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STABLE SYNCHRONIZATION OF MECHANICAL SYSTEM NETWORKS*

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Abstract. In this paper we address stabilization of a network of underactuated mechanical systems with unstable dynamics. The coordinating control law stabilizes the unstable dynamics with a term derived from the method of controlled Lagrangians and synchronizes the dynamics across the network with potential shaping designed to couple the mechanical systems. The coupled system is Lagrangian with symmetry, and energy methods are used to prove stability and coordinated behavior. Two cases of asymptotic stabilization are discussed; one yields convergence to synchronized motion staying on a constant momentum surface, and the other yields convergence to a relative equilibrium. We illustrate the results in the case of synchronization of n carts, each balancing an inverted pendulum.

Key words. coordinated control, mechanical systems, networks, synchronization, stabilization, energy shaping

AMS subject classifications. 70Q05, 70H33, 93D15

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1. Introduction.

Coordinated motion and cooperative control have become important topics of late because of growing interest in the possibility of faster data processing and more efficient decision-making by a network of autonomous systems. For example, mobile sensor networks are expected to provide better data about a distributed environment if the sensors can be made to cooperate towards optimal coverage and efficient coordination.

Much of the recent work explores coordination and cooperative control with very simple dynamical systems, e.g., single or double integrator models (see, e.g., [10, 17, 18]) or nonholonomic models (see, e.g., [4]). For example, in some of these and related works, stabilization of coordinated group dynamics is studied in the case of limited, time-varying communication topologies. These authors deliberately choose to focus on the coordination issues independently of issues in the stabilization of individual dynamics.

However, for networks of autonomous systems such as unmanned helicopters or underwater vehicles, stability of individual dynamics can be important and challenging, and it may not always be possible (or desirable) to decouple the stabilization problem of individual dynamics from the coordination problem. In [22] the authors consider stability of a group with dynamics that satisfy a leader-to-formation stability (LFS) condition based on input-to-state stability [20]. Examples include linear dynamical systems and kinematic nonholonomic robots; in the latter case feedback linearization is used for stabilization. Using the LFS property, the authors are able to quantify how leader inputs and disturbances affect group stability. In [6], an extension to the previous work of [5] on unmanned aerial vehicle motion planning is presented

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for identical multiple-vehicle stabilization and coordination. The single-vehicle motion planning is based on the interconnection of a finite number of suitably defined motion primitives. The problem is set in such a way that multiple-vehicle motion coordination primitives are obtained from the single-vehicle primitives. The technique is applied to motion planning for a group of small model helicopters.

Networks of rigid bodies are addressed in [8]. Reduction theory is applied in the case that control inputs depend only on relative configuration (relative orientation or position). The reduction results are used to study coordinated behavior of satellite and underwater vehicle network dynamics. Stability of a network of rotating rigid satellites and a network of coordinated underwater vehicles is proved in [14, 15].

In this paper, we investigate the problem of coordination of a network of underactuated mechanical systems with unstable dynamics. As a first step we make use of the method of controlled Lagrangians to stabilize the unstable dynamics of each mechanical system. The method of controlled Lagrangians and the equivalent interconnection and damping assignment passivity-based control (IDA-PBC) method use energy shaping for stabilization of underactuated mechanical systems (see [1, 19] and references therein). The method of controlled Lagrangians provides a control law for underactuated mechanical systems such that the closed-loop dynamics derive from a Lagrangian. The approach is to choose the control law to shape the controlled kinetic and potential energy for stability.

The class of underactuated mechanical systems we consider in this paper satisfies the *simplified matching conditions* (SMC) defined in [2, 1]. This class includes the planar or spherical inverted pendulum on a (controlled) cart. The goal of the development in this paper is to stabilize unstable dynamics for each individual mechanical system in the network and stably synchronize the actuated configuration variables across the network. For example, for a network of pendulum/cart systems, the problem is to stabilize each pendulum in the upright position while synchronizing the motion of the carts.

For stabilization of individual unstable dynamics we use the approach in [1]. To simultaneously synchronize the dynamics across the network, we show that potentials that couple the individual systems can be prescribed so that the complete coupled system still satisfies the SMC. Accordingly, we can choose potentials, find a Lagrangian for the coupled system, and prove Lyapunov stability of the stabilized and synchronized network. Since the controlled Lagrangian has a symmetry, we use Routh reduction and Routh criteria to prove stability.

We then design additional dissipative control terms and prove asymptotic stability. We show, on the one hand, how to apply a dissipative control term that yields convergence to synchronization staying on a constant momentum surface. In the pendulum/cart system example, this corresponds to a synchronized motion of the carts such that all the carts move together with a common velocity that is the sum of a constant plus an oscillation. Likewise, the pendula synchronize and oscillate at the same frequency as the carts. The oscillation frequency for the carts and pendula is determined by the control parameters. On the other hand, we show how to apply a dissipative control term that yields convergence to a relative equilibrium. In the example, this corresponds to steady, synchronized motion of n carts, each balancing its inverted pendulum.

In this paper we consider a homogeneous group of mechanical systems, i.e., no leaders, and a fixed, bidirectional, connected communication topology. Possibilities for extension include integration of the results with prior works cited above that address time-varying and directed communication topologies and/or the presence of leaders in the group.

The organization of the paper is as follows. In section 2, we define notation and the different kinds of stabilization studied. In section 3, we give a brief background on the class of underactuated mechanical systems that satisfy the SMC defined in [2, 1]. We discuss how unstable dynamics are stabilized with feedback control that preserves Lagrangian structure. In section 4, we study a network of n systems, each of which satisfies the SMC. We choose coupling potentials in section 5, and we prove stability and coordination of the network. Asymptotic stabilization is investigated in sections 6 and 7. We illustrate the theory with the example of n planar, inverted pendulum/cart systems in section 8. In section 9 we conclude with a few remarks.

2. Definitions. In [1] the method of controlled Lagrangians is used to derive a control law that asymptotically stabilizes a class of underactuated mechanical systems with otherwise unstable dynamics. This class of systems satisfies a set of "simplified matching conditions" and we denote such systems as *SMC systems*. SMC systems lack gyroscopic forces; the planar inverted pendulum on a cart and the spherical inverted pendulum on a 2D cart are two such systems.

Consider an underactuated mechanical system with an $(m+r)$-dimensional configuration space. Let x^α denote the coordinates for the unactuated directions with index α going from 1 to m. θ^a denotes the coordinates for the actuated directions with index a going from 1 to r. In the case of a network of n mechanical systems, each with the same $(m+r)$-dimensional configuration space, x_i^α and θ_i^a are the corresponding coordinates for the ith mechanical system, $i = 1, \ldots, n$. Beginning in section 5, we will assume that the configuration space for the actuated variables for each individual system is $\mathbb{R}^r$. Note that we only require the configuration space for the individual mechanical systems to be the same and do not require that each system be identical, e.g., the individual systems can have different mass and inertia values. We will need to make the assumption of individual systems being identical only in section 6.

