinear control methods have also been explored for adaptive and robust control of underwater vehicle motion (Cristi *et al.*, 1990; Fjellstad and Fossen, 1994; Yoerger and Slotine, 1985). Further, nonlinear methods have been used successfully for the related problem of attitude control and stabilization of a rigid spacecraft; see Tsiotras *et al.* (1995) for a survey.

In this paper we use a modern geometric framework to investigate the stabilizing effect of gravity on the motion of a six-degree-of-freedom, submerged rigid body in an ideal fluid. Fjellstad and Fossen (1994) also consider a six-degree-of-freedom vehicle model; however, this is a departure from a typical assumption that dynamics in the horizontal plane are decoupled from the remaining degrees of freedom. In order to address problems such as actuator failures, it is desirable to consider the coupled six-degree-of-freedom model, since we may need to take advantage of the coupling as means of compensating for the failures.

We model the underwater vehicle as a neutrally buoyant rigid body in an ideal fluid, so that Kirchhoff's equations of motion, based on potential flow theory, provide a dynamic model of the body. In the case that there are no external forces or torques acting on the body, these equations are Hamiltonian with respect to a Lie–Poisson structure (Arnold *et al.*, 1993).

We consider the case in which the center of buoyancy and center of gravity of the body are noncoincident, resulting in an orientation-

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dependent moment acting on the body. This is of practical interest, since underwater vehicles are built with the center of gravity below the center of buoyancy (i.e. bottom-heavy) for stability. We show that in this case the equations of motion are still Hamiltonian, and we derive the associated Lie–Poisson structure. The Lie–Poisson structure is determined by a reduction process that exploits symmetry in the system. Here symmetry corresponds to invariance of the equations of motion under reference coordinate frame translation in any direction and rotation about the direction of gravity.

The advantage of describing the equations of motion in a geometric framework is the availability of a variety of analytical tools that make use of the geometric structure as a means of determining nonlinear systems behavior. For example, in order to study nonlinear (Lyapunov) stability of motions of the body, we use the energy–Casimir method described in Marsden and Ratiu (1994). The energy–Casimir method provides a means to derive sufficient conditions for nonlinear stability of an equilibrium solution. Here equilibrium solutions, or relative equilibria, correspond to constant translations and rotations of the vehicle.

The results of our analysis provide conditions for stability that depend on the physical configuration of the vehicle, the equilibrium velocity of the vehicle, and the distance between the center of gravity and the center of buoyancy. These describe the relevant design parameters, which in some cases can be considered as control parameters. For example, suppose the vehicle needs to suddenly move quickly away from a particular location and suppose the distance between the centers of gravity and buoyancy is not sufficiently large to ensure stability. By moving masses around on board the vehicle, one could control this length on-line, potentially stabilizing the motion.

The results of this study are practically valid only in those circumstances in which potential flow theory is a sufficiently accurate model of the vehicle dynamics. This will be the case for streamlined motions, e.g. when an ellipsoidal vehicle is translating along (and/or rotating about) its major axis. In this case viscous effects are small and our ideal fluid model captures the dominating dynamic effects. In the case of non-streamlined motions, e.g. motion along the body's minor axis, the potential flow model does not capture effects that may be important, e.g. vortex shedding. Nonetheless, we note that it is actually the streamlined motions that are of most interest to us. First, these are the motions that we shall want the vehicle to perform, since, because resistance is minimal, they can be accomplished most efficiently. Secondly, as we show in this paper, these are the motions that are naturally unstable and require help from spin, gravity or external forces and torques in order to be stabilized, i.e. these are the motions that provide a need for nonlinear methods.

The results of this paper constitute a first step in laying the groundwork for addressing our larger goal of deriving control strategies for stabilization and motion control of underwater vehicles, including those that have a limited number of actuators. The Hamiltonian structure derived and used here provides a means of studying stability of submerged rigid-body dynamics in which we focus on the effect of hydrodynamics and gravity. It further provides a starting point to explore the larger picture that includes dissipation and external forces and torques. We note in this regard that there are recent relevant results on the effect of adding a small amount of dissipation to a Hamiltonian system (Bloch *et al.*, 1994) and on the use of the energy–Casimir method in deriving stabilizing controls for a rigid spacecraft (Bloch *et al.*, 1992). New results on feedback stabilization of underwater vehicle dynamics using the Hamiltonian structure and stability conditions derived here can be found in Leonard (1996b).

Kozlov (1989) has studied the stability and motion of a submerged rigid body that is not neutrally buoyant but falls as a result of a gravitational force that is larger than the buoyant force. In this work the motion of the body is restricted to the vertical plane, i.e. the six-degree-of-freedom problem is not considered.

This paper is organized as follows. In Section 2 we derive the equations of motion for the underwater vehicle with noncoincident centers using Newton–Euler balance laws, and use reduction to derive the associated Lie–Poisson structure. In Section 3 we apply the energy–Casimir method to study stability of equilibrium solutions. We apply the method first to the case of the body with coincident centers. This is the classical problem studied by Lamb (1932), for which he derived conditions for spectral stability using linearization. Using the energy–Casimir method, we prove conditions for nonlinear stability that are stronger than Lamb's result. Next, we apply the energy–Casimir method to the case with noncoincident centers to reveal conditions for the stabilizing effect of gravity. In Section 4 we summarize our results and discuss future work. For additional details on the work described in this paper see Leonard (1996a).

2. EQUATIONS OF MOTION

2.1. *Kinematics*

Following the development in Leonard (1995a), we identify the position and orientation of the underwater vehicle with SE(3), the group of rigid-body motions in $\mathbb{R}^3$. An element $X \in SE(3)$ is given by $X = (R, b)$, where $R \in SO(3)$ is a rotation matrix that describes the orientation of the vehicle and $b \in \mathbb{R}^3$ is a vector that describes the position of the vehicle. That is, if x_b is any point on the vehicle described with respect to a body-fixed coordinate frame and x_r is the same point expressed with respect to the inertial coordinate frame, then

$$\begin{pmatrix} x_r \\ 1 \end{pmatrix} = \begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_b \\ 1 \end{pmatrix} = \begin{pmatrix} Rx_b + b \\ 1 \end{pmatrix}.$$

Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ be the angular velocity of the vehicle in body-fixed coordinates. Let $v = (v_1, v_2, v_3)^T$ be the translational velocity of the origin of the body frame given in terms of body-fixed coordinates. Define $\hat{\cdot}: \mathbb{R}^3 \to \mathfrak{so}(3)$, where $\mathfrak{so}(3)$ is the space of 3×3 skew-symmetric matrices, by $\hat{\alpha}\beta = \alpha \times \beta$ for $\alpha, \beta \in \mathbb{R}^3$. Then, R and b satisfy

$$\begin{aligned} \dot{R} &= R\hat{\Omega}, \\ \dot{b} &= Rv. \end{aligned} \tag{1}$$

The pair $(\hat{\Omega}, v)$ is an element in the vector space $\mathfrak{se}(3)$ which is the space of infinitesimal rotations and translations in $\mathbb{R}^3$. This space becomes a Lie algebra on defining the Lie bracket on $\mathfrak{se}(3)$ as

$$[\cdot, \cdot]: \mathfrak{se}(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3),$$
$$[(\hat{\alpha}_i, \beta_i), (\hat{\alpha}_j, \beta_j)] \mapsto (\hat{\alpha}_i\hat{\alpha}_j - \hat{\alpha}_j\hat{\alpha}_i, \hat{\alpha}_i\beta_j - \hat{\alpha}_j\beta_i).$$

The Lie bracket is a formal way of describing how rotations and translations commute. Given $\{A_1, \ldots, A_6\}$, a basis for $\mathfrak{se}(3)$, we define the structure constants Γ_{ij}^k for this basis as

$$[A_i, A_j] = \sum_{k=1}^{6} \Gamma_{ij}^k A_k.$$

2.2. *Newton–Euler description of dynamics*

In this section we derive the dynamic equations of motion using Newton–Euler balance laws. The derivation follows Lamb (1932), with additional reference to Fossen (1994). We assume that the vehicle is submerged in an infinitely large volume of incompressible, irrotational and inviscid fluid at rest at infinity, where the motion of the fluid can be characterized by a single-valued velocity potential. We fix the orthonormal coordinate frame $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ to the vehicle with origin located at the vehicle's center of buoyancy. Then, as Kirchhoff showed, the kinetic energy of the fluid T_f is given as a quadratic function of body velocity. Let $W = [v^T \ \Omega^T]^T$; then

$$T_f = \tfrac{1}{2}W^T\Theta^f W, \quad \Theta^f = \begin{pmatrix} \Theta_{11}^f & \Theta_{12}^f \\ \Theta_{21}^f & \Theta_{22}^f \end{pmatrix},$$

where Θ^f is a 6×6 constant, symmetric matrix with elements determined by the configuration of the vehicle and the density of the fluid. Θ^f is diagonal in the case of a vehicle that has three mutually perpendicular planes of symmetry, e.g. a vehicle that can be approximated as an ellipsoid, if we set the body-fixed axes to be the principal axes of the displaced fluid.

Similarly, the kinetic energy of the vehicle alone, T_b, can be expressed as

$$T_b = \tfrac{1}{2}W^T\Theta^b W, \quad \Theta^b = \begin{pmatrix} mI & -m\hat{r}_G \\ m\hat{r}_G & J_b \end{pmatrix},$$

where m is the mass of the vehicle, I is the 3×3 identity matrix, J_b is the inertia matrix of the vehicle and r_G is the vector from the origin of the body-fixed frame to the vehicle center of gravity. The total kinetic energy of the body–fluid system $T = T_f + T_b$ is

$$\begin{aligned} T &= \tfrac{1}{2}W^T(\Theta^b + \Theta^f)W \\ &= \tfrac{1}{2}(\Omega^T J\Omega + 2\Omega^T Dv + v^T Mv), \end{aligned} \tag{2}$$

where

$$M = mI + \Theta_{11}^f, \quad J = J_b + \Theta_{22}^f, \quad D = m\hat{r}_G + \Theta_{21}^f.$$

The matrices Θ_{11}^f and Θ_{22}^f are sometimes referred to as *added mass* and *added inertia* matrices respectively. In Appendix B we provide formulas for the elements of these matrices.

