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# Stabilization of underwater vehicle dynamics with symmetry-breaking potentials

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**Abstract**

We show how to stabilize, in stages, arbitrary steady translations of an underwater vehicle with feedback that derives from a potential and deliberately breaks symmetry in the dynamics. First, rotational symmetry is broken to ensure stability in the momentum parameters. Then, translational symmetry is broken to prevent drift. Stability of the closed-loop system is proved using the energy-Casimir method. A resulting property of the control law is robustness to model parameter uncertainty. © 1997 Elsevier Science B.V.

*Keywords:* Stabilization; Underwater vehicle; Symmetry; Potentials; Energy-Casimir; Kirchhoff equations; Hamiltonian

## 1. Introduction

There has been recent interest in developing nonlinear control theory specialized to mechanical systems because traditional nonlinear control methods, which deal with a very large class of systems, do not exploit the rich structure of this smaller but important class of systems. One expects, for example, that making use of the geometric structure of mechanical systems may lead to improvements in robustness and efficiency in control as compared to methods such as feedback linearization which cancel nonlinearities. Recent work on control of mechanical systems includes [3, 6, 8, 9, 15, 16].

A problem of interest for mechanical systems is stabilization of a relative equilibrium, i.e., a steady motion. An example is stabilization of a spacecraft rotating steadily about a principal axis [1, 4, 7]. Another important example is stabilization of constant translations of an underwater vehicle. Here, a basic problem is that translation of a submerged rigid body along any but its short axis is unstable unless there are active or passive means to counter the destabilizing hydrodynamic forces.

Under circumstances in which viscous effects are small, the dominant dynamics of an underwater vehicle can be described by a model of a submerged rigid body in an ideal fluid, namely Kirchhoff's equations. This model describes a mechanical system with symmetry, i.e., the system is Hamiltonian on the group of rigid body motions and there is symmetry in, i.e., invariance to, translations and (possibly only some) rotations. Because of this geometric structure, one can use tools from geometric mechanics to study underwater vehicle dynamics. In [11] we described the geometry of uncontrolled underwater vehicle dynamics and used the energy-Casimir method to investigate stability of steady

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motions. The energy-Casimir method provides a step-by-step means to derive sufficient conditions for Lyapunov stability of a relative equilibrium for a mechanical system with symmetry.

The energy-Casimir method can also be used to design feedback stabilizing control laws. This was demonstrated in the context of stabilizing a rigid spacecraft spinning about its intermediate axis [4] and further generalized in [5, 6]. There, the authors design control laws that effectively change the kinetic energy in the Hamiltonian. That is, the controls are chosen so that the closed-loop system is still Hamiltonian but with a new Hamiltonian that reflects a modified kinetic energy. Because the closed-loop system is still Hamiltonian, the energy-Casimir method can be applied to the closed-loop system to determine conditions on the control gains that yield closed-loop stability.

In this paper, feedback stabilizing control laws are designed for the underwater vehicle problem by means of the energy-Casimir method (see also [10]). However, here, the control law is chosen in order to effectively change or add *potential* energy to the Hamiltonian rather than kinetic energy. Further, the potential energy terms added by means of control are designed to break system symmetry. Symmetry-breaking potentials can be used not only to stabilize the momentum variables of the system, but also, unlike modifications to kinetic energy, they can be used to prevent drift in the remaining symmetry directions, i.e., in the rotational and translational parameters. An example is the problem of stabilizing spin about the intermediate axis of a rigid spacecraft. In this case, the idea would be to provide a control law that effectively makes the rigid spacecraft look like a hanging top, i.e., to choose a control law that is the torque derived from a potential that looks similar to gravitational potential, such that rotational symmetry is broken in all directions except about the desired axis of rotation. By introducing attitude variables into the control law, this approach allows one not only to stabilize the spin but also to prescribe how the spacecraft is oriented in inertial space.

The structure of the control law is inspired by the kind of stabilizing (restoring) moment that is naturally produced when an underwater vehicle is bottom heavy, i.e., when the center of gravity is lower than the center of buoyancy. As shown in [11], a sufficiently low center of gravity ensures stability of a steadily rising or falling vehicle. Thus, to stabilize an otherwise unstable translation along an axis that is not parallel to gravity, the idea is to choose a feedback that mimics the naturally stabilizing moment. This means adding a control term that derives from a potential that mimics the potential energy due to buoyant and gravitational forces and so doing breaks rotational symmetry.