The goal of coordination is to synchronize the actuated variables θ_i^a with the variables θ_j^a for all $i, j = 1, \ldots, n$. We define stable synchronization of these variables as stabilization of $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$.

We define the following stability notions for the mechanical system network.

Definition 2.1 (SSRE). *A relative equilibrium of the mechanical system network dynamics is a* stable synchronized relative equilibrium (SSRE) *if it is defined by $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$, $x_i^\alpha = 0$ for all i, and if it is Lyapunov stable. This implies that the unactuated dynamics are stable and the actuated dynamics are stably synchronized.*

Definition 2.2 (ASSRE). *A relative equilibrium of the mechanical system network dynamics is an* asymptotically stable synchronized relative equilibrium (ASSRE) *if it is SSRE and asymptotically stable.*

Definition 2.3 (ASSM). *An asymptotically stable solution of the mechanical system network dynamics is an* asymptotically stable synchronized motion (ASSM) *if it is defined by $x_i^\alpha - x_j^\alpha = 0$ and $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$ and the dynamics of the network evolve on a constant momentum surface.*

We note that an ASSRE is a special case of an ASSM. In the example of the network of pendulum/cart systems, the relative equilibrium of interest corresponds to the carts moving together at the same constant speed with each pendulum at rest in the upright position. In section 8 we asymptotically stabilize this synchronized relative equilibrium as well as a family of synchronized motions that exhibit a synchronized steady motion plus an oscillation of the carts and pendula.

3. SMC. Let the Lagrangian for an individual mechanical system be given by

$$L(x^\alpha, \theta^a, \dot{x}^\beta, \dot{\theta}^b) = \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{\theta}^a + \frac{1}{2} g_{ab}\dot{\theta}^a \dot{\theta}^b - V(x^\alpha, \theta^a),$$

where summation over indices is implied, g is the kinetic energy metric, and V is the potential energy. It is assumed that the actuated directions are symmetry directions for the kinetic energy; that is, we assume $g_{\alpha\beta}$, $g_{\alpha a}$, g_{ab} are all independent of θ^a. The equations of motion for the mechanical system with control inputs u_a are given by

$$\mathcal{E}_{x^\alpha}(L) = 0,$$
$$\mathcal{E}_{\theta^a}(L) = u_a,$$

where $\mathcal{E}_q(L)$ denotes the Euler–Lagrange expression corresponding to a Lagrangian L and generalized coordinates q, i.e.,

$$\mathcal{E}_q(L) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}. \tag{3.1}$$

For such a system, following [1], the SMC are

- $g_{ab} =$ constant;
- $\frac{\partial g_{\alpha a}}{\partial x^\beta} = \frac{\partial g_{\beta a}}{\partial x^\alpha}$;
- $\frac{\partial^2 V}{\partial x^\alpha \partial \theta^a} g^{ad} g_{\beta d} = \frac{\partial^2 V}{\partial x^\beta \partial \theta^a} g^{ad} g_{\alpha d}$.

Satisfaction of these SMC allows for a structured feedback shaping of kinetic *and* potential energy. In particular, a control law $u_a = u_a^{\mathrm{cons}}$ is given in [1] such that the closed-loop system is a Lagrangian system. The controlled Lagrangian L_c, parametrized by constant parameters κ and ρ and by a potential term V_ϵ, is given by

$$\begin{aligned} L_c(x^\alpha, \theta^a, \dot{x}^\beta, \dot{\theta}^b) = \frac{1}{2}\left(g_{\alpha\beta} + \rho(\kappa+1)\left(\kappa + \frac{\rho-1}{\rho}\right) g_{\alpha a} g^{ab} g_{b\beta}\right)\dot{x}^\alpha \dot{x}^\beta + \rho(\kappa+1) g_{\alpha a}\dot{x}^\alpha \dot{\theta}^a \\ + \frac{1}{2}\rho g_{ab}\dot{\theta}^a\dot{\theta}^b - V(x^\alpha, \theta^b) - V_\epsilon(x^\alpha, \theta^b), \end{aligned}$$

where V_ϵ must satisfy

$$-\left(\frac{\partial V}{\partial \theta^a} + \frac{\partial V_\epsilon}{\partial \theta^a}\right)\left(\kappa + \frac{\rho - 1}{\rho}\right) g^{ad} g_{\alpha d} + \frac{\partial V_\epsilon}{\partial x^\alpha} = 0. \tag{3.2}$$

The results in [1] further give conditions on ρ, κ, and V_ϵ that ensure stability of the equilibrium in the full state space. Without loss of generality, we assume that the equilibrium of interest is the origin. We further assume that it is a *maximum* of the original potential energy V (the case when the origin is a minimum can be handled similarly). The inverted pendulum systems fall into this category. In this case, $\kappa > 0$ and $\rho < 0$ and the potential V_ϵ can be chosen such that the energy function E_c for the controlled Lagrangian has a maximum at the origin of the full state space. Asymptotic stability is obtained by adding a dissipative term u_a^{diss} to the control law, i.e.,

$$u_a = u_a^{\mathrm{cons}} + \frac{1}{\rho} u_a^{\mathrm{diss}},$$

which drives the controlled system to the maximum value of the energy E_c.

In [1], it is also shown how to select new, useful coordinates $(x^\alpha, y^a, \dot{x}^\alpha, \dot{y}^a)$. In particular, for any SMC system, there exists a function $h^a(x^\alpha)$ defined on an open subset of the configuration space of the unactuated variables such that

$$\frac{\partial h^a}{\partial x^\alpha} = \left(\kappa + \frac{\rho - 1}{\rho}\right) g^{ac} g_{\alpha c}, \qquad h^a(0) = 0.$$

The new coordinates are defined as

$$(x^\alpha, y^a) = (x^\alpha, \theta^a + h^a(x^\alpha)).$$

Note that if the origin is an equilibrium in the original coordinates, it is also an equilibrium in the new coordinates. In these coordinates, the closed-loop Lagrangian takes the form

$$\begin{aligned} L_c &= \frac{1}{2}\left(g_{\alpha\beta} - \left(\kappa + \frac{\rho-1}{\rho}\right) g_{\alpha a} g^{ab} g_{b\beta}\right) \dot{x}^\alpha \dot{x}^\beta + g_{\alpha a}\dot{x}^\alpha \dot{y}^a + \frac{1}{2}\rho g_{ab}\dot{y}^a\dot{y}^b \\ &\quad - V(x^\alpha, y^a - h^a(x^\alpha)) - V_\epsilon(y^a) \qquad (3.3) \\ &= \frac{1}{2}\tilde{g}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \tilde{g}_{\alpha a}\dot{x}^\alpha\dot{y}^a + \frac{1}{2}\tilde{g}_{ab}\dot{y}^a\dot{y}^b - V(x^\alpha, y^a - h^a(x^\alpha)) - V_\epsilon(y^a), \qquad (3.4) \end{aligned}$$

where

$$\begin{aligned} \tilde{g}_{\alpha\beta} &= \left(g_{\alpha\beta} - \left(\kappa + \frac{\rho-1}{\rho}\right) g_{\alpha a} g^{ab} g_{b\beta}\right), \\ \tilde{g}_{\alpha a} &= g_{\alpha a}, \\ \tilde{g}_{ab} &= \rho g_{ab}\,. \qquad (3.5) \end{aligned}$$

Further, after adding dissipation u_a^{diss}, the Euler–Lagrange equations in the new coordinates become

$$\begin{aligned} \mathcal{E}_{x^\alpha}(L_c) &= 0, \\ \mathcal{E}_{y^a}(L_c) &= u_a^{\mathrm{diss}}\,. \end{aligned}$$

4. Matching for a network of SMC systems. In this section we examine a network of n systems, each of which satisfies the SMC. We determine what control design freedom remains under the constraint that the complete network dynamics are Lagrangian and satisfy the simplified matching conditions.