The *impulse* of the body–fluid system, as defined by Lord Kelvin, is that required to counteract the impulsive pressures acting on the surface of the vehicle and to generate the momentum of the vehicle itself. While this impulse is not equivalent to the total momentum of the system (roughly speaking, it excludes the infinite term in the momentum), it can be shown to vary as the momentum of a finite dynamical system would vary under the influence of external forces and torques.

Let p and π be the linear and angular components of the impulse respectively with respect to the inertial orthonormal coordinate frame. Let P and Π be the analogous components given with respect to the body-fixed frame. Then

$$p = RP, \tag{3}$$
$$\pi = R\Pi + b \times p. \tag{4}$$

Let f_i, $i = 1, \ldots, k$, be the external forces applied to the vehicle given in inertial coordinates, and let ρ_i be the vector from the origin of the inertial frame to the line of action of the force f_i, $i = 1, \dots, k$. Let $f = \sum_{i=1}^{k} f_i$, and let $F = R^{\mathrm{T}} f$, i.e. the applied forces given in body coordinates. Similarly, let τ be the net external torque applied to the vehicle in inertial coordinates and $\mathcal{T} = R^{\mathrm{T}}\tau$. Then

$$\dot{p} = f, \tag{5}$$

$$\dot{\pi} = \tau + \sum_{i=1}^{k} \rho_i \times f_i. \tag{6}$$

Differentiating (3) and (4) with respect to time and using (1), (5) and (6), we find that

$$\dot{P} = P \times \Omega + F, \tag{7}$$

$$\dot{\Pi} = \Pi \times \Omega + P \times v + \sum_{i=1}^{k} (R^{\mathrm{T}}(\rho_i - b)) \times R^{\mathrm{T}} f_i + \mathcal{T}. \tag{8}$$

P and Π can be computed from the total energy T given by (2) as

$$P = \frac{\partial T}{\partial v} = Mv + D^{\mathrm{T}}\Omega, \tag{9}$$

$$\Pi = \frac{\partial T}{\partial \Omega} = J\Omega + Dv. \tag{10}$$

Using (9) and (10) for P and Π in (7) and (8) gives Kirchhoff's equations of motion for a rigid body in an ideal fluid.

It is interesting to note that, because of the fluid, the mass matrix M is not a multiple of the identity (unless the body is symmetric, as in the case of a sphere). As a result, Mv is generically not parallel to v, so that the term $Mv \times v$ in (8) is nontrivial.

In this paper we study a neutrally buoyant, ellipsoidal vehicle and assume that the principal axes of the displaced fluid are the same as the principal axes of the body. Neutral buoyancy means that the buoyant force is exactly equal and opposite to the gravitational force. Our assumptions imply that $D = m\hat{r}_G$ and J and M are each diagonal.

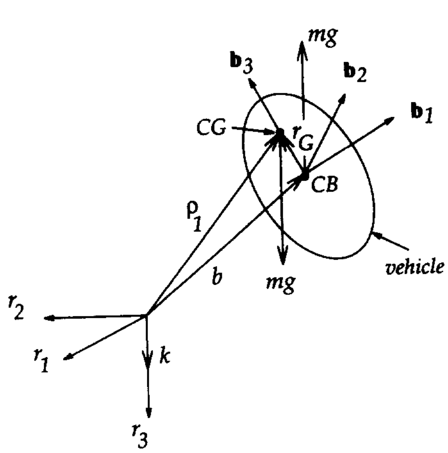

Fig. 1. Vehicle with noncoincident centers of gravity and buoyancy.

In the special case that the center of buoyancy is coincident with the center of gravity, i.e. $r_G = 0$, $D = 0$, so that

$$P = Mv, \quad \Pi = J\Omega,$$

and gravity does not appear in the dynamic equations of motion.

In the more general case, we let the centers of buoyancy and gravity be noncoincident as illustrated in Fig. 1. We assume that the center of gravity (CG) lies along the $\mathbf{b}_3$ axis a distance l from the center of buoyancy (CB) such that $r_G = le_3$ (and so $D = Ml\hat{e}_3$). When $l > 0$ the vehicle is *bottom-heavy*, and when $l < 0$ the vehicle is *top-heavy*. Further, while the buoyant force acts through the origin of the body-fixed frame, the gravitational force does not. This means that gravity exerts an orientation-dependent moment on the vehicle. Let k be the unit vector that points in the direction of gravity with respect to the inertial frame. Define

$$\Gamma = R^{\mathrm{T}} k$$

as the direction of gravity with respect to body-fixed coordinates. The gravitational force is given by $f_1 = mgk$. Let ρ_1 be the vector from the origin of the inertial frame to the center of gravity of the vehicle. Then the moment applied to the body due to gravity is given in body coordinates by

$$R^{\mathrm{T}}(\rho_1 - b) \times R^{\mathrm{T}} f_1 = r_G \times R^{\mathrm{T}} mgk = -mgl(\Gamma \times e_3).$$

Thus the equations of motion for this case are

$$\dot{P} = P \times \Omega + F, \tag{11}$$

$$\dot{\Pi} = \Pi \times \Omega + P \times v - mgl(\Gamma \times e_3) + \sum_{i=2}^{k} (R^{\mathrm{T}}(\rho_i - b)) \times R^{\mathrm{T}} f_i + \mathcal{T}, \tag{12}$$

$$\dot{\Gamma} = \Gamma \times \Omega, \tag{13}$$

where gravitational terms have been explicitly represented.

2.3. *Lie–Poisson description of dynamics*

In this section we derive the Lie–Poisson description of the dynamics of the underwater vehicle with noncoincident centers. We show the dynamics to be Hamiltonian with respect to a Lie–Poisson structure, which is a generalization of the canonical Hamiltonian structure. The derivation uses reduction, for which our main reference is Marsden and Ratiu (1994). Reduction is a way of exploiting symmetry in the dynamics so that one can study system dynamics by considering a reduced (i.e. simplified) set of equations of motion. For the underwater vehicle, the configuration space is $Q = \mathrm{SE}(3)$ and the phase space is $T^*Q = T^*\mathrm{SE}(3)$, the cotangent bundle of SE(3), which is a twelve-dimensional space.

2.3.1. *Coincident centers of buoyancy and gravity.* In the preparation for our derivation in the noncoincident centers case, we first describe the Lie–Poisson structure for the neutrally buoyant vehicle with coincident centers and no external forces or torques (Arnold *et al.*, 1993). In the case where the vehicle is ellipsoidal, the Lagrangian L: $T\mathrm{SE}(3) \to \mathbb{R}$ is given from (2) by

$$L(R, b, R\hat{\Omega}, Rv) = \tfrac{1}{2}(\Omega^{\mathrm{T}} J\Omega + v^{\mathrm{T}} Mv).$$

SE(3) acts on itself by left translation, $\mathscr{L}_{(\bar{R},\bar{b})}$: $\mathrm{SE}(3) \to \mathrm{SE}(3)$, according to

$$\mathscr{L}_{(\bar{R},\bar{b})}(R, b) = (\bar{R}R, \bar{R}b + \bar{b}), \quad (\bar{R}, \bar{b}), \quad (R, b) \in \mathrm{SE}(3).$$

Under the left action of SE(3), the Lagrangian becomes

$$\begin{aligned} &L(T\mathscr{L}_{(\bar{R},\bar{b})}(R, b, R\hat{\Omega}, Rv)) \\ &= L(\bar{R}R, \bar{R}b + \bar{b}, \bar{R}R\hat{\Omega}, \bar{R}Rv) \\ &= \tfrac{1}{2}(\Omega^{\mathrm{T}} J\Omega + v^{\mathrm{T}} Mv) = L(R, b, R\hat{\Omega}, Rv), \end{aligned}$$

where $T\mathscr{L}_{(\bar{R},\bar{b})}$: $T\mathrm{SE}(3) \to T\mathrm{SE}(3)$ is the derivative of the left translation. This shows the Lagrangian (and likewise the Hamiltonian and the equations of motion) to be invariant under rotations or translations of the inertial frame, and so we say the system has SE(3) symmetry. Accordingly, Lie–Poisson reduction of the dynamics by the action of SE(3) induces a Lie–Poisson system on $\mathfrak{se}(3)^*$, the dual of the Lie algebra $\mathfrak{se}(3)$. That is, because of symmetry, the equations of motion can be reduced from those on the twelve-dimensional phase space $T^*\mathrm{SE}(3)$ to the six-dimensional space $\mathfrak{se}(3)^*$. Let $\mu = (\Pi, P) \in \mathfrak{se}(3)^*$; then the reduced Hamiltonian on $\mathfrak{se}(3)^*$ is

$$H(\mu) = H(\Pi, P) = \tfrac{1}{2}(\Pi^{\mathrm{T}} A\Pi + P^{\mathrm{T}} CP), \tag{14}$$

where we define

$$A = J^{-1}, \quad C = M^{-1}.$$

Given two differentiable functions G,K on $\mathfrak{se}(3)^*$, the Lie–Poisson bracket on $\mathfrak{se}(3)^*$ that makes $\mathfrak{se}(3)^*$ a Poisson manifold is

$$\{G, K\}(\mu) = \nabla G^{\mathrm{T}} \Lambda(\mu) \nabla K, \tag{15}$$

where

$$\Lambda(\mu) = \Lambda(\Pi, P) = \begin{pmatrix} \hat{\Pi} & \hat{P} \\ \hat{P} & 0 \end{pmatrix}. \tag{16}$$

The Lie–Poisson equations of motion are given by

$$\dot{\mu}_i = \{\mu_i, H\}(\mu), \quad i = 1, \ldots, 6. \tag{17}$$

Equivalently, by (14) and (15),

$$\begin{pmatrix} \dot{\Pi} \\ \dot{P} \end{pmatrix} = \Lambda(\mu) \nabla H(\mu) = \begin{pmatrix} \Pi \times \Omega + P \times v \\ P \times \Omega \end{pmatrix}, \tag{18}$$

which are the equations of motion (7) and (8) derived using Newton–Euler balance laws.

A *Casimir function* C_i on $\mathfrak{se}(3)^*$ is one in which ∇C_i is in the nullspace of Λ, or, equivalently, $\{C_i, K\}(\mu) = 0$ for any function K. This implies that Casimir functions are conserved quantities along equations of the form (17) for any H, i.e. along the equations (18) in particular. The nullspace of Λ given by (16) has rank 2 (for $P \neq 0$), and two independent Casimirs are given by

$$C_1(\Pi, P) = \Pi \cdot P, \quad C_2(\Pi, P) = \|P\|^2.$$

Further, any Casimir can be expressed as a smooth function of these two.