As indicated above, even when stabilization has been achieved there still may be the possibility of drift in the remaining symmetry directions. This is particularly of issue in the case of systems with a noncompact configuration space, such as the space of rigid body motions, as was shown in [12]. For example, the stable, bottom-heavy rising vehicle may drift in translation in directions transverse to gravity. However, this too can be prevented with the addition of potentials (e.g., resembling spring potentials) that break the appropriate translational symmetry.

The control laws developed in this paper provide Lyapunov stability of an arbitrary constant translation of an underwater vehicle with arbitrary prescribed orientation. With the addition of dissipative control, one could then achieve asymptotic stability (drag forces for the underwater vehicle problem will help in this regard). Further, we note that the control laws developed in this paper are robust to uncertainties in model parameters in the spirit of the work of Zhao and Posbergh [17].

The paper is organized as follows. In Section 2, we describe underwater vehicle dynamics. Stability is discussed in Section 3 and stabilizing control laws derived from symmetry-breaking potentials are presented in Section 4. We give conclusions in Section 5.

## 2. Underwater vehicle dynamics

We describe underwater vehicle dynamics for a six degree-of-freedom vehicle modeled as a neutrally buoyant, submerged rigid body in an infinitely large volume of irrotational, incompressible, inviscid fluid that is at rest at infinity. We do not make any assumptions on the shape of the vehicle nor do we require the mass of the vehicle to be distributed uniformly, i.e., we assume that the center of buoyancy and center of gravity are *not* coincident. This latter assumption is of practical interest since underwater vehicles are typically built with a relatively low center of gravity for stability.

We fix an orthonormal coordinate frame $(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$ to the body with origin at the body's center of buoyancy and let $(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ be an inertial frame with $\boldsymbol{r}_3$ pointing "down", i.e., in the direction of gravity. The configuration space of the vehicle is the Euclidean group, $SE(3)$, which globally describes rigid body positions and orientations in three-dimensional space. An element in $SE(3)$ is given by $X = (R, b)$. $R \in SO(3)$ is the rotation matrix that maps body coordinates into inertial coordinates and describes the orientation of the vehicle. $b$ is the vector from the origin of the inertial frame to the origin of the body frame and describes the position of the vehicle. If we define $\Omega$ and $v$, respectively, to be angular and translational velocity of the vehicle given in body coordinates, then

$$\dot{R} = R\hat{\Omega}, \quad \dot{b} = Rv,$$

where $\hat{\Omega}y = \Omega \times y$ for $y \in \mathbb{R}^3$.

We note that $SE(3) = SO(3) \circledS \mathbb{R}^3$ is an example of a semidirect product. If $G$ is a Lie group that acts linearly on a vector space $V$ and $G \times V$ is given the group structure defined by $(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, g_1 v_2 + v_1)$, with $(g_1, v_1), (g_2, v_2) \in G \times V$, then $G \times V$ is a Lie group called a semidirect product and denoted $G \circledS V$.

Let $\{e_1, e_2, e_3\}$ be the standard Euclidean basis for $\mathbb{R}^3$ and define

$$\Delta = R^{\mathrm{T}} e_1, \quad \Sigma = R^{\mathrm{T}} e_2, \quad \Gamma = R^{\mathrm{T}} e_3,$$

so that $R^{\mathrm{T}} = (\Delta, \Sigma, \Gamma)$. $\Gamma$ is the unit vector pointing in the direction of gravity expressed in body coordinates. Let the vector from the center of buoyancy to the center of gravity be $lr$ where $l$ is a scalar and $r$ is a unit vector.

Define $\Pi$ and $P$, respectively, to be the angular and linear components of the *impulse* of the system. Roughly, impulse refers to the finite part of the momentum of the system, but we will use the terms impulse and momentum interchangeably. Let

$$\xi = \begin{pmatrix} \Omega \\ v \end{pmatrix}, \quad \nu = \begin{pmatrix} \Pi \\ P \end{pmatrix}, \quad \mathbb{M} = \begin{pmatrix} J & D \\ D^{\mathrm{T}} & M \end{pmatrix}.$$

Then the relationship between velocity and momentum is given by

$$\nu = \mathbb{M}\xi.$$

$J$ is the matrix that is the sum of the body inertia matrix plus the *added inertia matrix* associated with the potential flow model of the fluid. Similarly, $M$ is the sum of the mass matrix for the body alone, i.e., the mass of the body $m$ multiplied by the identity matrix, plus the *added mass matrix* associated with the fluid (note that $M$ itself is not a multiple of the identity unless the body is symmetric). $D$ accounts for cross terms. The body axes can always be chosen so that $M$ is diagonal, i.e., one chooses them to be the principal axes of the added mass ellipsoid.