Consider n SMC systems and let the ith system have dynamics described by Lagrangian L_i, where

$$L_i(x_i^\alpha, \theta_i^a, \dot{x}_i^\beta, \dot{\theta}_i^b) = \frac{1}{2} g^i_{\alpha\beta}\dot{x}_i^\alpha\dot{x}_i^\beta + g^i_{\alpha a}\dot{x}_i^\alpha\dot{\theta}_i^a + \frac{1}{2}g^i_{ab}\dot{\theta}_i^a\dot{\theta}_i^b - V_i(x_i^\alpha, \theta_i^a), \qquad (4.1)$$

and the index i on every variable refers to the ith system.

The Lagrangian for the total (uncontrolled, uncoupled) system is $L = \sum_{i=1}^n L_i = \frac{1}{2}\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} - \sum_{i=1}^n V_i(x_i^\alpha, \theta_i^a)$, where $\boldsymbol{x} = (x_1^\alpha, \ldots, x_n^\beta, \theta_1^a, \ldots, \theta_n^b)^T$, and

$$M = \left(\begin{array}{cc|cc} g^1_{\alpha\beta} & 0 & g^1_{\alpha a} & 0 \\ & \ddots & & \ddots \\ 0 & g^n_{\alpha\beta} & 0 & g^n_{\alpha a} \\ \hline g^1_{a\alpha} & 0 & g^1_{ab} & 0 \\ & \ddots & & \ddots \\ 0 & g^n_{a\alpha} & 0 & g^n_{ab} \end{array}\right).$$

Since each system satisfies the SMC, g^i_{ab} = constant for each $i = 1, \ldots, n$. It can be easily verified that the SMC are satisfied for the total system L, since they are satisfied for each individual system.

For the total system, the symmetry coordinates are $(\theta^a_1, \ldots, \theta^b_n)$. As in [1], we can find a control law and a change of coordinates $\boldsymbol{x} = (x^\alpha_1, \ldots, x^\beta_n, \theta^a_1, \ldots, \theta^b_n) \mapsto \boldsymbol{x}' = (x^\alpha_1, \ldots, x^\beta_n, y^a_1, \ldots, y^b_n)$ such that the closed-loop system is equivalent to another Lagrangian system with

$$L'_c = \frac{1}{2}(\dot{\boldsymbol{x}}')^T M_c \dot{\boldsymbol{x}}' - V'_\epsilon(\boldsymbol{x}') \tag{4.2}$$

and

$$M_c = \left(\begin{array}{ccc|ccc} \tilde{g}^1_{\alpha\beta} & & 0 & \tilde{g}^1_{\alpha a} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \tilde{g}^n_{\alpha\beta} & 0 & & \tilde{g}^n_{\alpha a} \\ \hline \tilde{g}^1_{a\alpha} & & 0 & \tilde{g}^1_{ab} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \tilde{g}^n_{a\alpha} & 0 & & \tilde{g}^n_{ab} \end{array}\right) := \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{12}^T & M_{22} \end{array}\right), \tag{4.3}$$

$$V'_\epsilon = \sum_{i=1}^n \Big(V_i(x^\alpha_i, y^a_i - h^a_i(x^\alpha_i)) + V_{\epsilon i}(x^\alpha_i, y^a_i) \Big).$$

Here, $\tilde{g}^i_{\alpha\beta}$, $\tilde{g}^i_{\alpha a}$, and $\tilde{g}^i_{\alpha a}$ are defined as in (3.5) with all variables replaced with those corresponding to the ith system, e.g., $\tilde{g}^i_{ab} = \rho_i g^i_{ab}$, etc.

The control gains κ_i and ρ_i and control potentials $V_{\epsilon i}$ can be chosen such that the mass matrix M_c is negative definite and the potential V'_ϵ has a maximum when the configuration of each system, i.e., (x^α_i, θ^a_i), is at the origin. This means the control law brings each system independently to the origin without coordination.

To determine what additional freedom exists in the choice of the control, notably in the choice of control potentials $V_{\epsilon i}$, such that the network system satisfies the SMC, we specialize to a network of SMC systems which each satisfy the following condition.

AS1. *The potential energy for each system in the original coordinates satisfies* $V_i(x^\alpha_i, \theta^a_i) = V_{1i}(x^\alpha_i) + V_{2i}(\theta^a_i)$.

The inverted pendulum examples satisfy this assumption in the general case that the cart moves on an inclined plane. In the case that the cart moves in the horizontal plane, $V_2 = 0$.

As shown in [1], given the assumption **AS1**, $V_{\epsilon i}$ in the new coordinates for $i = 1, \ldots, n$ can be chosen to take the form

$$V_{\epsilon i}(x^\alpha_i, y^a_i) = -V_{2i}(y^a_i - h^a_i(x^\alpha_i)) + \bar{V}_{\epsilon i}(y^a_i),$$

where $\bar{V}_{\epsilon i}$ is an arbitrary function and $h^a_i(x^\alpha_i)$ satisfies

$$\frac{\partial h^a_i}{\partial x^\alpha_i} = \left(\kappa_i + \frac{\rho_i - 1}{\rho_i}\right) g^{ac}_i g^i_{\alpha c}, \quad h^a_i(0) = 0. \tag{4.4}$$

We show next that a more general potential V_ϵ can be used in V'_ϵ in place of the sum of potentials $V_{\epsilon i}(x^\alpha_i, y^a_i)$.