Coadjoint orbits of $\mathfrak{se}(3)^*$ are smooth immersed submanifolds of $\mathfrak{se}(3)^*$ on which the Casimir functions are constant. Therefore flows of (17) do not leave the coadjoint orbit on which they start. The coadjoint orbit through $\mu = (\Pi, P)$, where $P \neq 0$, is diffeomorphic to $T^*S^2_{\|P\|}$, the cotangent bundle to the two-sphere of radius $\|P\|$, which is a four-dimensional manifold.

2.3.2. *Noncoincident centers of buoyancy and gravity.* The Lie–Poisson description of the neutrally buoyant vehicle with noncoincident centers and no external forces of torques (see Fig. 1) is derived in this section. In this case the Lagrangian is the kinetic energy of the system minus the potential energy due to gravity, i.e.

$$L_k(R, b, R\hat{\Omega}, Rv) = \tfrac{1}{2}(\Omega^{\mathrm{T}} J\Omega + 2\Omega^{\mathrm{T}} Dv + v^{\mathrm{T}} Mv + 2mgl(k \cdot Re_3)),$$

where the subscript k indicates that the Lagrangian is smoothly parametrized by k. Under the left action of SE(3), the Lagrangian becomes

$$\begin{aligned} &L_k(T\mathscr{L}_{(\bar{R},\bar{b})}(R, b, R\hat{\Omega}, Rv)) \\ &= L_k(\bar{R}R, \bar{R}b + \bar{b}, \bar{R}R\hat{\Omega}, \bar{R}Rv) \\ &= \tfrac{1}{2}(\Omega^{\mathrm{T}} J\Omega + 2\Omega^{\mathrm{T}} Dv + v^{\mathrm{T}} Mv + 2mgl(\bar{R}^{\mathrm{T}} k \cdot Re_3)). \end{aligned}$$

So,

$$L_k(T\mathscr{L}_{(\bar{R},\bar{b})}(R, b, R\hat{\Omega}, Rv)) = L_k(R, b, R\hat{\Omega}, Rv) \Leftrightarrow \bar{R}^{\mathrm{T}} k = k,$$

i.e. the Lagrangian is left-invariant under the action of the group

$$G_k = \{(R, b) \in \mathrm{SE}(3) \mid R^{\mathrm{T}}k = k\} = \mathrm{SE}(2) \times \mathbb{R}. \tag{19}$$

Gravity has broken the full SE(3) symmetry of the coincident-centers case, leaving the (left) symmetry group of this system to be $\mathrm{SE}(2) \times \mathbb{R}$. This means that the Lagrangian (and consequently the equations of motion) are unchanged if we translate the inertial frame in any direction and rotate it about k, the direction of gravity. Because of gravity, reduction by SE(3) as in the coincident-centers case is no longer possible. However, following Marsden *et al.* (1984) we can reduce the Hamiltonian system on $T^*\mathrm{SE}(3)$, which can also be parametrized by k and is also necessarily left-invariant under the action of $\mathrm{SE}(2) \times \mathbb{R}$, to a Hamiltonian system on the dual of the Lie algebra of a semidirect product. In this case we shall reduce the equations of motion from those on the twelve-dimensional phase space $T^*\mathrm{SE}(3)$ to a nine-dimensional space.

Define ρ: $\mathrm{SE}(3) \to \mathrm{Aut}(\mathbb{R}^3)$ by $\rho(R, b) = R$, where $\mathrm{Aut}(\mathbb{R}^3)$ is the space of automorphisms of $\mathbb{R}^3$. Then $S = \mathrm{SE}(3) \times_\rho \mathbb{R}^3$ is a semidirect product, where group multiplication is given by

$$\begin{aligned}((R, b), w)((R', b'), w') &= ((R, b)(R', b'), \rho(R, b)w' + w) \\ &= ((RR', Rb' + b), Rw' + w).\end{aligned}$$

Let ρ^* be the associated right representation of SE(3) on $\mathbb{R}^{3*}$. Then

$$\rho^*(R, b)k = k \Leftrightarrow R^{\mathrm{T}}k = k,$$

so that G_k as defined by (19) is the stabilizer of $k \in \mathbb{R}^{3*}$ under ρ^*, i.e.

$$G_k = \{(R, b) \in \mathrm{SE}(3) \mid \rho^*(R, b)k = k\}.$$

By Theorem 3.4 of Marsden *et al.* (1984), the Hamiltonian dynamics on $T^*\mathrm{SE}(3)$ can be reduced to a Lie–Poisson system on the nine-dimensional space $\mathfrak{s}^*$, the dual of the Lie algebra $\mathfrak{s}$ of S. $\mathfrak{s} = \mathfrak{se}(3) \times_{\rho'} \mathbb{R}^3$, where ρ' is the induced Lie algebra representation, i.e. for $(\alpha, \beta) \in \mathfrak{se}(3)$, $\rho'(\alpha, \beta) = \hat{\alpha}$. By performing matrix commutation of two elements in $\mathfrak{s}$ given in matrix form, we find that the Lie bracket on $\mathfrak{s}$ is given by

$$\begin{aligned}&[(\alpha, \beta, \gamma), (\alpha', \beta', \gamma')] \\ &\quad= (\alpha \times \alpha', \alpha \times \beta' - \alpha' \times \beta, \alpha \times \gamma' - \alpha' \times \gamma).\end{aligned}$$

The triplet $\mu = (\Pi, P, \Gamma)$ is an element in $\mathfrak{s}^*$, where

$$\Pi = J\Omega + Dv, \quad P = Mv + D^{\mathrm{T}}\Omega, \quad \Gamma = R^{\mathrm{T}}k. \tag{20}$$

Alternatively, we can express Ω and v in terms of Π and P:

$$\Omega = A\Pi + B^{\mathrm{T}}P, \quad v = CP + B\Pi, \tag{21}$$

where

$$\begin{aligned}A &= (J - DM^{-1}D^{\mathrm{T}})^{-1}, \\ B &= -CD^{\mathrm{T}}J^{-1} = -M^{-1}D^{\mathrm{T}}A, \\ C &= (M - D^{\mathrm{T}}J^{-1}D)^{-1}.\end{aligned} \tag{22}$$

From this, we can compute the reduced Hamiltonian on $\mathfrak{s}^*$ as

$$\begin{aligned}H(\mu) = \tfrac{1}{2}(&\Pi^{\mathrm{T}}A\Pi + 2\Pi^{\mathrm{T}}B^{\mathrm{T}}P + P^{\mathrm{T}}CP \\ &- 2mgl(\Gamma \cdot e_3))).\end{aligned} \tag{23}$$

The Lie–Poisson bracket on $\mathfrak{s}^*$ can be found as follows. Let $\{B_1, \ldots, B_9\}$ be the standard basis for $\mathfrak{s}$ such that if B_i^b, $i = 1, \ldots, 9$, is the dual basis for $\mathfrak{s}^*$, then μ is given in local coordinates as

$$\mu = \sum_{i=1}^{9} \mu_i B_i^b,$$

where $(\mu_1, \ldots, \mu_9) = (\Pi_1, \ldots, \Gamma_3)$. The Poisson tensor Λ is computed according to

$$[\Lambda(\mu)]_{ij} = -\sum_{k=1}^{9} \Gamma_{ij}^k \mu_k,$$

where Γ_{ij}^k are the structure constants for the chosen basis of $\mathfrak{s}$. Explicitly, we find

$$\Lambda(\mu) = \Lambda(\Pi, P, \Gamma) = \begin{pmatrix} \hat{\Pi} & \hat{P} & \hat{\Gamma} \\ \hat{P} & 0 & 0 \\ \hat{\Gamma} & 0 & 0 \end{pmatrix}. \tag{24}$$

Thus the Lie–Poisson bracket of two differentiable functions G and K in dual coordinates on $\mathfrak{s}^*$ is

$$\{G, K\}(\mu) = \nabla G^{\mathrm{T}}\Lambda(\mu)\nabla K,$$

and the equations of motion are given by

$$\begin{aligned}\begin{pmatrix} \dot{\Pi} \\ \dot{P} \\ \dot{\Gamma} \end{pmatrix} &= \Lambda(\mu)\nabla H(\mu) \\ &= \begin{pmatrix} \Pi \times \Omega + P \times v - mgl\Gamma \times e_3 \\ P \times \Omega \\ \Gamma \times \Omega \end{pmatrix}.\end{aligned} \tag{25}$$

These are the equations of motion (11)–(13) derived using the Newton–Euler balance laws.

The nullspace of the Poisson tensor Λ given by

(24) has rank 3 for P, $\Gamma \neq 0$ and $P \nparallel \Gamma$. Three independent Casimir functions on $\mathfrak{s}^*$ are

$$C_1(\Pi, P, \Gamma) = P \cdot \Gamma,$$
$$C_2(\Pi, P, \Gamma) = \|P\|^2,$$
$$C_3(\Pi, P, \Gamma) = \|\Gamma\|^2.$$

Further, any Casimir can be expressed as a smooth function of these three. Note that

$$C_1(\Pi, P, \Gamma) = \|P\| \, \|\Gamma\| \cos\theta = \sqrt{C_2 C_3} \cos\theta,$$

where θ is the angle between P and Γ.

Coadjoint orbits of $\mathfrak{s}^*$ are smooth immersed submanifolds of $\mathfrak{s}^*$ on which the Casimir functions are constant. For details of the coadjoint action and orbits for the Poisson structure on $\mathfrak{s}^*$ see Appendix A and Leonard (1996a). The coadjoint orbit through $\mu = (\Pi, P, \Gamma)$, where P, $\Gamma \neq 0$ and $P \nparallel \Gamma$ (i.e., $\theta \neq 0$), is diffeomorphic to the six-dimensional manifold $T^*SO(3)$. When (Π, P, Γ) is such that $P \parallel \Gamma$ (i.e. $\theta = 0$) and P and Γ are not both 0, then Λ loses rank, i.e. the nullspace of Λ has rank 5. Points such as these are called *nongeneric*. In this case, in addition to the three Casimirs defined above, $\Pi \cdot P$ (and therefore $\Pi \cdot \Gamma$) are constants of motion, called *sub-Casimirs*. The coadjoint orbit through (Π, P, Γ), where P and Γ are not both zero and $P \parallel \Gamma$, is diffeomorphic to the four-dimensional manifold T^*S^2.