Kirchhoff showed that the kinetic energy of the body–fluid system is given by $\frac{1}{2}\xi \cdot \mathbb{M}\xi$ where $\mathbb{M}$ is positive definite so that the dynamic equations of motion (Kirchhoff's equations) become

$$\begin{aligned} \dot{\Pi} &= \Pi \times \Omega + P \times v + \mathscr{T}, \\ \dot{P} &= P \times \Omega + \mathscr{F}, \end{aligned} \tag{1}$$

where $\mathscr{T}$ and $\mathscr{F}$ are external torques and forces. In the absence of external forces and torques (this would require coincident centers of buoyancy and gravity), these dynamics can be viewed as Lie–Poisson dynamics on $\mathfrak{se}(3)^*$ [2, 11]; $\mathfrak{se}(3)^*$ is the dual of the Lie algebra of $SE(3)$, and $\nu = (\Pi, P)^{\mathrm{T}}$ is an element in $\mathfrak{se}(3)^*$.

The Lie–Poisson dynamics on $\mathfrak{se}(3)^*$ can be interpreted as reduced dynamics starting from the dynamics on the system phase space $T^*SE(3)$, reduced by the symmetry group $SE(3)$. Given a differentiable manifold $Y$, $T^*Y$ is the cotangent bundle of $Y$, i.e., the disjoint union of the cotangent spaces to $Y$ at the points $y \in Y$. Thus, $T^*SE(3)$ is the collection of all configuration and conjugate momentum pairs for a rigid body in three-dimensional space. Symmetry in this context means that the Hamiltonian that describes the dynamics in $T^*SE(3)$ is invariant to actions of $SE(3)$, i.e., one can translate the inertial frame or rotate it in any direction and not affect the equations of motion. The equations of motion (1) are equivalent to the Lie–Poisson equations $\dot{\nu}_i = \{\nu_i, H_c\}$ where the bracket is the reduced Poisson bracket on $se(3)^*$, and the reduced Hamiltonian $H_c$ is equal to the kinetic energy, i.e.,

$$H_c(\nu) = \tfrac{1}{2}\nu \cdot \mathbb{M}^{-1}\nu.$$

In the case of interest where the center of buoyancy and the center of gravity are noncoincident (and there are no other external forces or torques), Kirchhoff's equations need to include the torque due to the buoyant and gravitational force pair as

$$\dot{\Pi} = \Pi \times \Omega + P \times v - mgl\Gamma \times r,$$
$$\dot{P} = P \times \Omega, \qquad (2)$$
$$\dot{\Gamma} = \Gamma \times \Omega.$$

These equations can also be viewed as Lie–Poisson dynamics; however, this time on $\mathfrak{w}^*$, the dual of the Lie algebra of the double semidirect product $W = SO(3) ⓢ (\mathbb{R}^3 \times \mathbb{R}^3)$ [11]. $W$ is called a double semidirect product since from [11], $W = SE(3) ⓢ \mathbb{R}^3$, i.e., a semidirect product of a semidirect product and a vector space. The Lie–Poisson dynamics on $\mathfrak{w}^*$ can be interpreted as reduced dynamics starting from the dynamics on $T^*SE(3)$, reduced by the symmetry group $SE(2) \times \mathbb{R}$. This smaller symmetry group (relative to the case of coincident centers) reflects the fact that the torque due to gravity and buoyancy has broken some of the rotational symmetry, i.e., one can still translate the inertial frame but now only rotate it about the direction of gravity without affecting the equations of motion. Let $\mu = (\Pi, P, \Gamma) \in \mathfrak{w}^*$. The Poisson bracket on $\mathfrak{w}^*$ is $\{F, K\}(\mu) = \nabla F^{\mathrm{T}} \Lambda(\mu) \nabla K$ where $F$ and $K$ are smooth functions on $\mathfrak{w}^*$ and the Poisson tensor $\Lambda$ is given by

$$\Lambda(\mu) = \begin{pmatrix} \hat{\Pi} & \hat{P} & \hat{\Gamma} \\ \hat{P} & 0 & 0 \\ \hat{\Gamma} & 0 & 0 \end{pmatrix}.$$