PROPOSITION 4.1. *Under assumption* **AS1**, *the potential* $V'_\epsilon = V + V_\epsilon$ *satisfies the SMC with*

$$V = \sum_{i=1}^{n} (V_{1i}(x_i^\alpha)) + V_{2i}(y_i^a - h_i^a(x_i^\alpha)),$$

$$V_\epsilon = -\left(\sum_{i=1}^{n} V_{2i}(y_i^a - h_i^a(x_i^\alpha))\right) + \tilde{V}_\epsilon(y_1^a, \ldots, y_n^a) \tag{4.5}$$

and $\tilde{V}_\epsilon$ *an arbitrary function.*

Proof. Recall that the potential $V'_\epsilon = V + V_\epsilon$ given by (4.5) satisfies the SMC if (3.2) holds. Following [1], we can use the definition of $h_i^a(x_i^\alpha)$ given by (4.4) to write the SMC (3.2) for the potential as

$$\frac{\partial V_\epsilon}{\partial x_i^\alpha} = \frac{\partial V}{\partial y_i^a} \frac{\partial h_i^a(x_i^\alpha)}{\partial x_i^\alpha}, \quad i = 1, \ldots, n. \tag{4.6}$$

By a direct computation, one can check that each side of (4.6) is equal to $\frac{\partial V_{2i}}{\partial v_i^a}\frac{\partial v_i^a}{\partial x_i^\alpha}$, where $v_i^a = y_i^a - h_i^a(x_i^\alpha)$. ∎

Proposition 4.1 implies that *we can couple the n vehicles in the network using the freedom in our choice of* $\tilde{V}_\epsilon = \tilde{V}_\epsilon(y_1^a, \ldots, y_n^a)$, *and the network dynamics will still satisfy the simplified matching conditions.* This result is completely independent of the degree of coupling; i.e., it extends from a network of uncoupled systems to a network of completely connected systems.

5. Stable coordination of an SMC network. In this section we make use of Proposition 4.1 to design coupling potentials $\tilde{V}_\epsilon$ for stable coordination of the network of SMC systems. We prove that the relative equilibrium of interest is an SSRE. Recall from section 2 that to be an SSRE, a relative equilibrium should be defined by $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$ and $x_i^\alpha = 0$ for all i and should be Lyapunov stable. We note that this is equivalent to showing that $y_i^a - y_j^a = 0$ for all $i \neq j$ and $x_i^\alpha = 0$ for all i is Lyapunov stable. In the remainder of the paper we assume that the configuration space for the actuated variables for each individual system is $\mathbb{R}^r$.

To synchronize the actuated variables we use the results of Proposition 4.1 and design coupling potentials for stabilization of $y_i^a - y_j^a = 0$ for all $i \neq j$. Note that the condition $y_i^a - y_j^a = 0$ for all $i \neq j$ by itself is necessary but not sufficient for $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$ *and* $x_i^\alpha = 0$ for all i. We have $y_i^a - y_j^a = 0$ for all $i \neq j$ under more general conditions, e.g., if $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$ and $h_i(x_i^\alpha) = h_j(x_j^\alpha) \neq 0$, $i \neq j$. This more general case makes possible interesting synchronized dynamics, when we add dissipation for asymptotic stability, as will be discussed in section 6.

We choose $\tilde{V}_\epsilon$ such that the closed-loop potential V'_ϵ, defined in Proposition 4.1, has a maximum when $x_i^\alpha = 0$ and $y_i^a - y_j^a = 0$ for all $i \neq j$. This is possible since from (4.5), the closed-loop potential is $V'_\epsilon = \sum_{i=1}^{n}(V_{1i}(x_i)) + \tilde{V}_\epsilon(y_1^a, \ldots, y_n^a)$ and the V_{1i} are assumed to already be maximized at $x_i^\alpha = 0$. We choose in this paper $\tilde{V}_\epsilon$ to be quadratic in $(y_i^a - y_j^a)$ with a maximum at $y_i^a - y_j^a = 0$ for all $i \neq j$. In this case, consider a graph with one node corresponding to each individual system in the network. There is an (undirected) edge between nodes k and l if the term $(y_k^a - y_l^a)$ appears in the quadratic function $\tilde{V}_\epsilon$. Then, V'_ϵ has a strict maximum when $x_i^\alpha = 0$ and $y_i^a - y_j^a = 0$ for all $i \neq j$ if the (undirected) graph is connected. Figure 5.1 illustrates an example of a connected, undirected communication graph for four vehicles.

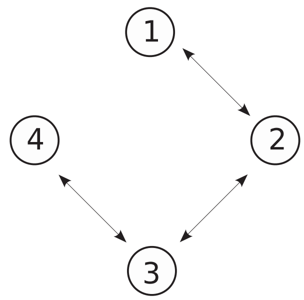

FIG. 5.1. *Connected, undirected communication graph for four vehicles.*

With coupling of the individual systems using terms that depend only on $y_i^a - y_j^a$, the network system has a translational symmetry. Specifically, the system dynamics are invariant under translation of the center of mass of the network. Consider a new set of coordinates given by

$$\boldsymbol{x}_c = (x_1^\alpha, \dots, x_n^\beta, z_1^a, \dots, z_n^b)^T, \tag{5.1}$$

where

$$z_i^a = y_1^a - y_{i+1}^a, \quad i = 1, \dots, n-1,$$
$$z_n^b = y_1^b + \cdots + y_n^b.$$

In this coordinate system, the controlled Lagrangian for the total system (with abuse of notation for V'_ϵ) is

$$\tilde{L}_c = \frac{1}{2}\dot{\boldsymbol{x}}_c^T \tilde{M}_c \dot{\boldsymbol{x}}_c - V'_\epsilon(\boldsymbol{x}_r), \tag{5.2}$$

where $\boldsymbol{x}_r = (x_1^\alpha, \dots, x_n^\beta, z_1^a, \dots, z_{n-1}^b)^T$ and

$$\tilde{M}_c = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & \tilde{M}_{22} \end{pmatrix}. \tag{5.3}$$

The transformation which takes the coordinates $\boldsymbol{x}_c$ to the coordinates $\boldsymbol{x}' = (x_1^\alpha, \dots, x_n^\beta, y_1^c, \dots, y_n^d)$ is given by the matrix

$$B = \begin{bmatrix} I_{mn\times mn} & 0 \\ 0 & B_{22} \end{bmatrix}, \tag{5.4}$$

where

$$B_{22} = \frac{1}{n}\begin{bmatrix} I_{r\times r} & I_{r\times r} & \dots & I_{r\times r} \\ (1-n)I_{r\times r} & I_{r\times r} & \dots & I_{r\times r} \\ \vdots & \vdots & \dots & \vdots \\ I_{r\times r} & \dots & (1-n)I_{r\times r} & I_{r\times r} \end{bmatrix} \tag{5.5}$$

and $I_{l\times l}$ denotes an $l \times l$ identity matrix and B_{22} is an $rn \times rn$ matrix. The expression for $\tilde{M}_c$ in terms of M_c from (4.3) is

$$\tilde{M}_c = B^T M_c B. \tag{5.6}$$

We can compute the block elements in $\tilde{M}_c$ to be

$$\tilde{M}_{11} = M_{11}, \tag{5.7}$$

$$\tilde{M}_{12} = \frac{1}{n}\begin{pmatrix} \tilde{g}^1_{\alpha a} & \tilde{g}^1_{\alpha a} & \dots & \tilde{g}^1_{\alpha a} & \tilde{g}^1_{\alpha a} \\ (1-n)\tilde{g}^2_{\alpha a} & \tilde{g}^2_{\alpha a} & \dots & \tilde{g}^2_{\alpha a} & \tilde{g}^2_{\alpha a} \\ & & \ddots & & \\ \tilde{g}^{n-1}_{\alpha a} & \tilde{g}^{n-1}_{\alpha a} & \dots & \tilde{g}^{n-1}_{\alpha a} & \tilde{g}^{n-1}_{\alpha a} \\ \tilde{g}^n_{\alpha a} & \tilde{g}^n_{\alpha a} & \dots & (1-n)\tilde{g}^n_{\alpha a} & \tilde{g}^n_{\alpha a} \end{pmatrix}, \tag{5.8}$$