3. STABILITY

We analyze the stability of relative equilibria of the underwater vehicle equations of motion using the energy–Casimir method for the vehicle both with coincident and with noncoincident centers of buoyancy and gravity. A *relative equilibrium* for the underwater vehicle is a solution in the phase space $T^*SE(3)$ that corresponds to a fixed point μ_e for the reduced dynamic equations, i.e., for equations (18) or (25). In the case of coincident centers $\mu_e \in \mathfrak{se}(3)^*$, while for noncoincident centers $\mu_e \in \mathfrak{s}^*$. Here relative equilibria correspond to constant translations and rotations.

Using the energy–Casimir method, we can study stability of the relative equilibrium in the phase space $T^*SE(3)$ by studying stability of the corresponding equilibrium μ_e in the reduced space. The energy–Casimir method provides sufficient conditions for nonlinear stability of an equilibrium μ_e in reduced space. Here nonlinear stability refers to stability in the sense of Lyapunov, i.e. for every neighborhood V of μ_e there is a neighborhood U of μ_e such that trajectories that start in U never leave V. This is distinct from spectral stability, which requires only that the linearization of the reduced dynamics at μ_e have no eigenvalues with positive real part. We recall that for Hamiltonian systems eigenvalues are symmetrically distributed under reflection about the real and imaginary axes, so that spectral stability implies that all eigenvalues of the linearization are on the imaginary axis. Thus spectral stability does not imply Lyapunov stability, and so linearization techniques are not sufficient for proving stability of motions. Linearization can, however, be used to prove instability of μ_e, i.e. divergent behavior in the reduced dynamics near μ_e, since an eigenvalue with positive real part (spectral instability) does imply instability in the sense of Lyapunov.

Proving Lyapunov stability of μ_e in the reduced space using the energy–Casimir method implies stability of the corresponding relative equilibrium in the phase space $T^*SE(3)$ modulo the symmetry group G (e.g. here $G = SE(3)$ or $G = SE(2) \times \mathbb{R}$). Stability modulo G is just the usual notion of Lyapunov stability, except that one allows arbitrary drift along the orbits of G (Patrick, 1992). For example, stability in $T^*SE(3)$ modulo SE(3) means that there is the possibility that there will be drift in the rotational and translational parameters but not in the linear and angular momentum parameters.

However, for the underwater vehicle problem, stronger results are possible (Leonard and Marsden, 1996). Indeed, Lyapunov stability of μ_e in the reduced space will typically imply stability of the corresponding relative equilibrium in $T^*SE(3)$ modulo a smaller group than the symmetry group. In particular, there will be the possibility of drift in the translational parameters (i.e. in the directions corresponding to the noncompact part of the symmetry group), but drift in the rotational parameters will be possible only about the axis of rotation and not away from the axis of rotation. An extension to energy methods, such as the energy–Casimir method, that can be used to prove this kind of stronger stability conclusion is presented in Leonard and Marsden (1996). This work makes use of reduction by stages and a result for compact symmetry groups due to Patrick (1992). We note that Lamb (1932) does not extend his classical stability analysis of a submerged rigid body to the full twelve-dimensional phase space $T^*SE(3)$, i.e. he does not indicate whether or not translation and rotation parameters drift.

The energy–Casimir method gives a step-by-step procedure for constructing a Lyapunov function to prove stability. For a generic equilibrium of a Lie–Poisson system, the Lyapunov function is a function of the Hamiltonian, the Casimir functions and any other conserved quantities. Let $H_{\Phi,\phi} = H + \Phi(C_i) + \phi_j(c_j)$ where H is the Hamiltonian, C_i are the Casimirs, c_j are other constants of motion, and $\Phi(\cdot)$ and $\phi_j(\cdot)$ are smooth functions to be determined. The method then consists in finding $\Phi(\cdot)$ and $\phi_j(\cdot)$ such that μ_e is a critical point of $H_{\Phi,\phi}$. The condition that the second derivation of $H_{\Phi,\phi}$ be definite (positive or negative) at μ_e is sufficient for nonlinear stability of μ_e. See Holm *et al.* (1985), Krishnaprasad (1989) and Marsden and Ratiu (1994) for further details. Stability of a relative equilibrium is sometimes referred to as *relative stability*.

Aeyels (1992) gives an alternative interpretation of the energy–Casimir method in terms of a topological criterion sufficient for Lyapunov stability of an equilibrium point μ_e for a system with constants of motion. Given that $c_1, \ldots, c_k$ are constants of motion, he shows that if μ_e is a locally isolated solution of $c_j(\mu) = c_j(\mu_e)$, $j = 1, \ldots, k$, then μ_e is Lyapunov-stable. The energy–Casimir method is a sufficient condition for this topological criterion.

Here we emphasize the perspective from geometric mechanics, since it is the Lie–Poisson structure that provides a means to find the Casimirs that are the conserved quantities that we need to use to prove stability. The significance of Lie–Poisson structure and Casimir functions in the energy–Casimir method can be further appreciated in recent related work on feedback stabilization of underwater vehicle dynamics (Leonard, 1996b).

3.1. *Coincident centers of buoyancy and gravity*

In this section we analyze stability of an ellipsoidal, neutrally buoyant vehicle with coincident centers of gravity and buoyancy using the Lie–Poisson structure described in Section 2.3.1. Additional details of the computations can be found in Leonard (1996a). We assume that the principal axes of the vehicle and the principal axes of the displaced fluid are coincident, so that

$$J = \mathrm{diag}\,(I_1, I_2, I_3),$$
$$M = \mathrm{diag}\,(m_1, m_2, m_3).$$

Recall that the equations of motion with no external forces or torques are

$$\dot{\Pi} = \Pi \times \Omega + P \times v,$$
$$\dot{P} = P \times \Omega,$$

where $\Pi = J\Omega$ and $P = Mv$.

Of interest is the stability of three sets of two-parameter families of equilibrium solutions, each family corresponding to constant translation along and rotation about one of the principal axes of the vehicle. (We note that there may be other nontrivial equilibrium solutions for particular orderings of the mass and inertia parameters, but we do not consider those here.) Without loss of generality (because of symmetry), we shall study the two-parameter family of equilibrium solutions

$$\Pi_e = (0, 0, \Pi_3^0)^T, \quad P_e = (0, 0, P_3^0)^T,$$

which corresponds to constant translation along and rotation about the $\mathbf{b}_3$ axis (i.e. the third principal axis). We take $P_3^0 \neq 0$, so that the equilibrium solution is generic, i.e. it lives on a maximal (four-dimensional) coadjoint orbit. We shall further assume (for ease of computation) that the $\mathbf{b}_3$ axis is an axis of symmetry, i.e. $I_1 = I_2$ and $m_1 = m_2$. In this case we have $\dot{\Pi}_3 = 0$, so that any function $\phi(\Pi_3)$ is conserved.

To begin, we consider conserved quantities of the form

$$H_{\Phi,\phi} = H + \Phi(\Pi \cdot P, \|P\|^2) + \phi(\Pi_3),$$

where, from (14), $H = \frac{1}{2}(\Pi^T A \Pi + P^T C P)$, and $\Phi(\cdot)$, and $\phi(\cdot)$ are to be chosen as necessary. Define

$$\dot{\Phi} = \frac{\partial \Phi}{\partial(\Pi \cdot P)}, \quad \Phi' = \frac{\partial \Phi}{\partial(\|P\|^2)}.$$

When evaluated at the equilibrium solution, the first derivative of $H_{\Phi,\phi}$ is zero if and only if

$$\dot{\Phi}(\Pi_3^0 P_3^0, (P_3^0)^2) = -\left(\frac{\Pi_3^0}{I_3} + \phi'(\Pi_3^0)\right)\frac{1}{P_3^0}, \quad (26)$$

$$2\Phi'(\Pi_3^0 P_3^0, (P_3^0)^2) = -\frac{1}{m_3} + \left(\frac{\Pi_3^0}{I_3} + \phi'(\Pi_3^0)\right)\frac{\Pi_3^0}{(P_3^0)^2}. \quad (27)$$

Next we check for definiteness of the second variation of $H_{\Phi,\phi}$ at the equilibrium point. Let

$$a = \phi''(\Pi_3^0), \quad b = 4\Phi''(\Pi_3^0 P_3^0, (P_3^0)^2),$$
$$c = \ddot{\Phi}(\Pi_3^0 P_3^0, (P_3^0)^2), \quad d = 2\dot{\Phi}'(\Pi_3^0 P_3^0, (P_3^0)^2).$$

The matrix of the second derivative at the equilibrium point is

$$\begin{pmatrix} \frac{1}{I_2} & 0 & 0 & \dot{\Phi} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 & 0 & \dot{\Phi} & 0 \\ 0 & 0 & a_{33} & 0 & 0 & a_{36} \\ \dot{\Phi} & 0 & 0 & \frac{1}{m_2} + 2\Phi' & 0 & 0 \\ 0 & \dot{\Phi} & 0 & 0 & \frac{1}{m_2} + 2\Phi' & 0 \\ 0 & 0 & a_{36} & 0 & 0 & a_{66} \end{pmatrix} \quad (28)$$

where

$$a_{33} = \frac{1}{I_3} + a + c(P_3^0)^2,$$
$$a_{36} = \dot{\Phi} + cP_3^0\Pi_3^0 + d(P_3^0)^2,$$
$$a_{66} = \frac{1}{m_3} + 2\Phi' + c(\Pi_3^0)^2 + dP_3^0 + b(P_3^0)^2,$$

and evaluation of partial derivatives of Φ at the equilibrium point is implied.

If the matrix (28) is definite, it is positive-definite, since the first principal determinant is positive. If we take $a = c = d = 0$ and $b > 0$ sufficiently large, positive-definiteness is assured if and only if

$$\frac{1}{I_2}\left(\frac{1}{m_2} + 2\Phi'\right) - \dot{\Phi}^2 > 0. \tag{29}$$

Define

$$e = \frac{\Pi_3^0}{I_3} + \phi'(\Pi_3^0)$$

and note that we have the freedom to choose e to be as large as necessary, since we can choose $\phi'(\Pi_3^0)$ arbitrarily. Using (26) and (27), the condition (29) becomes

$$-\frac{1}{(P_3^0)^2}e^2 + \frac{1}{I_2}\frac{\Pi_3^0}{(P_3^0)^2}e + \frac{1}{I_2}\left(\frac{1}{m_2} - \frac{1}{m_3}\right) > 0. \tag{30}$$

The left-hand side of (30) is a polynomial in e. Since the leading coefficient is negative, this polynomial will be positive for some e if and only if the discriminant is positive, i.e.

$$\left(\frac{\Pi_3^0}{P_3^0}\right)^2 > 4I_2\left(\frac{1}{m_3} - \frac{1}{m_2}\right). \tag{31}$$

Thus, by the energy–Casimir method, (31) is a sufficient condition for nonlinear stability of the equilibrium solution (nonzero P_3^0).