The equations of motion (2) are equivalent to $\dot{\mu}_i = \{\mu_i, H\}$, i.e., $\dot{\mu} = \Lambda(\mu)\nabla H(\mu)$, with reduced Hamiltonian $H$ equal to the kinetic plus potential energy, i.e.,

$$H(\mu) = \tfrac{1}{2} v \cdot \mathbb{M}^{-1} v - mgl\Gamma \cdot r.$$

Three independent Casimir functions $C_i : \mathfrak{w}^* \to \mathbb{R}$ are $C_1 = P \cdot \Gamma$, $C_2 = \|P\|^2$, $C_3 = \|\Gamma\|^2$. These are functions which Poisson commute with any function $K$ on $\mathfrak{w}^*$, i.e., $\{C_i, K\} = 0$, and thus, are conserved quantities along the equations of motion. A point in reduced space is called *generic* if the Poisson tensor has maximum rank; otherwise, it is called *nongeneric*. Here, $\mu$ is generic as long as $P \nparallel \Gamma$.

## 3. Stability

The main tool that we use to analyze stability is the energy-Casimir method, applicable to Lie Poisson systems. We need such tools for Hamiltonian systems since linearization techniques cannot be used to prove stability. This is because a stable Hamiltonian system will have all its eigenvalues on the imaginary axis.

The energy-Casimir method provides sufficient conditions for nonlinear (Lyapunov) stability of a generic equilibrium $\mu_e$ in the reduced dynamics. This establishes nonlinear stability of the corresponding relative equilibrium $z_e$ in phase space modulo the symmetry group $G$ used in reduction. Stability modulo $G$ is just the usual notion of Lyapunov stability except that one allows arbitrary drift along the orbits of $G$. For the uncontrolled underwater vehicle, a *relative equilibrium* $z_e$ is a solution of the dynamics in $T^*SE(3)$ which corresponds to an *equilibrium point* $\mu_e = (\Pi_e, P_e, \Gamma_e) \in \mathfrak{w}^*$, i.e., relative equilibria are steady translations and rotations.

The energy-Casimir method consists of finding a (Lyapunov) function $H_{\Phi,\phi} = H + \Phi(C_i) + \phi_j(c_j)$, where $H$ is the Hamiltonian, $C_i$ are Casimirs, $c_j$ are other constants of motion and $\Phi(\cdot)$ and $\phi_j(\cdot)$ are smooth functions to be determined. The condition that $\Phi(\cdot)$ and $\phi_j(\cdot)$ can be found so that $\mu_e$ is a critical point of $H_{\Phi,\phi}$ and the second derivative of $H_{\Phi,\phi}$ is definite at $\mu_e$, is sufficient for Lyapunov stability of $\mu_e$ [14].

In [12], stability theorems were derived that extend the energy-Casimir method to predict stability modulo a group smaller than the symmetry group and to allow for nongeneric equilibria such as the steadily rising or falling vehicle. In general, steady translation with no spin along any but the short axis of a vehicle is unstable. However, the extended theorems of [12] helped to show that steady translation along even the long axis of an ellipsoidal vehicle can be made stable by design if the body is translating parallel to gravity (i.e., rising or falling) and the center of gravity is sufficiently far below the center of buoyancy. The stabilizing moment due to gravity and buoyancy derives from a potential that breaks rotational symmetry except about the direction of gravity. Stability is modulo $SE(2) \times \mathbb{R}$, i.e., the body is stable in the momentum parameters but may rotate about the axis of motion or may drift in any translational direction.

## 4. Stabilization

Inspired by the naturally stabilizing effect of gravity and buoyancy in the case of a rising or falling underwater vehicle, we derive control laws in this section using symmetry-breaking potentials that stabilize arbitrary constant vehicle translations without spin. The desired translation does not have to be a steady motion for the uncontrolled dynamics; it will be turned into a stable steady motion for the closed-loop system. The desired motion will be denoted by variables with subscript d, e.g., the desired velocity in body coordinates $v_d \in \mathbb{R}^3$ is assumed to be given and the desired body angular velocity is taken to be $\Omega_d = 0$ (no spin). It will also be of interest to consider a desired orientation of the vehicle given by $R_d \in SO(3)$.