$$\tilde{M}_{22} = \frac{1}{n^2}B_{22}^T M_{22} B_{22}, \tag{5.9}$$

where M_{11} and M_{22} are as defined in (4.3). From (5.5) and (4.3), we can calculate the lowermost diagonal $r \times r$ block of $\tilde{M}_{22}$ to be

$$\tilde{g}_{ab} = \frac{1}{n^2}\sum_{i=1}^{n}(\tilde{g}^i_{ab}). \tag{5.10}$$

Thus, we can define $\bar{M}_{22} = \tilde{g}_{ab}$ and $\bar{M}_{11}$ and $\bar{M}_{12}$ in terms of $\tilde{M}_c$ such that

$$\begin{pmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{12}^T & \bar{M}_{22} \end{pmatrix} = \tilde{M}_c.$$

Then, we can rewrite (5.2) as

$$\tilde{L}_c = \frac{1}{2}\begin{pmatrix} \dot{\boldsymbol{x}}_r^T & \dot{\mathbf{z}}_n^T \end{pmatrix}\begin{pmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{12}^T & \bar{M}_{22} \end{pmatrix}\begin{pmatrix} \dot{\boldsymbol{x}}_r \\ \dot{\mathbf{z}}_n \end{pmatrix} - V'_\epsilon(\boldsymbol{x}_r),$$

where $\mathbf{z}_n = (z_n^a)^T$.

Note that in these coordinates z_n^a is the symmetry variable. We are interested in the relative equilibria given by

$$\mathbf{v}_{RE} := \begin{pmatrix} \boldsymbol{x}_r \\ \dot{\boldsymbol{x}}_r \\ \dot{\mathbf{z}}_n \end{pmatrix}, \tag{5.11}$$

where

$$\boldsymbol{x}_r = \mathbf{0}, \quad \dot{\boldsymbol{x}}_r = \mathbf{0}, \quad \dot{z}_n^d = \zeta^d,$$

and ζ^d corresponds to (n times) the constant velocity of the center of mass of the network.

Definition 5.1 (amended potential [13]). *The* amended potential *for the Lagrangian system with Lagrangian* (5.2) *is defined by*

$$V_\mu(\boldsymbol{x}_r) = V'_\epsilon(\boldsymbol{x}_r) + \frac{1}{2}\tilde{g}^{cd}\mu_c\mu_d,$$

where V'_ϵ is given by (4.5) *and $\tilde{g}_{ab}$ is given by* (5.10)*. If J_a is the momentum conjugate to z_n^a, then μ_a is J_a evaluated at the relative equilibrium corresponding to $\dot{z}_n^a = \zeta^a$, i.e.,*

$$J_a = \frac{\partial \tilde{L}_c}{\partial \dot{z}_n^a} = (\bar{M}_{12}^T\dot{\boldsymbol{x}}_r + \bar{M}_{22}\dot{\mathbf{z}}_n)_a, \qquad \mu_a = \left.\frac{\partial \tilde{L}_c}{\partial \dot{z}_n^a}\right|_{\boldsymbol{x}_r=0,\dot{\boldsymbol{x}}_r=0,\dot{z}_n^a=\zeta^a} = \tilde{g}_{ab}\zeta^b. \tag{5.12}$$

By the Routh criteria, the relative equilibrium is stable if the second variation of

$$E_\mu := \frac{1}{2}\dot{\boldsymbol{x}}_r^T(\bar{M}_{11} - \bar{M}_{12}\bar{M}_{22}^{-1}\bar{M}_{12}^T)\dot{\boldsymbol{x}}_r + V_\mu(\boldsymbol{x}_r), \tag{5.13}$$

evaluated at the origin, is definite. Also, if $R^\mu(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r)$ is defined as

$$R^\mu := \frac{1}{2}\dot{\boldsymbol{x}}_r^T(\bar{M}_{11} - \bar{M}_{12}\bar{M}_{22}^{-1}\bar{M}_{12}^T)\dot{\boldsymbol{x}}_r - V_\mu(\boldsymbol{x}_r), \tag{5.14}$$

then the reduced Euler–Lagrange equations can be written as

$$\mathcal{E}_{x_r^\alpha} R^\mu = 0.$$

The Routhian R^μ plays the role of a Lagrangian for the reduced system in variables $(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r)$. Since $\tilde{g}^i_{ab}$ is a constant for each $i \in \{1, 2, \ldots, n\}$, the second term in the amended potential V_μ does not contribute to the second variation. It follows that the relative equilibrium with momentum μ_a is stable if the matrix $(\bar{M}_{11} - \bar{M}_{12}\bar{M}_{22}^{-1}\bar{M}_{12}^T)$ evaluated at the origin is negative definite, since the potential V'_ϵ is already maximum at the equilibrium. But $(\bar{M}_{11} - \bar{M}_{12}\bar{M}_{22}^{-1}\bar{M}_{12}^T)$ is negative definite because it is the Schur complement of the negative definite matrix $\tilde{M}_c$ [9].

THEOREM 5.2 (SSRE). *Consider a network of n SMC systems, each satisfying assumption* **AS1**. *Suppose for each system that the origin is an equilibrium and that the original potential energy is maximum at the origin. Consider the kinetic energy shaping defined in section 4 and potential energy coupling defined above with a connected graph so that the closed-loop dynamics derive from the Lagrangian $\tilde{L}_c$ given by* (5.2) *and the potential energy V'_ϵ is maximized at the relative equilibrium* (5.11). *The corresponding control law for the ith mechanical system is*

(5.15)
$$\begin{aligned} u_{a,i} = u^{\text{cons}}_{a,i} = &- \kappa_i \left\{ g^i_{\beta a,\gamma} - g^i_{\delta a}A^{\delta\alpha}_i \left[g^i_{\alpha\beta,\gamma} - \frac{1}{2}g^i_{\beta\gamma,\alpha} - (1+\kappa_i)g^i_{\alpha d}g^{da}_i g^i_{\beta a,\gamma} \right] \right\} \dot{x}^\beta_i \dot{x}^\gamma_i \\ &+ \kappa_i g^i_{\delta a}A^{\delta\alpha}_i \frac{\partial V_i}{\partial x^\alpha_i} + \frac{\partial V_i}{\partial \theta^a_i} - \frac{1}{\rho_i}\left(1 + \kappa_i g^i_{\delta a}A^{\delta\alpha}_i g^i_{\alpha d}g^{db}_i\right)\frac{\partial V'_\epsilon}{\partial \theta^a_i}, \end{aligned}$$

where $A^i_{\alpha\beta} = g^i_{\alpha\beta} - (1+\kappa_i)g^i_{\alpha d}g^{da}_i g^i_{\beta a}$, $\rho_i < 0$, *and*

$$\kappa_i + 1 > \max\left\{\lambda \middle| \det\left(g^i_{\alpha\beta} - \lambda g^i_{\alpha a}g^{ab}_i g^i_{b\beta}\right)|_{x^\alpha_i = 0} = 0\right\}.$$