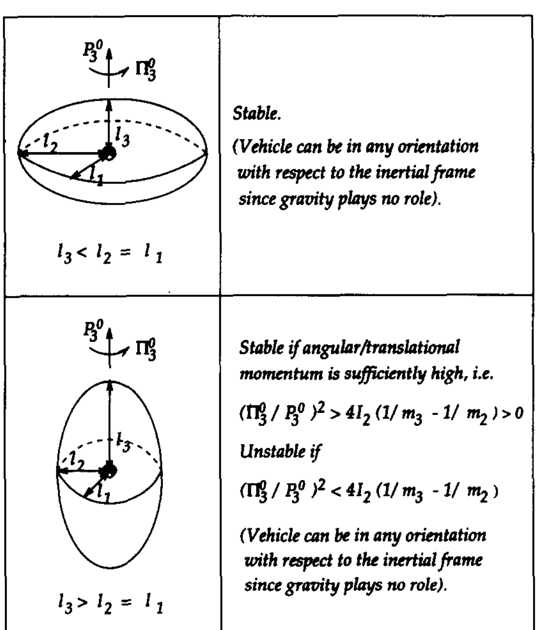

Fig. 2. Summary of stability with coincident centers (Theorem 1).

Let l_1, l_2 and l_3 be the lengths of the semiaxes of the ellipsoidal vehicle, where l_i corresponds to the semiaxis along $\mathbf{b}_i$. We have assumed that $l_1 = l_2$. Suppose that $l_3 < l_2$, i.e. the vehicle is an oblate spheroid. Then $m_3 > m_2$ (see Appendix B for details). Thus the right-hand side of the condition (31) is negative, so the equilibrium solution is stable for all values of Π_3^0 and P_3^0. That is, the vehicle moving in the direction of its minor axis is stable. Now suppose that $l_3 > l_2$, i.e. the vehicle is a prolate spheroid. Then $m_3 < m_2$, and the right-hand side of (31) is positive. Thus the equilibrium solution is stable if the ratio of angular momentum to linear momentum is large enough to satisfy (31). These results are summarized in Fig. 2.

We use spectral analysis of the linearized equations to find conditions for instability of the equilibrium solutions. When evaluated at (Π_e, P_e), the linearized equations evolve on the tangent space to the coadjoint orbit in $\mathfrak{se}(3)^*$ through (Π_e, P_e) given by

$$\{\delta\Pi, \delta P \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \delta\Pi \cdot P_e + \Pi_e \cdot \delta P = 0$$
$$\text{and } \delta P \cdot P_e = 0\} \approx \{\delta\Pi_1, \delta\Pi_2, \delta P_1, \delta P_2\} = \mathbb{R}^4.$$

That is, at (Π_e, P_e), $\delta\dot{\Pi}_3 = \delta\dot{P}_3 = 0$, i.e. the linearization always has at least two eigenvalues fixed at 0. A computation shows that at least one of the other four eigenvalues of the matrix of the linearization will have positive real part if and only if (31) holds with the direction of the inequality sign reversed.

The condition (31) for stability in this case agrees with that described by Lamb (1992). However, Lamb derives this condition using linearization, and thus can only conclude spectral stability when the condition is met. The result derived here proves nonlinear (Lyapunov) stability of the equilibrium when the condition is met. This is stronger than Lamb's result. We can state the results as the following theorem.

Theorem 1. Consider an ellipsoidal, neutrally buoyant vehicle with coincident centers of gravity and buoyancy and $\mathbf{b}_3$ an axis of symmetry. Constant (nonzero) translation along and rotation about the $\mathbf{b}_3$ axis is (nonlinearly) stable if

$$\left(\frac{\Pi_3^0}{P_3^0}\right)^2 > 4I_2\left(\frac{1}{m_3} - \frac{1}{m_2}\right).$$

If this relation holds with the direction of the inequality sign reversed, the motion is unstable.

Note that, based on our analysis, we cannot conclude stronger than spectral stability for the case in which the condition in Theorem 1 holds with equality. For this, one would need to examine higher-order derivations. Because of symmetry, the analogous theorem holds for constant translation along and rotation about the $\mathbf{b}_1$ or $\mathbf{b}_2$ axis with the appropriate substitutions for I_2, m_2 and m_3. See Fig. 2 for a summary.

3.2. *Noncoincident centers of buoyancy and gravity*

Next, we analyze stability in the case with noncoincident centers of buoyancy and gravity using the Lie–Poisson structure described in Section 2.3.2. Additional detail on the computations can be found in Leonard (1996a). We assume that the vehicle is an ellipsoid with semiaxes l_1, l_2 and l_3, where l_i lies along the $\mathbf{b}_i$ axis. We assume that the principal axes of the vehicle and the principal axes of the displaced fluid are coincident and $r_G = le_3$, so that

$$J = \mathrm{diag}\,(I_1, I_2, I_3),$$
$$M = \mathrm{diag}\,(m_1, m_2, m_3),$$
$$D = ml\hat{e}_3.$$

From (22), this implies that

$$A = \mathrm{diag}\left(\bar{a}_1, \bar{a}_2, \frac{1}{I_3}\right), \quad C = \mathrm{diag}\left(\bar{c}_1, \bar{c}_2, \frac{1}{m_3}\right),$$
$$B = \begin{pmatrix} 0 & \bar{b}_1 & 0 \\ -\bar{b}_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\bar{a}_1 = \frac{m_2}{m_2 I_1 - m^2 l^2}, \quad \bar{a}_2 = \frac{m_1}{m_1 I_2 - m^2 l^2},$$
$$\bar{c}_1 = \frac{I_2}{m_1 I_2 - m^2 l^2}, \quad \bar{c}_2 = \frac{I_1}{m_2 I_1 - m^2 l^2},$$
$$\bar{b}_1 = -\frac{ml}{m_1 I_2 - m^2 l^2}, \quad \bar{b}_2 = -\frac{ml}{m_2 I_1 - m^2 l^2}.$$

We have $m_2 I_1 - m^2 l^2 > 0$ and $m_1 I_2 - m^2 l^2 > 0$. Thus $\bar{a}_1$, $\bar{a}_2$, $\bar{c}_1$, $\bar{c}_2 > 0$ and $\bar{b}_1$, $\bar{b}_2 < 0$.

Recall from (20), (21) and (25) that the equations of motion are

$$\dot{\Pi} = P \times v + \Pi \times \Omega - mgl(\Gamma \times e_3),$$
$$\dot{P} = P \times \Omega,$$
$$\dot{\Gamma} = \Gamma \times \Omega,$$

where

$$\Pi = J\Omega + Dv, \quad P = Mv + D^{\mathrm{T}}\Omega, \quad \Gamma = R^{\mathrm{T}}k,$$
$$\Omega = A\Pi + B^{\mathrm{T}}P, \quad v = CP + B\Pi.$$

We classify the equilibrium solutions into two cases: the first when $\Omega_e = 0$ (no spin) and the second when $\Omega_e \neq 0$ (spin). The subscript 'e' denotes evaluation at the equilibrium solution (Π_e, P_e, Γ_e).

Equilibrium solutions with no spin. When $\Omega_e = 0$, $v_e = (v_{e1}, v_{e2}, v_{e3})^{\mathrm{T}}$, the equilibrium solutions (Π_e, P_e, Γ_e) satisfy

$$P_e = Mv_e, \quad \Pi_e = ml\hat{e}_3 v_e,$$
$$P_e \times v_e - mgl(\Gamma_e \times e_3) = 0. \tag{32}$$

Equation (32) is equivalent to

$$v_{e1}v_{e2} = 0, \quad \Gamma_{e3} \text{ arbitrary},$$
$$(m_2 - m_3)v_{e2}v_{e3} - mgl\Gamma_{e2} = 0,$$
$$(m_3 - m_1)v_{e3}v_{e1} + mgl\Gamma_{e1} = 0.$$

There are two sets of no-spin equilibria satisfying these conditions, namely, those corresponding to $v_{e1} = 0$ and those corresponding to $v_{e2} = 0$. When we impose the constraint that $\|\Gamma_e\| = 1$, each of these sets is a two-parameter family of equilibrium points. For example, the set corresponding to $v_{e1} = 0$ is parametrized by v_{e2} and v_{e3} as follows:

$$\Pi_e = \begin{pmatrix} -mlv_{e2} \\ 0 \\ 0 \end{pmatrix}, \quad P_e = \begin{pmatrix} 0 \\ m_2 v_{e2} \\ m_3 v_{e3} \end{pmatrix},$$
$$\Gamma_e = \begin{pmatrix} 0 \\ \dfrac{(m_2 - m_3)v_{e2}v_{e3}}{mgl} \\ \sqrt{1 - \left(\dfrac{(m_2 - m_3)v_{e2}v_{e3}}{mgl}\right)^2} \end{pmatrix}, \tag{33}$$

where $v_e = (0, v_{e2}, v_{e3})^{\mathrm{T}}$. Equilibrium points in this family typically live on a generic six-dimensional coadjoint orbit of $\mathfrak{s}^*$, since typically $P_e \nparallel \Gamma_e$. When $v_{e3} = 0$, $P_e \perp \Gamma_e$, and the set of equilibria described by (33) reduces to a single-parameter family, parametrized by v_{e2}, taking the form

$$\Pi_e = \begin{pmatrix} -mlv_{e2} \\ 0 \\ 0 \end{pmatrix}, \quad P_e = \begin{pmatrix} 0 \\ m_2 v_{e2} \\ 0 \end{pmatrix}, \quad \Gamma_e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{34}$$

where $v_e = (0, v_{e2}, 0)^{\mathrm{T}}$. The no-spin equilibrium solutions given by (34) correspond to a vehicle with its $\mathbf{b}_3$ axis aligned with the direction of gravity and constant translation along its second principal axis $\mathbf{b}_2$ (perpendicular to the direction of gravity).