We assume that control torques $u^\tau \in \mathbb{R}^3$ and forces $u^f \in \mathbb{R}^3$ can be provided by available actuators such as propellers, so that the system dynamics with control become

$$\begin{aligned}
\dot{\Pi} &= \Pi \times \Omega + P \times v - mgl\Gamma \times r + u^\tau, \\
\dot{P} &= P \times \Omega + u^f, \\
\dot{R} &= R\hat{\Omega}, \\
\dot{b} &= Rv.
\end{aligned} \tag{3}$$

*4.1. Stabilization in momentum space*

Stabilization of the desired steady motion is achieved in this section modulo the possibility of translational drift. Let

$$\hat{k} = R_d v_d / \|v_d\|$$

be the unit vector that describes the desired vehicle velocity with respect to the inertial frame. We assume that $\hat{k} \neq e_3$. In the case that $R_d v_d \| e_3$, the desired motion corresponds to a rising or falling vehicle. This case is discussed in Remark 4.3. Define $\Theta$ as the vector $\hat{k}$ in body coordinates, i.e.,

$$\Theta = R^T \hat{k}.$$

Consider the control law

$$\begin{aligned}
u^\tau &= \Theta \times (M - \alpha I) v_d \|v_d\| - mgl\Gamma \times (\beta \Gamma_d - r), \\
u^f &= 0,
\end{aligned} \tag{4}$$

where $\alpha$ and $\beta$ are scalar control gains to be determined. As shown in the next lemma, this control law derives from a symmetry-breaking potential. The first term in the control law comes from a potential that breaks all remaining rotational symmetry. The second term comes from a potential that adds to the existing gravitational potential as needed to hold the vehicle in the desired orientation (no symmetry broken with this term).

**Lemma 4.1.** *The closed-loop dynamics* (3) *with control given by* (4) *describe a system that is Hamiltonian on $T^*SE(3)$ with symmetry group $\mathbb{R}^3$. Reduction by this symmetry group yields Lie–Poisson dynamics on $\mathfrak{s}^*$, the dual of the Lie algebra of the semidirect product $S = SO(3)$ⓢ$(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$. The reduced space $\mathfrak{s}^*$ is parametrized by $\mu_s = (\Pi, P, \Gamma, \Theta)$ and the reduced dynamics are*

$$\begin{aligned}
\dot{\Pi} &= \Pi \times \Omega + P \times v + \Theta \times (M - \alpha I) v_d \|v_d\| \\
&\quad - mgl\beta\Gamma \times \Gamma_d, \\
\dot{P} &= P \times \Omega, \\
\dot{\Gamma} &= \Gamma \times \Omega, \\
\dot{\Theta} &= \Theta \times \Omega.
\end{aligned} \tag{5}$$

*These dynamics correspond to $\dot{\mu}_s = \Lambda_s(\mu_s)\nabla H_s(\mu_s)$ where*

$$\Lambda_s(\mu_s) = \begin{pmatrix} \hat{\Pi} & \hat{P} & \hat{\Gamma} & \hat{\Theta} \\ \hat{P} & 0 & 0 & 0 \\ \hat{\Gamma} & 0 & 0 & 0 \\ \hat{\Theta} & 0 & 0 & 0 \end{pmatrix}$$

*and*

$$\begin{aligned}
H_s(\mu_s) &= \tfrac{1}{2} v \cdot \mathbb{M}^{-1} v + \Theta \cdot (M - \alpha I) v_d \|v_d\| \\
&\quad - mgl\beta\Gamma \cdot \Gamma_d.
\end{aligned} \tag{6}$$

*Six independent Casimirs $C_i$: $\mathfrak{s}^* \to \mathbb{R}$ are $P \cdot \Gamma$, $P \cdot \Theta$, $\Gamma \cdot \Theta$, $\|P\|^2$, $\|\Gamma\|^2$, $\|\Theta\|^2$.*

**Proof.** By analogy to the description of the open-loop system dynamics [11], the closed-loop system can be described as a system on $SE(3)$ with kinetic energy $T = \frac{1}{2}\xi \cdot \mathbb{M}\xi$ and potential energy

$$V = R^T\hat{k} \cdot (M - \alpha I) v_d \|v_d\| - mgl\beta(R^T e_3) \cdot r.$$

The Lagrangian $L: TSE(3) \to \mathbb{R}$ is $L(R, b, R\hat{\Omega}, Rv) = T - V$. $L$ is left-invariant under the action of the group

$$G = \{(R, b) \in SE(3) \mid R^T e_3 = e_3,\ R^T\hat{k} = \hat{k}\} = \mathbb{R}^3$$

since $e_3 \nparallel \hat{k}$. The corresponding Hamiltonian system on $T^*SE(3)$ is also necessarily left-invariant to the action of $\mathbb{R}^3$. Reduction by $\mathbb{R}^3$ is possible using the semidirect product reduction theorem of [13]. This yields the Lie–Poisson dynamics of the lemma. □

One can check that the desired motion in reduced space,

$$(\Pi_{\rm d}, P_{\rm d}, \Gamma_{\rm d}, \Theta_{\rm d}) = (Dv_{\rm d}, Mv_{\rm d}, R_{\rm d}e_3, v_{\rm d}/\|v_{\rm d}\|) \in \mathfrak{s}^*, \quad (7)$$

is a steady motion for the closed-loop dynamics (5). By Lemma 4.1, the closed-loop dynamics are Lie–Poisson, and so we can apply the energy-Casimir method to the closed-loop system to determine conditions on the control gains $\alpha$, $\beta$ that make the desired motion (7) a stable relative equilibrium.