Then, the relative equilibrium (5.11) *is an SSRE for any ζ^d.*

Proof. Since $(\bar{M}_{11} - \bar{M}_{12}\bar{M}_{22}^{-1}\bar{M}_{12}^T)$ evaluated at the origin is negative definite, the second variation of E_μ evaluated at the origin is definite. Hence, the relative equilibrium (5.11) is stable for the total network system independent of momentum value μ_a. ∎

6. Asymptotic stability of the constant momentum solution. In this section we investigate asymptotic stabilization of the coordinated network to a solution corresponding to a constant momentum $J_a = \mu_a$. We prove that the solution is an ASSM. Recall from section 2 that an ASSM is an asymptotically stable solution of the mechanical system network defined by $x^\alpha_i = x^\alpha_j$ and $\theta^a_i = \theta^a_j$ for all $i \neq j$ and dynamics that evolve on a constant momentum surface. An ASSM describes a fully synchronized motion, i.e., one in which each degree of freedom is synchronized across the whole network. If $x_i^\alpha = x_j^\alpha = 0$, then the solution is a relative equilibrium. However, in general, an ASSM is *not* a relative equilibrium. For example, in the case of a network of pendulum/cart systems presented in section 8, the ASSM corresponds to periodic solutions (synchronized oscillations of pendula and carts). In this section we prove a control law that yields ASSM where the constant value of the momentum is given by the initial conditions. Equivalently, given an arbitrary momentum value μ_a, initial conditions on the corresponding momentum surface converge to the ASSM on the same momentum surface. In the pendulum/cart example of section 8, we show that control gains can be used to determine the frequency of the periodic solution (ASSM). We discuss at the end of the section a second case in which a momentum value is prescribed and a control term is added to drive the ASSM to the prescribed constant momentum surface.

In this section we apply no dissipative control in the x_i^α directions for all i and as our Case I below we use no control in the z_n^a direction. Recall that for our closed-loop system, z_n^a is the symmetry direction. If there is no control applied in this direction, J_a remains a constant; i.e., the system evolves on a constant momentum surface. On this surface, E_μ as defined in (5.13) is a conserved quantity and can be chosen as a Lyapunov function to prove stability. By choosing appropriate dissipation in the nonsymmetry directions $z_1^a, \ldots, z_{n-1}^b$, we prove that solutions on a constant momentum surface, corresponding to $x_i^\alpha - x_j^\alpha = 0$ and $\theta_i^a - \theta_j^a = 0$ for all $i \neq j$, are asymptotically stable, i.e., they are ASSM.

Let the control input for the ith mechanical system be

$$u_{a,i} = u_{a,i}^{\text{cons}} + \frac{1}{\rho_i} u_{a,i}^{\text{diss}}, \tag{6.1}$$

where $u_{a,i}^{\text{cons}}$ is the "conservative" control term given by (5.15) and $u_{a,i}^{\text{diss}}$ is the dissipative control term to be designed. The Euler–Lagrange equations in the original coordinates for the ith uncontrolled system are

$$\mathcal{E}_{x_i^\alpha}(L_i) = 0; \quad \mathcal{E}_{\theta_i^a}(L_i) = u_{a,i}^{\text{cons}} + \frac{1}{\rho_i} u_{a,i}^{\text{diss}},$$

where L_i is given by (4.1).

In the new coordinates given by (5.1), we have for $i = 1, \ldots, n$,

$$\mathcal{E}_{x_i^\alpha}(\tilde{L}_c) = 0; \quad \mathcal{E}_{z_i^a}(\tilde{L}_c) = \frac{1}{n} \tilde{u}_{a,i}^{\text{diss}}, \tag{6.2}$$

where $\tilde{L}_c$ is given by (5.2) and

$$\tilde{u}_{a,i}^{\text{diss}} = \sum_{j=1, j\neq i+1}^{n} u_{a,j}^{\text{diss}} - (n-1) u_{a,i+1}^{\text{diss}}, \quad i = 1, \ldots, n-1,$$
$$\tilde{u}_{a,n}^{\text{diss}} = \sum_{j=1}^{n} u_{a,j}^{\text{diss}}.$$

Case I. $\tilde{u}_{a,n}^{\text{diss}} = 0$.

Let $\tilde{E}_c$ be the energy function for the Lagrangian $\tilde{L}_c$. Given momentum value μ_a, let $\xi^b = \tilde{g}^{ab}\mu_a$. Then, the function $\tilde{E}_c^\xi$ defined by

$$\tilde{E}_c^\xi = \tilde{E}_c - J_a \xi^a$$

has the property that its restriction to the level set $J_a = \mu_a = \tilde{g}_{ab}\xi^b$ of the momentum gives E_μ (5.13). We can use this fact to calculate the time derivative of E_μ as follows. From (6.2), we get

$$\frac{d}{dt}\tilde{E}_c = \frac{1}{n}\sum_{i=1}^{n}(\dot{z}_i^a \tilde{u}_{a,i}^{\text{diss}}). \tag{6.3}$$

Using (6.3) and the fact that $\frac{d}{dt}J_a = \frac{1}{n}\tilde{u}_{a,n}^{\text{diss}}$, we get

$$\frac{d}{dt}\tilde{E}_c^\xi = \frac{1}{n}\sum_{i=1}^{n}(\dot{z}_i^a \tilde{u}_{a,i}^{\text{diss}}) - \left(\frac{1}{n}\tilde{u}_{a,n}^{\text{diss}}\xi^a\right). \tag{6.4}$$

The expression for the time derivative of E_μ is obtained by restricting $\frac{d}{dt}\tilde{E}_c^\xi$ to the set $J_a = \mu_a$. This and (5.12) give us

$$\begin{aligned}
\frac{d}{dt}E_\mu &= \frac{1}{n}\sum_{i=1}^{n-1}(\dot{z}_i^a \tilde{u}_{a,i}^{\text{diss}}) + \frac{1}{n}\tilde{u}_{a,n}^{\text{diss}}(\,\dot{z}_n^a|_{J_b=\mu_b} - \xi^a) \\
&= \frac{1}{n}\sum_{i=1}^{n-1}(\dot{z}_i^a \tilde{u}_{a,i}^{\text{diss}}) + \frac{1}{n}\tilde{u}_{a,n}^{\text{diss}}(\tilde{g}^{ab}(\mu_b - (\bar{M}_{12}^T\dot{\boldsymbol{x}}_r)_b) - \xi^a) \\
&= \frac{1}{n}\sum_{i=1}^{n-1}(\dot{z}_i^a \tilde{u}_{a,i}^{\text{diss}}) + \frac{1}{n}\tilde{u}_{a,n}^{\text{diss}}(-\tilde{g}^{ab}(\bar{M}_{12}^T\dot{\boldsymbol{x}}_r)_b).
\end{aligned}$$