Equilibrium solutions with spin. When $\Omega_e \neq 0$, (Π_e, P_e, Γ_e) satisfy

$$\Pi_e \times \Omega_e + P_e \times v_e - mgl(\Gamma_e \times e_3) = 0,$$
$$P_e \times \Omega_e = 0, \tag{35}$$
$$\Gamma_e \times \Omega_e = 0. \tag{36}$$

The conditions (35) and (36) imply that

$$P_e \parallel \Gamma_e \parallel \Omega_e,$$

and consequently that these equilibrium solutions are nongeneric in that they live on a four-dimensional coadjoint orbit of $\mathfrak{s}^*$. Since $\Gamma_e \parallel \Omega_e$, the spin axis will always coincide with the direction of gravity. One set of equilibrium solutions with spin is a two-parameter family, parameterized by Ω_{e3} and v_{e3}, of the form

$$\Pi_e = \begin{pmatrix} 0 \\ 0 \\ I_3\Omega_{e3} \end{pmatrix}, \quad P_e = \begin{pmatrix} 0 \\ 0 \\ m_3 v_{e3} \end{pmatrix}, \quad \Gamma_e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{37}$$

where $\Omega_e = (0, 0, \Omega_{e3})^T$ and $v_e = (0, 0, v_{e3})^T$. This corresponds to constant translation along and rotation about the $\mathbf{b}_3$ axis, which coincides with the direction of gravity, i.e. a rising or falling vehicle. A second set of equilibria is characterized by $P_e = 0$. In the general case in which $m_1 \neq m_2$, $I_1 \neq I_2$, i.e. the body is not symmetric about its $\mathbf{b}_3$ axis, this second set of equilibria must satisfy $\Omega_{e1}\Omega_{e2} = 0$. For either $\Omega_{e1} \neq 0$ or $\Omega_{e2} \neq 0$, the set of equilibria is a one-parameter family when we impose the additional constraint that $\|\Gamma_e\| = 1$. For example, when $\Omega_{e1} \neq 0$, the family of equilibria is parametrized by Ω_{e1} and given by

$$\Pi_e = \begin{pmatrix} \frac{1}{\bar{a}_1}\Omega_{e1} \\ 0 \\ I_3\Omega_{e3} \end{pmatrix}, \quad P_e = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Gamma_e = \frac{(1/\bar{a}_1 - I_3)\Omega_{e3}}{mgl} \begin{pmatrix} \Omega_{e1} \\ 0 \\ \Omega_{e3} \end{pmatrix},$$

where $\Omega_e = (\Omega_{e1}, 0, \Omega_{e3})^T$, $v_e = (0, ml\Omega_{e1}/m_2, 0)^T$, and Ω_{e3} is computed such that $\|\Gamma_e\| = 1$. Further equilibrium solutions with spin may be possible, depending upon the relative magnitudes of the mass and inertia parameters.

3.2.1. *Stability of equilibrium solution with no spin.* In this section we study the equilibrium solution given by (34), which we shall denote as

$$\Pi_e = (\Pi_1^0, 0, 0)^T, \quad P_e = (0, P_2^0, 0)^T, \quad \Gamma_e = (0, 0, 1)^T,$$

where $P_2^0 = m_2 v_{e2}$ and $\Pi_1^0 = -mlv_{e2} = -mlP_2^0/m_2$. We shall interpret $\Pi_1^0 = \Pi_1^0(P_2^0)$, so that we keep in mind that this is a single-parameter family of equilibria. It is useful to study stability of this family of equilibria, because it corresponds to translation along a principal axis perpendicular to the direction of gravity, which is a practical motion of interest. Stability of the more general two-parameter family of no-spin equilibrium solutions can be analyzed in a similar manner.

We begin by considering conserved quantities of the form

$$H_\Phi = H + \Phi(P \cdot \Gamma, \|P\|^2, \|\Gamma\|^2),$$

where H is given by (23). Define

$$\dot{\Phi} = \frac{\partial \Phi}{\partial (P \cdot \Gamma)}, \quad \Phi' = \frac{\partial \Phi}{\partial (\|P\|^2)}, \quad \Phi^\dagger = \frac{\partial \Phi}{\partial (\|\Gamma\|^2)}.$$

When evaluated at the equilibrium solution, the first derivative of H_Φ is zero if and only if

$$2\Phi'(\Pi_1^0 P_2^0, (P_2^0)^2, 1) = -1/m_2, \tag{38}$$

$$\dot{\Phi}(\Pi_1^0 P_2^0, (P_2^0)^2, 1) = 0, \tag{39}$$

$$2\Phi^\dagger(\Pi_1^0 P_2^0, (P_2^0)^2, 1) = mgl. \tag{40}$$

Next we check for definiteness of the second variation of H_Φ at the equilibrium solution. Let

$$a = \ddot{\Phi}, \quad b = 2\dot{\Phi}', \quad c = 4\Phi'',$$
$$d = 2\dot{\Phi}^\dagger, \quad e = 4\Phi'^\dagger, \quad f = 4\Phi^{\dagger\dagger}$$

with partial derivatives evaluated at the equilibrium. In the case that we choose $b = d = e = f = 0$ and we use the conditions (38)–(40), for positive-definiteness of the second variation of H_Φ we need

$$\frac{m_2 - m_1}{m_2(m_1 I_2 - m^2 l^2)} > 0, \quad c(P_2^0)^2 \bar{a}_1 > 0$$

$$\frac{1}{m_3} - \frac{1}{m_2} + a > 0, \quad mgl > 0,$$

$$(mgl + a(P_2^0)^2)\left(\frac{1}{m_3} - \frac{1}{m_2} + a\right) - (aP_2^0)^2 > 0.$$

We note that $b = d = e = f = 0$ is the best simplifying choice, since the above inequalities would have been necessary regardless, and the choice introduces no extra conditions for positive-definiteness. The last inequality is satisfied by taking $a > 0$ sufficiently large and

$$mgl > \left(\frac{1}{m_2} - \frac{1}{m_3}\right)(P_2^0)^2. \tag{41}$$

Therefore if we choose $a > 0$ sufficiently large and $c > 0$ then positive-definiteness is assured if and only if (41) holds and

$$mgl > 0, \quad (P_2^0)^2 > 0, \quad m_2 > m_1. \tag{42}$$

By the energy–Casimir method (41) and (42) are sufficient conditions for nonlinear stability of the equilibrium solution.

These conditions can be interpreted as follows. Recall that l_i is the length of the semiaxis of the ellipsoidal vehicle along the $\mathbf{b}_i$ axis. For stable (nonzero) translation along the $\mathbf{b}_2$ axis, if $l_2 < l_3$ then we also need $l_2 < l_1$ and $mgl > 0$, i.e. the motion should be along the minor axis and the center of gravity should be below the center of buoyancy. If $l_2 > l_3$, we also need $l_2 < l_1$ and mgl

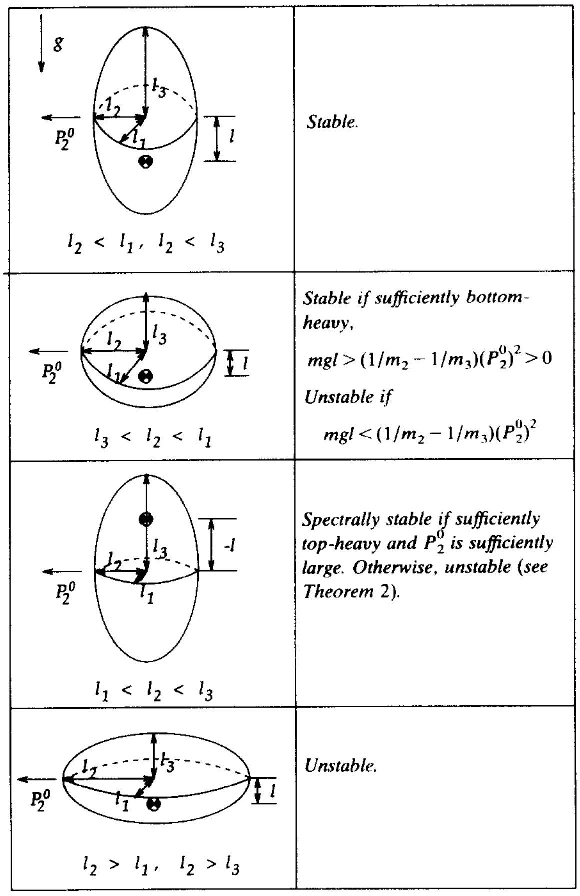

Fig. 3. Summary of stability with noncoincident centers and no spin (Theorem 2).

should satisfy (41), i.e. the motion should be along the intermediate axis, the vertical $\mathbf{b}_3$ axis should be the minor axis and the center of gravity should be *sufficiently* far below the center of buoyancy. These results are summarized in Fig. 3.

If one of the conditions (41) or (42) fails then the equilibrium solution is formally unstable. Using a spectral analysis of the linearized equations, we now investigate when the equilibrium solution is in fact unstable due to failure of one of these conditions. When evaluated at (Π_e, P_e, Γ_e), the linearized equations evolve on the tangent space to the coadjoint orbit in $\mathfrak{s}^*(3)$ through (Π_e, P_e, Γ_e) given by

$$\{(\delta\Pi, \delta P, \delta\Gamma) \in \mathbb{R}^9 \mid \delta P \cdot \Gamma_e + \delta\Gamma \cdot P_e = 0,$$
$$\delta P \cdot P_e = 0,\ \delta\Gamma \cdot \Gamma_e = 0\} \approx \mathbb{R}^6.$$

This means that the linearization has three eigenvalues fixed at 0. The characteristic polynomial of the matrix of the linearization is

$$\lambda^3\left(\lambda^2 + \bar{a}_1\left(mgl - (P_2^0)^2\left(\frac{1}{m_2} - \frac{1}{m_3}\right)\right)\right)$$
$$\times (\lambda^4 + q_1\lambda^2 + q_2),$$

where

$$q_1 = \frac{m_1 - m_2}{m_2}\frac{(P_2^0)^2}{I_3}\left(-\frac{1}{m_2} + \left(\frac{1}{m_2} - \frac{1}{m_1}\right)\bar{a}_2 I_2\right)$$
$$+ \bar{a}_2 mgl, \tag{43}$$

$$q_2 = -\bar{a}_2 mgl\left(\frac{1}{m_2} - \frac{1}{m_1}\right)\frac{(P_2^0)^2}{I_3}. \tag{44}$$

From the polynomial factor of degree two, we see that the chacteristic polynomial has an unstable eigenvalue if

$$mgl < \left(\frac{1}{m_2} - \frac{1}{m_3}\right)(P_2^0)^2.$$

From the polynomial factor of degree four, we compute that the characteristic polynomial has an unstable eigenvalue if (i) $m_2 > m_1$ and $mgl < 0$; (ii) $m_2 < m_1$ and $mgl > 0$; or (iii) $m_2 < m_1$ and $mgl < 0$ unless $mgl < 0$ satisfies $q_1 > 2\sqrt{q_2}$ and (41) (these conditions require that $m_2 > m_3$ and that the translational velocity v_{e2} be high). In this latter exceptional case the equilibrium solution will be spectrally stable. This is very curious, since it implies that, for an appropriate vehicle configuration, constant translation along the intermediate axis ($m_1 > m_2 > m_3$) of the vehicle looking like an inverted pendulum is spectrally stable (although not necessarily nonlinearly stable). It is likely that, with the introduction of dissipation as in Bloch *et al.* (1994), this top-heavy vehicle motion will be unstable.