**Theorem 4.2.** *The equilibrium solution* (7) *is stable for the closed-loop system* (5) *if* $\alpha$ *and* $\beta$ *satisfy*

$$\alpha I - M > 0 \quad \textit{and} \quad l\beta > 0. \quad (8)$$

*Stability in* $T^*SE(3)$ *is modulo* $\mathbb{R}^3$.

**Proof.** The theorem follows by application of the energy-Casimir method to the conserved quantity

$$H_\Phi = H_{\rm s} + \Phi(P\cdot\Gamma, P\cdot\Theta, \Gamma\cdot\Theta, \|P\|^2, \|\Gamma\|^2, \|\Theta\|^2),$$

where $H_{\rm s}$ is given by (6). □

**Remark 4.3.** The case of a vehicle with coincident centers is an interesting special case in which stabilization by breaking rotational symmetry may be handled in two stages. Recall that the uncontrolled vehicle with coincident centers has full rotational and translational symmetry. To stabilize a desired steady translation, first consider applying the control torque given by (4) with $l = 0$. This will break rotational symmetry except about the desired axis of motion. The closed-loop system will have $SE(2)\times\mathbb{R}$ symmetry and the reduced dynamics will be Lie–Poisson on $\mathfrak{w}^*$. One can then use the energy-Casimir method to determine conditions on $\alpha$ for stability. If the desired translation is along one of the body axes, i.e., $P_{\rm d}\,\|\,\Theta_{\rm d} = v_{\rm d}/\|v_{\rm d}\|$, then the equilibrium is nongeneric and analogous to the rising/falling bottom-heavy vehicle. As in the case of the rising/falling vehicle, using the extended stability theorems of [12] one could find $\alpha$ such that the dynamics are stable modulo $SE(2)\times\mathbb{R}$, i.e., there may be rotational drift about (but not away from) the desired axis of motion as well as translational drift. If rotational drift is unacceptable, one could include a second control torque term derived from a potential that breaks the remaining rotational symmetry. In the case of the bottom-heavy rising/falling vehicle the first control torque term can be provided by gravity and buoyancy with a sufficiently low center of gravity.

**Remark 4.4.** One can also use control laws derived from (gyroscopic) terms that do not break symmetry, but do stabilize the desired motion modulo a possibly larger group. Such an alternative is described in [10]. In this case, the control law is a function of $P$ instead of $\Theta$. Having such an alternative gives one a choice of which dynamic variables one needs to measure for feedback.

**Remark 4.5.** The feedback control law (4) is robust to uncertainty in the model parameters. That is, we can choose values for control gains $\alpha$ and $\beta$ so that even if we are incorrect in our knowledge of the model parameters, the resulting motion of the closed-loop system will stay close to the desired motion. This can be understood as follows. First, suppose that we choose our control law using a nominal model, i.e., our best guess at the model parameters. We will denote the nominal parameters with a subscript 0. The control law looks like

$$u^\tau = \Theta\times(M_0 - \alpha I)v_{\rm d}\|v_{\rm d}\| - m_0 g l_0 \Gamma\times(\beta\Gamma_{\rm d} - r_0).$$

The closed-loop system takes the same form as (5) except that

$$\dot\Pi = \Pi\times\Omega + P\times v + \Theta\times(M - \alpha I)v_{\rm d}^*\|v_{\rm d}^*\| - m_0 g l_0 \beta\Gamma\times\Gamma_{\rm d}^*, \quad (9)$$

where

$$v_{\rm d}^*\|v_{\rm d}^*\| = (M - \alpha I)^{-1}(M_0 - \alpha I)v_{\rm d}\|v_{\rm d}\|,$$

$$\Gamma_{\rm d}^* = \frac{1}{\beta}\left(\frac{ml}{m_0 l_0}r - r_0\right) + \Gamma_{\rm d},$$