Here, $\bar{M}_{12}^T\dot{\boldsymbol{x}}_r$ is a covariant vector just like a momentum. Hence, its components are denoted by subscripts. Since $\tilde{u}_{a,n}^{\text{diss}}$ is chosen to be zero, we get

$$\frac{d}{dt}E_\mu = \frac{1}{n}\sum_{i=1}^{n-1}(\dot{z}_i^a \tilde{u}_{a,i}^{\text{diss}}). \tag{6.5}$$

Expressing $\tilde{u}_{a,i}^{\text{diss}}$ in terms of $u_{a,i}^{\text{diss}}$, we can write the expression for $\dot{E}_\mu$ as

$$n\frac{d}{dt}E_\mu = u_{a,1}^{\text{diss}}\left(\sum_{j=1}^{n-1}\dot{z}_j^a\right) + \sum_{j=2}^{n-1}u_{a,j}^{\text{diss}}\left(-(n-1)\dot{z}_{j-1}^a + \sum_{k=1,k\neq j-1}^{n-1}\dot{z}_k^a\right) \tag{6.6}$$

and choose

$$\begin{aligned}
u_{a,1}^{\text{diss}} &= d_{ab}\left(\sum_{j=1}^{n-1}\dot{z}_j^b\right), \\
u_{a,j}^{\text{diss}} &= d_{ab}\left(-(n-1)\dot{z}_{j-1}^b + \sum_{k=1,k\neq j-1}^{n-1}\dot{z}_k^b\right), \\
& j = 2,\ldots,n-1,
\end{aligned} \tag{6.7}$$

where d_{ab} is a positive definite control gain matrix, possibly dependent on x_i^α, $i = 1,\ldots,n$, and z_i^a, $j = 1,\ldots,n-1$. With the dissipative control term (6.7), $\frac{d}{dt}E_\mu \geq 0$.

We note that this dissipative control term requires that each individual system can measure the variables $\dot{z}_i^a$ of all other vehicles. Recall that for Lyapunov stability the interconnection among individual systems need only be *connected* for the coupling potential $\tilde{V}_\epsilon$ which is a function of the y_k^a, $k = 1, \ldots, n$. That is, for Lyapunov stability, each individual system need only measure its relative position with respect to some subset of the other individual systems. However, for ASSM we require *complete* interconnection in the dissipative control term which is a function of the variables $\dot{z}_n$. That is, each individual system feedbacks relative velocity with respect to every other individual system. Figure 6.1 illustrates a complete interconnected graph for the case of four vehicles. Complete interconnection is not needed for stabilization of group dynamics in the simpler dynamical models used more typically in the literature, as described in section 1. It is hoped that the interconnection limitation here in stabilization of networks of underactuated mechanical systems can likewise be overcome in future work.

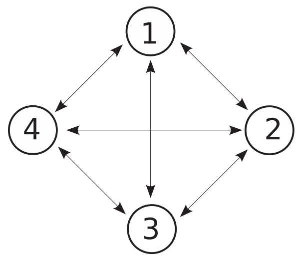

FIG. 6.1. *Complete interconnected communication graph for four vehicles.*

We next study convergence of the system using the LaSalle invariance principle [11]. For $c > 0$, let $\Omega_c = \{(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r) | E_\mu \geq c\}$. Ω_c is a compact and positive invariant set with integral curves starting in Ω_c and staying in Ω_c for all $t \geq 0$. Define the LaSalle surface

$$\mathcal{E} = \left\{ (\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r) \,\middle|\, \frac{d}{dt} E_\mu = 0 \right\} .$$

On this surface, $u_{a,j}^{\text{diss}} = 0$, $i = 1, \ldots, n$, which implies that $\dot{z}_i^a = 0$ for $i = 1, \ldots, n-1$. Let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{E}$. By the LaSalle invariance principle, solutions that start in Ω_c approach $\mathcal{M}$. The relative equilibrium (5.11) is contained in $\mathcal{M}$; however, there are other solutions in this set.

We now proceed to analyze in more detail the structure of solutions on the LaSalle surface $\mathcal{E}$. Using the condition $\dot{z}_i^a = 0$ for $i = 1, \ldots, n-1$, we get $\dot{y}_i^a = \dot{y}_j^a$ for all $i, j \in \{1, \ldots, n\}$. This gives $y_i^a - y_j^a = \text{constant}$. Since we have chosen $\tilde{V}_\epsilon$ to be a quadratic function of the terms $y_i^a - y_j^a$, we get $\frac{\partial \tilde{V}_\epsilon}{\partial y_i^a} = \text{constant} =: \Delta_a^i$. The equations of motion for the y_i^a restricted to the LaSalle surface are $\mathcal{E}_{y_i^a}(L_c') = 0$, where L_c' is given by (4.2). Equivalently,

$$\ddot{y}_i^a + \frac{d}{dt}\left(\tilde{g}_i^{ab} g_{\alpha b}^i \dot{x}_i^\alpha\right) = -\tilde{g}_i^{ab} \frac{\partial \tilde{V}_\epsilon}{\partial y_i^b} = -\tilde{g}_i^{ab} \Delta_b^i \,. \tag{6.8}$$

As illustrated in [1], for SMC systems, there is a function $l_i^a(x_i^\alpha)$ for each vehicle i defined on an open set of the configuration space for the ith vehicle's unactuated variables such that

$$\frac{\partial l_i^a}{\partial x_i^\alpha} = \tilde{g}_i^{ac} g_{\alpha c}^i \,. \tag{6.9}$$

We can assume, by shrinking Ω_c if necessary, that (6.9) holds in Ω_c.

Let K_c be the projection of Ω_c onto the coordinates $(\boldsymbol{x}_n, \dot{\boldsymbol{x}}_n)$, where $\boldsymbol{x}_n = (x_1^\alpha, \ldots, x_n^\alpha)$. Then, since l_i^a is continuous and K_c is compact, there exist constants m_i and n_i such that

$$m_i \leq ||l_i(x_i)|| \leq n_i \tag{6.10}$$

for all x_i^α such that $\boldsymbol{x}_n \in \Omega_c$. Using (6.8), (6.9), and the condition $\dot{y}_i^a = \dot{y}_j^a$ on $\mathcal{E}$, we get

$$\frac{d}{dt}(\dot{l}_i^a - \dot{l}_j^a) = \tilde{g}_j^{ab}\Delta_b^j - \tilde{g}_i^{ab}\Delta_b^i\,. \tag{6.11}$$

Therefore, on $\mathcal{E}$,

$$l_i^a - l_j^a = \frac{1}{2}(\tilde{g}_j^{ab}\Delta_b^j - \tilde{g}_i^{ab}\Delta_b^i)t^2 + \nu_1^a t + \nu_2^a \tag{6.12}$$

for some constant vectors ν_1^a and ν_2^a. The only way (6.10) can also be satisfied is if $\tilde{g}_j^{ab}\Delta_b^j - \tilde{g}_i^{ab}\Delta_b^i = 0$ and $\nu_1^a = 0$.