We can state the results of this stability analysis of the no-spin equilibrium solution as the following theorem.

Theorem 2. (*Stability of bottom-heavy vehicle*). Consider an ellipsoidal, neutrally buoyant vehicle with noncoincident centers of gravity and buoyancy. Constant (nonzero) translation along the $\mathbf{b}_2$ axis is (nonlinearly) stable if

$$m_2 > m_1,\quad mgl > 0,\quad mgl > \left(\frac{1}{m_2} - \frac{1}{m_3}\right)(P_2^0)^2.$$

If these relations hold with the direction of at least one of the inequality signs reversed, the translation is spectrally unstable except under the following special conditions, for which the translation is spectrally stable:

$$m_2 < m_1,\quad mgl < 0,$$
$$mgl > \left(\frac{1}{m_2} - \frac{1}{m_3}\right)(P_2^0)^2,\quad q_1 > 2\sqrt{q_2},$$

where q_1 and q_2 are defined by (43) and (44).

Note that the nonlinear stability result refers to a bottom-heavy vehicle while the spectral stability result refers to a top-heavy vehicle. Note also that equality conditions are not covered by our analysis. Because of symmetry the analogous theorem holds for constant translation along the $\mathbf{b}_1$ axis with the roles of m_1 and m_2, etc., swapped. See Fig. 3 for a summary.

3.2.2. *Stability of equilibrium solution with spin.* In this section we examine the two-parameter family of equilibrium solutions given by (37) which corresponds to a vehicle rising (or falling) along and rotating about the vertical axis and is denoted as

$$\Pi_e = (0, 0, \Pi_3^0)^T, \quad P_e = (0, 0, P_3^0)^T, \quad \Gamma_e = (0, 0, 1)^T, \tag{45}$$

where $\Pi_3^0 = I_3\Omega_{e3}$ and $P_3^0 = m_3 v_{e3}$. We shall assume (for ease of computation) that the $\mathbf{b}_3$ axis is an axis of symmetry, i.e. $I_1 = I_2$ and $m_1 = m_2$. From the equations of motion, (25), this implies that $\dot{\Pi}_3 = 0$, so that any function $\phi(\Pi_3)$ is conserved.

This equilibrium solution is nongeneric, since the equilibrium lives on a four-dimensional coadjoint orbit (rather than a generic six-dimensional coadjoint orbit). As a result, one needs to be careful with applying the energy–Casimir method in this case (Krishnaprasad, 1989).

We consider conserved quantities of the form

$$H_{\Phi,\phi} = H + \Phi(P \cdot \Gamma, \|P\|^2, \|\Gamma\|^2) + \phi(\Pi_3),$$

where H is given by (23) and $\Phi(\cdot)$ and $\phi(\cdot)$ can be chosen as necessary. A straightforward computation as in the previous cases shows that the equilibrium solution (45) is a critical point of $H_{\Phi,\phi}$ and the second variation of $H_{\Phi,\phi}$ evaluated at the equilibrium is (positive-) definite if and only if

$$mgl > \left(\frac{1}{m_3} - \frac{1}{m_2}\right)(P_3^0)^2. \tag{46}$$

Interpret $H_{\Phi,\phi}$ as a Lyapunov function, which we know to be constant along the equations of motion. If (46) holds, the equilibrium solution is a strict minimum of $H_{\Phi,\phi}$, and is therefore stable by Lyapunov's direct method. Thus (46) is a sufficient condition for stability of the equilibrium solution.

The condition (46) can be interpreted as follows. If $l_3 < l_1 = l_2$, i.e. the vehicle is an oblate spheroid, then $m_3 > m_2$ so that the right-hand side of (46) is negative. Therefore vertical motion along the minor axis is stable if the vehicle is bottom-heavy or even slightly top-heavy, i.e. mgl may be slightly negative. If $l_3 > l_1$, i.e. the vehicle is a prolate spheroid, then $m_3 < m_2$ and the right-hand side of (46) is positive. Thus vertical motion along the major axis is stable if the vehicle is *sufficiently* bottom heavy, i.e. if mgl is sufficiently large.

From the linearization of the equations of motion at the equilibrium solution, we find that there is at least one eigenvalue with positive real part, i.e., the motion is unstable, if

$$mgl < \left(\frac{1}{m_3} - \frac{1}{m_2}\right)(P_3^0)^2 - \frac{\bar{a}_2}{4}(\Pi_3^0)^2. \tag{47}$$

When neither (46) nor (47) is satisfied, i.e.

$$\left(\frac{1}{m_3} - \frac{1}{m_2}\right)(P_3^0)^2 - \frac{\bar{a}_2}{4}(\Pi_3^0)^2 < mgl < \left(\frac{1}{m_3} - \frac{1}{m_2}\right)(P_3^0)^2, \tag{48}$$

then the system is spectrally stable.

However, it has been shown in Leonard and Marsden (1996) that the motion is actually nonlinearly stable if (48) holds. The proof makes use of sub-Casimirs and extensions to energy methods to account for the possibility of drift in noncompact directions. In the case of motion along the long axis this implies stability for an insufficiently bottom-heavy vehicle if the angular momentum Π_3^0 is sufficiently high. We note that the condition

$$mgl = \left(\frac{1}{m_3} - \frac{1}{m_2}\right)(P_3^0)^2$$

corresponds to a passing of eigenvalues on the imaginary axis.

We can state the results of this stability analysis of the equilibrium solution with spin as the following theorem.

Theorem 3. (*Stability of rising/falling bottom-heavy vehicle* (Leonard and Marsden, 1996).) Consider an ellipsoidal, neutrally buoyant vehicle with noncoincident centers of gravity and buoyancy and $\mathbf{b}_3$ an axis of symmetry. Constant translation along and rotation about the $\mathbf{b}_3$ axis is (nonlinearly) stable if

$$mgl > \left(\frac{1}{m_3} - \frac{1}{m_2}\right)(P_3^0)^2 - \frac{\bar{a}_2}{4}(\Pi_3^0)^2.$$

If this relation holds with the direction of the inequality sign reversed, the motion is unstable.

See Fig. 4 for a summary. For further details and a more general investigation of stability of nongeneric relative equilibrium solutions see Leonard and Marsden (1996).

3.2.3. *Discussion.* We make a few remarks on interpretation of the stability results (see Figs 2–4 for a summary).

Remark 4. Theorems 1–3 provide conditions for

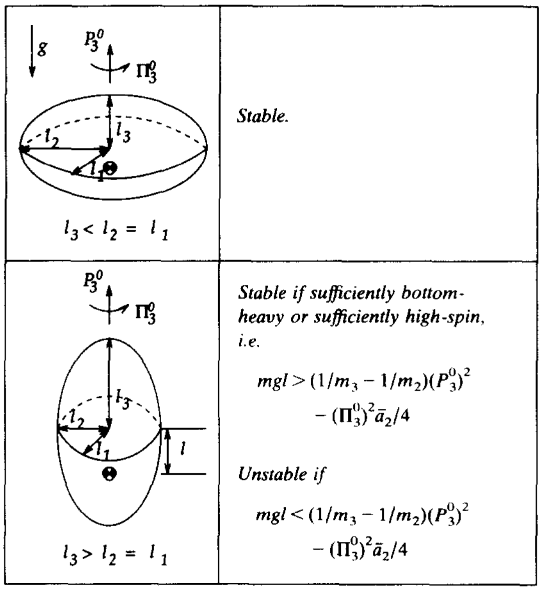

Fig. 4. Summary of stability with noncoincident centers and spin (Theorem 3).

stability in terms of equilibrium velocity and system parameters such as masses and moments of inertia for the body-plus-fluid system and the distance between the centers of gravity and buoyancy. These parameters can be considered as design parameters. For example, given the desired operating range of the vehicle in terms of velocity, one can design the vehicle for good stability properties by applying the conditions of the theorems. Alternatively, these parameters could be interpreted as control parameters, i.e. parameters to be changed on-line if necessary for stability. For example, redistribution of mass inside the vehicle will affect the center of gravity without affecting the center of buoyancy. Consequently, one could change the distance between the two centers as a means of preventing unstable behavior in circumstances that required unusually high-velocity motions. For example, to stabilize a vehicle that must suddenly rise very quickly, one could lower mass to the bottom of the vehicle so that the vehicle is sufficiently bottom-heavy.

Remark 5. As noted in Section 1, the potential flow model that we use in this paper is only practically valid for streamlined motions, i.e. for motions with small viscous effects. For an ellipsoidal body these correspond to translations along or rotations about the body's major axis. In the case of a vehicle with coincident centers of gravity and buoyancy the practically relevant stability result is the condition for stability of translation along the body's major axis. From Theorem 1, pure translation along the major axis is unstable, but can become stable by spinning the body about this axis (assumed to be an axis of symmetry) such that the ratio of angular velocity to translational velocity is sufficiently high. In the case of a bottom-heavy vehicle, from Theorem 3, pure translation along the major axis of the vehicle moving parallel to gravity is stable with sufficient spin velocity and/or a sufficiently low center of gravity. However, from Theorem 2, pure translation along the major axis of the vehicle moving perpendicular to gravity is unstable no matter how far the center of gravity is located below the center of buoyancy. It is therefore of interest to find control forces and torques that will stabilize this motion.