i.e., $v_{\rm d}$, $\Gamma_{\rm d}$ are replaced by $v_{\rm d}^*$, $\Gamma_{\rm d}^*$ and $mgl$ is replaced by $m_0 g l_0$. It follows that the motion

$$(\Pi, P, \Gamma, \Theta) = (Dv_{\rm d}^*, Mv_{\rm d}^*, \Gamma_{\rm d}^*, v_{\rm d}^*/\|v_{\rm d}^*\|) \in \mathfrak{s}^* \quad (10)$$

is a relative equilibrium of the closed-loop dynamics (5) with (9). For the generic case in which $Mv_{\rm d}^*$, $v_{\rm d}^*$, $\Gamma_{\rm d}^*$ are not all parallel, from Theorem 4.2, (10) is a stable equilibrium for the closed-loop system (5) with (9) if we choose $\alpha$ and $\beta$ to satisfy (8). This can be done if an upper bound on the size of $M$ (e.g., maximum singular value of $M$) and the sign of $l$ are known. Note, further, that $v_d^* \to v_d$ as $\alpha \to \infty$ and $\Gamma_d^* \to \Gamma_d$ as $\beta \to \infty$. Thus, we can choose $\alpha$ and $\beta$ large enough to ensure that the stabilized motion (10) is close enough to the desired motion (7).

### 4.2. Preventing translational drift

In this section we augment the control law from the previous section in order to prevent translational drift away from the desired motion. The added control term derives from a potential that breaks translational symmetry in directions transverse to the direction of desired motion. Let $J = \mathrm{diag}(1,1,0)$ and let $Q \in SO(3)$ satisfy

$$QR_d v_d = e_3 \|v_d\|.$$

$Q$ maps inertial frame vectors into vectors described in terms of a frame in which the third axis corresponds to the desired direction of motion. Define.

$$\bar{b} = Qb, \quad \tilde{b} = JQb =: J\bar{b},$$

i.e., $\bar{b}$ describes $b$ is this new frame and $\tilde{b}$ replaces the third component of $\bar{b}$ with zero. Thus,

$$\dot{\tilde{b}} = JQRv,$$

and so $\tilde{b}(t) = \tilde{b}(0)$ when $R = R_d$ and $v = v_d$. Without loss of generality, we will take $b(0) = 0$ so that the desired translational displacement is given by $\tilde{b}_d = \tilde{b}(0) = 0$, i.e., the vehicle should translate only in the desired direction of motion.

Consider the control law

$$\begin{aligned} u^r &= \Theta \times (M - \alpha I)v_d \|v_d\| - mgl\Gamma \times (\beta\Gamma_d - r), \\ u^f &= -R^T Q^T JK\tilde{b}, \end{aligned} \tag{11}$$

where $K$ is a $3\times 3$ control gain matrix and $\alpha$, $\beta$ are scalar control gains, all to be determined. As shown in the next lemma, the new control force derives from a symmetry-breaking potential that looks like a spring potential and breaks translational symmetry.

**Lemma 4.6.** *The closed-loop dynamics* (3) *with control given by* (11) *are Hamiltonian on* $T^*SE(3)$ *with symmetry group* $\mathbb{R}$. *Reduction by this symmetry group yields Lie–Poisson dynamics on* $\mathfrak{n}^*$, *the dual of the Lie algebra of the semidirect product* $N = SE(3) \circledS (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$. *The reduced space* $\mathfrak{n}^*$ *is parametrized by* $\mu_n = (\Pi, P, \Delta, \Sigma, \Gamma, \tilde{b})$ *and the reduced dynamics are*

$$\begin{aligned} \dot{\Pi} &= \Pi \times \Omega + P \times v + \Theta \times (M - \alpha I)v_d \|v_d\| \\ &\quad - mgl\beta\Gamma \times \Gamma_d, \\ \dot{P} &= P \times \Omega - R^T Q^T JK\tilde{b}, \\ \dot{\Delta} &= \Delta \times \Omega, \\ \dot{\Sigma} &= \Sigma \times \Omega, \\ \dot{\Gamma} &= \Gamma \times \Omega, \\ \dot{\tilde{b}} &= JQRv, \end{aligned} \tag{12}$$

*where* $R^T = (\Delta, \Sigma, \Gamma)$ *and* $\Theta = R^T R_d v_d / \|v_d\| = R^T Q^T e_3 = Q_{31}\Delta + Q_{32}\Sigma + Q_{33}\Gamma$. *These dynamics correspond to* $\dot{\mu}_n = \Lambda_n(\mu_n)\nabla H_n(\mu_n)$, *where*

$$\Lambda_n(\mu_n) = \begin{pmatrix} \hat{\Pi} & \hat{P} & \hat{\Delta} & \hat{\Sigma} & \hat{\Gamma} & 0 \\ \hat{P} & 0 & 0 & 0 & 0 & -R^T Q^T J \\ \hat{\Delta} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Sigma} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & JQR & 0 & 0 & 0 & 0 \end{pmatrix}$$

*and*

$$\begin{aligned} H_n(\mu_n) &= \tfrac{1}{2} v \cdot \mathbb{M}^{-1} v + \Theta \cdot (M - \alpha I)v_d \|v_d\| \\ &\quad - mgl\beta\Gamma \cdot \Gamma_d + \tfrac{1}{2}\tilde{b} \cdot K\tilde{b}. \end{aligned} \tag{13}$$

*Seven independent Casimirs* $C_i\colon \mathfrak{n}^* \to \mathbb{R}$ *are* $\Delta \cdot \Sigma, \Delta \cdot \Gamma, \Sigma \cdot \Gamma, \|\Delta\|^2, \|\Sigma\|^2, \|\Gamma\|^2, e_3^T QRP$.

**Proof.** Here, the closed-loop system can be described as a system on $SE(3)$ with kinetic energy $T = \frac{1}{2}\xi \cdot \mathbb{M}\xi$ and potential energy

$$\begin{aligned} V &= R^T\hat{k} \cdot (M - \alpha I)v_d\|v_d\| - mgl\beta(R^T e_3) \cdot r \\ &\quad + \tfrac{1}{2}(JQb) \cdot KJQb. \end{aligned}$$

The Lagrangian $L\colon TSE(3) \to \mathbb{R}$ is $L(R, b, R\hat{\Omega}, Rv) = T - V$. $L$ is left-invariant under the action of the group

$$G = \{(R, b) \in SE(3) \mid R^T e_3 = e_3,\ R^T\hat{k} = \hat{k},\ JQb = 0\} = \mathbb{R}.$$

The corresponding Hamiltonian system on $T^*SE(3)$ is also necessarily left-invariant to the action of $\mathbb{R}$. Reduction by $\mathbb{R}$ is possible using the semidirect product reduction theorem of [13]. This yields the Lie–Poisson dynamics of the lemma. $\square$

One can check that desired motion in reduced space,

$$(\Pi_\mathrm{d}, P_\mathrm{d}, \Delta_\mathrm{d}, \Sigma_\mathrm{d}, \Gamma_\mathrm{d}, \tilde{b}_\mathrm{d}) = (Dv_\mathrm{d}, Mv_\mathrm{d}, R_\mathrm{d}e_1, R_\mathrm{d}e_2, R_\mathrm{d}e_3, 0) \in \mathfrak{n}^*, \tag{14}$$

is a steady motion for the closed-loop dynamics (12). By Lemma 4.6, the closed-loop dynamics are Lie–Poisson, and so we can apply the energy-Casimir method to the closed-loop system to determine conditions on the control gains that make the desired motion (14) a stable relative equilibrium.

**Theorem 4.7.** *The equilibrium solution* (14) *is stable for the closed-loop system* (12) *if* $\alpha$, $\beta$ *and* $K$ *satisfy*

$$\alpha I - M > 0, \quad l\beta > 0 \quad \text{and} \quad K > 0. \tag{15}$$

*Stability in* $T^*SE(3)$ *is modulo* $\mathbb{R}$.

**Proof.** The theorem follows by application of the energy-Casimir method to the conserved quantity

$$H_\Phi = H_\mathrm{n} + \Phi(\Delta \cdot \Sigma, \Delta \cdot \Gamma, \Sigma \cdot \Gamma, \|\Delta\|^2, \|\Sigma\|^2, \|\Gamma\|^2, e_3^\mathrm{T} QRP),$$

where $H_\mathrm{n}$ is given by (13). □

From Theorem 4.7, since stability is modulo $\mathbb{R}$, there is no longer the possibility of drift in directions transverse to the direction of motion.

## 5. Conclusions

A technique which uses symmetry-breaking potentials to stabilize relative equilibria has been proposed and applied to the dynamics of an underwater vehicle. Future plans include coupling the method of symmetry-breaking potentials with the method of modifying the kinetic energy which together will allow more flexibility in the nature of the control law and the number and type of actuators needed for control. It is also of interest to consider extensions to tracking.

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