4. CONCLUSIONS

Underwater vehicles for deep sea exploration typically have different performance requirements compared with more traditional submarines. Accordingly, they may be quite differently shaped and equipped. This has consequences for stable and accurate control of motion. On the one hand, requirements for the submersible may preclude including surface such as fins and a tail that on a submarine provide stability. On the other hand, one may have greater opportunity and flexibility in designing various other features such that the vehicle is more easily and stably maneuvered and more efficiently controlled. We believe that nonlinear methods can be used to tackle some of the relevant issues.

In this paper we have provided some groundwork for studying the six-degree-of-freedom underwater vehicle modeled as a rigid body. We have investigated stability of the vehicle submerged in an ideal fluid, focusing on the effects of hydrodynamics and gravity. We have derived the Hamiltonian structure and applied the energy–Casimir method to find conditions for nonlinear stability of equilibrium solutions corresponding to constant translations and rotations.

Since viscous effects are left out of the model, the results are only practically valid for those equilibrium solutions corresponding to streamlined motions. These, however, are the solutions of interest, since low resistance is desirable for high-efficiency motion. We have briefly mentioned the potentially *destabilizing* effect of dissipation; however, a deeper investigation of the effect of dissipation is warranted. For example, it is of interest to know, in the cases where we have shown stability, when the addition of dissipation yields asymptotic stability. Results along the lines of Bloch *et al.* (1994) may provide direction.

For an ellipsoidal bottom-heavy vehicle, motion along the major axis in the plane perpendicular to the direction of gravity was shown to be unstable independent of how far we drop the center of gravity relative to the center of buoyancy. As a result, in order to stabilize this motion, we need to consider applying external forces and torques. In recent work we investigate the use of external forces and torques for stabilization (Leonard, 1996b). In this work the energy–Casimir method proves as useful as it was in deriving a stabilizing control law for the rigid spacecraft (Bloch *et al.*, 1992).

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APPENDIX A. COADJOINT ACTIONS AND ORBITS FOR $SE(3) \times_\rho \mathbb{R}^3$

From Appendix A of Leonard (1996a), the *coadjoint action* of S on $\mathfrak{s}^* = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ has the expression

$$\mathrm{Ad}^*_{(R,b,w)^{-1}}(x, y, z) = (Rx + \hat{b}Ry + \hat{w}Rz, Ry, Rz).$$

The *coadjoint orbit* through $\mu \in \mathfrak{s}^*$, denoted by O_μ, is defined as

$$O_\mu = \{\mathrm{Ad}^*_{(R,b,w)^{-1}}(\mu) \mid (R, b, w) \in S\}.$$

Since $\mathfrak{s}^*$ is nine-dimensional, there can be zero-, two-, four-, six- and eight-dimensional coadjoint orbits. It turns out that there are no eight-dimensional coadjoint orbits. The only zero-dimensional orbit is the origin. We find the four- and six-dimensional coadjoint orbits by applying the semidirect product reduction theorem (Marsden *et al.*, 1984). Let $\bar{G} = SO(3)$ and $\bar{V} = \mathbb{R}^3 \times \mathbb{R}^3$, and let $\bar{\rho}$: $SO(3) \to \mathrm{Aut}(\mathbb{R}^3 \times \mathbb{R}^3)$ be defined by

$$\bar{\rho}(R) \cdot (b, w) = (Rb, Rw), \quad (R, b, w) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Then $\bar{S} = SO(3) \times_{\bar{\rho}} (\mathbb{R}^3 \times \mathbb{R}^3)$ is a semidirect product and $\bar{S} = S$. The dual of $\bar{V} = \mathbb{R}^3 \times \mathbb{R}^3$ can be identified with $\bar{V}$ itself. $\bar{G} = SO(3)$ acts on the dual of $\bar{V}$ as

$$R \cdot (\beta, \gamma) = (R\beta, R\gamma), \quad \beta, \gamma \in \mathbb{R}^3 \times \mathbb{R}^3.$$

The isotropy for the action of $SO(3)$ on $\bar{V}^*$ at $(\beta, \gamma) \in \bar{V}^* = \bar{V}$ is then given by

$$\bar{G}_{(\beta,\gamma)} = \{R \in SO(3) \mid (R\beta, R\gamma) = (\beta, \gamma), \beta, \gamma \in \mathbb{R}^3 \times \mathbb{R}^3\}.$$

In the generic case in which $\beta \times \gamma \neq 0$ (i.e. $\beta \nparallel \gamma$ and $\beta, \gamma \neq 0$),

$$\bar{G}_{(\beta,\gamma)} = I \in SO(3),$$

which is trivially an Abelian group. Following Marsden (1992), the coadjoint orbit through generic $\mu = (\alpha, \beta, \gamma) \in \mathfrak{so}(3)^* \times \mathbb{R}^3 \times \mathbb{R}^3$ is diffeomorphic to

$$T^*(\bar{G}/\bar{G}_{(\beta,\gamma)}) = T^*SO(3),$$

which is a six-dimensional space. In the case that $\beta \parallel \gamma$ (or $\beta = 0$ or $\gamma = 0$),

$$\bar{G}_{(\beta,\gamma)} = \{R \in SO(3) \mid R\beta = \beta\} = S^1,$$

which is again an Abelian group. The coadjoint orbit through $\mu = (\alpha, \beta, \gamma) \in \mathfrak{so}(3)^* \times \mathbb{R}^3 \times \mathbb{R}^3$, $\beta \parallel \gamma$, is then diffeomorphic to

$$T^*(\bar{G}/\bar{G}_{(\beta,\gamma)}) = T^*(SO(3)/S^1) = T^*S^2,$$

which is a four-dimensional space.

APPENDIX B. ADDED MASS AND INERTIA

In this appendix we give the formulas for the elements of the added mass and inertia matrices for a submerged ellipsoidal body. We assume, as in Section 3, that the inerita matrix for the body, J_b, is diagonal. Then

$$M = \operatorname{diag}(m_1, m_2, m_3) = mI + M_{\text{fluid}},\qquad J = \operatorname{diag}(I_1, I_2, I_3) = J_b + J_{\text{fluid}}, \tag{B.1}$$

where

$$M_{\text{fluid}} = \Theta_{11}^f = \operatorname{diag}(\bar{A}, \bar{B}, \bar{C}),$$
$$J_{\text{fluid}} = \Theta_{22}^f = \operatorname{diag}(\bar{P}, \bar{Q}, \bar{R}).$$

Following Lamb (1932), we define

$$\alpha_0 = l_1 l_2 l_3 \int_0^\infty \frac{d\lambda}{(l_1^2+\lambda)\Delta},\quad \beta_0 = l_1 l_2 l_3 \int_0^\infty \frac{d\lambda}{(l_2^2+\lambda)\Delta},$$
$$\gamma_0 = l_1 l_2 l_3 \int_0^\infty \frac{d\lambda}{(l_3^2+\lambda)\Delta},\quad \Delta = \sqrt{(l_1^2+\lambda)(l_2^2+\lambda)(l_3^2+\lambda)}, \tag{B.2}$$

where l_i is the length of the semiaxis of the ellipoidal body along the axis $\mathbf{b}_i$, $i = 1, 2, 3$. Then

$$\bar{A} = \frac{\alpha_0}{2-\alpha_0}\rho_0 V,$$
$$\bar{B} = \frac{\beta_0}{2-\beta_0}\rho_0 V,$$
$$\bar{C} = \frac{\gamma_0}{2-\gamma_0}\rho_0 V,$$
$$\bar{P} = \frac{1}{5}\frac{(l_2^2-l_3^2)^2(\gamma_0-\beta_0)}{2(l_2^2-l_3^2)+(l_2^2+l_3^2)(\beta_0-\gamma_0)}\rho_0 V,$$
$$\bar{Q} = \frac{1}{5}\frac{(l_3^2-l_1^2)^2(\alpha_0-\gamma_0)}{2(l_3^2-l_1^2)+(l_3^2+l_1^2)(\gamma_0-\alpha_0)}\rho_0 V,$$
$$\bar{R} = \frac{1}{5}\frac{(l_1^2-l_2^2)^2(\beta_0-\alpha_0)}{2(l_1^2-l_2^2)+(l_1^2+l_2^2)(\alpha_0-\beta_0)}\rho_0 V, \tag{B.3}$$

where $V = \frac{4}{3}\pi l_1 l_2 l_3$ is the volume of the vehicle and ρ_0 is the density of the fluid.

From these formulas, one can show that $l_i < l_j$ implies that $m_i > m_j$. For example, by (B.1) and (B.3),

$$m_1 - m_3 = \bar{A} - \bar{C} = \frac{2(\alpha_0-\gamma_0)}{(2-\alpha_0)(2-\gamma_0)}\rho_0 V.$$

If $l_1 < l_3$ then, by (B.2), $\alpha_0 > \gamma_0$, and so $m_1 > m_3$. Similarly, if $l_1 > l_3$ then $\alpha_0 < \gamma_0$, and so $m_1 < m_3$.

The ordering of the moments of inertia, I_1, I_2 and I_3, is not so straightforward. Fix l_1 and l_2 and suppose that $l_1 > l_2 > l_3$. For different values of l_3 the ordering of the inertia terms depends upon the ratio l_3/l_2. When l_3/l_2 is close to one, $I_3 > I_2 > I_1$. However, when $l_3/l_2 \ll 1$, i.e. the short semiaxis is very short, $I_2 > I_1 > I_3$. For intermediate values of l_3/l_2, $I_2 > I_3 > I_1$. Note that it will always be true that $I_2 > I_1$; however, I_3 can be the largest, intermediate or smallest moment of inertia. For physical intuition consider an ellipsoid that rotates about its long axis. When this is an axis of symmetry, i.e. $l_2 = l_3$, there is no added inertia about the long axis ($\bar{P} = 0$). So, I_1 is equal to the moment of inertia of the body alone. Now suppose we shrink l_3, i.e. we 'flatten' the ellipsoid. In this case it is harder to spin it about the long axis. In fact, the added inertia $\bar{P}$ about this axis can grow arbitrarily large (and thus I_1 grows very large) as l_3 decreases. At the same time I_2 grows even faster. See Kochen *et al.* (1964) for parametric plots of added inertia terms.