To simplify our calculations, we assume that the n individual mechanical systems are identical. In this case, $\tilde{g}_j^{ab} = \tilde{g}_j^{ab}$ for any $i, j \in \{1, \ldots, n\}$. This gives $\Delta_a^i = \Delta_a^j$ for any $i, j \in \{1, \ldots, n\}$, and so for a connected network with potential V'_ϵ having a maximum at $x_i^\alpha = 0$ and $y_i^a = y_j^a$ for all $i \neq j$, we get that $y_i^a = y_j^a$ on $\mathcal{E}$ for all $i, j \in \{1, \ldots, n\}$.

Using the definition (6.9) and the assumption that the individual systems are identical, the fact that $\dot{l}_i^a - \dot{l}_j^a = 0$ on $\mathcal{E}$ yields

$$g_{\alpha b}^i \dot{x}_i^\alpha = g_{\alpha b}^j \dot{x}_j^\alpha\,, \tag{6.13}$$

where $g_{\alpha b}^k = g_{\alpha b}(x_k^\alpha)$ for all $k = 1, \ldots, n$. Therefore, on the LaSalle surface $\mathcal{E}$, we see that solutions are of the form $(\boldsymbol{x}_n(t), \dot{\boldsymbol{x}}_n(t), y_1^a(t), \ldots, y_n^b(t), \dot{y}_1^c(t), \ldots, \dot{y}_n^d(t))$, where $y_i^a(t) = y_j^a(t)$ for any $i, j \in \{1, \ldots, n\}$, $J_a = \mu_a$, and condition (6.13) holds. Since $z_n^a = \sum_{i=1}^n y_i^a$ and the individual systems are identical, we have

$$\begin{aligned} J_a = \frac{\partial \tilde{L}_c}{\partial \dot{z}_n^a} &= \sum_{i=1}^n (g_{\alpha a}^i \dot{x}_i^\alpha + \tilde{g}_{ab}\dot{y}_i^b) \\ &= \tilde{g}_{ab}\sum_{i=1}^n (\tilde{g}^{bc} g_{\alpha c}^i \dot{x}_i^\alpha + \dot{y}_i^b) \\ &= n\tilde{g}_{ab}(\tilde{g}^{bc} g_{\alpha c}^i \dot{x}_i^\alpha + \dot{y}_i^b) \end{aligned}$$

for any $i \in \{1, \ldots, n\}$, where we have used the facts that $\dot{y}_i^a = \dot{y}_j^a$ and (6.13) holds on $\mathcal{E}$. Therefore, for each i we get

$$\dot{y}_i^a = \frac{1}{n}\tilde{g}^{ab}\mu_b - \tilde{g}^{ab} g_{\alpha b}^i \dot{x}_i^\alpha\,. \tag{6.14}$$

Substituting (6.14) into the closed-loop equations for the Lagrangian L'_c (4.2), we get the following equations for the x_i^α variables:

$$\frac{d}{dt}\frac{\partial L^\mu}{\partial \dot{x}_i^\alpha} = \frac{\partial L^\mu}{\partial x_i^\alpha}, \tag{6.15}$$

where

$$
\begin{aligned}
L^\mu &= \sum_{i=1}^n \left(\frac{1}{2} (\tilde{g}^i_{\alpha\beta} - \tilde{g}^{ab} g^i_{\alpha a} g^i_{\beta b}) \dot{x}^\alpha_i \dot{x}^\beta_i - V_{1i}(x^\alpha_i) \right) \\
&= \sum_{i=1}^n \left(\frac{1}{2} (g^i_{\alpha\beta} - (\kappa+1) g^{ab} g^i_{\alpha a} g^i_{\beta b}) \dot{x}^\alpha_i \dot{x}^\beta_i - V_{1i}(x^\alpha_i) \right),
\end{aligned} \tag{6.16}
$$

and V_{1i} is defined by assumption **AS1**. Here, $\kappa_i = \kappa$ for all $i = 1, \ldots, n$.

L^μ is just the Routhian R^μ for a mechanical system with abelian symmetry variables without a linear term in velocity and without the amended part of the potential. This follows because, for SMC systems, these latter terms do not contribute to the dynamics of the reduced system. We also see that the x^α_i dynamics completely decouple from the x^α_j dynamics on the LaSalle surface $\mathcal{E}$ for all i and j. The y^a_i dynamics given by (6.14) can be thought of as a reconstruction of dynamics in the symmetry variables, obtained after solving the reduced dynamics in the x^α_i variables. We now make the following assumption.

> **AS2.** *Consider two solutions $(x^\alpha(t), y^a(t))$ and $(\tilde{x}^\alpha(t), \tilde{y}^a(t))$ of the Euler–Lagrange equations corresponding to the Lagrangian given by* (3.3)*. If $y^a(t) = \tilde{y}^a(t)$ and $g_{\alpha a}(x^\alpha(t))\dot{x}^\alpha(t) = g_{\alpha a}(\tilde{x}^\alpha(t))\dot{\tilde{x}}^\alpha(t)$, then $x^\alpha(t) = \tilde{x}^\alpha(t)$.*

Note that checking this condition does not require extensive computation since we already know the expression for the closed-loop Lagrangian. Consider two solutions $x^\alpha(t)$ and $\tilde{x}^\alpha(t)$ such that $g_{\alpha a}(x^\alpha(t))\dot{x}^\alpha(t) = g_{\alpha a}(\tilde{x}^\alpha(t))\dot{\tilde{x}}^\alpha(t)$. This is equivalent to $l^a(x^\alpha) = l^a(\tilde{x}^\alpha) + c^a$, where l^a is defined by (6.9) and c^a is a constant; i.e., $x^\alpha(t)$ and $\tilde{x}^\alpha(t)$ are two solutions in $(x^\alpha, \dot{x}^\alpha)$ space satisfying the Euler–Lagrange equation corresponding to the Lagrangian L^μ given by (6.16) and differing by a constant. For mechanical systems with symmetries, it may be possible to prove that c^a is zero, as is done for the pendulum/cart case in section 8. Then, **AS2** is equivalent to assuming that the function l^a is injective, i.e., $g_{\alpha a}$ is one-to-one in a neighborhood about the equilibrium. For the pendulum/cart example in section 8, this holds in the neighborhood defined by pendulum angles which are above the horizontal plane. As mentioned in [1], **AS2** is equivalent to the *(local) strong inertial coupling* assumption in [21] and *internal/external convertible system* in [7].

Using (6.13) and the fact that $y^a_i = y^a_j$ on the LaSalle surface, we get from **AS2** that $x^\alpha_i = x^\alpha_j$ and $\theta^a_i = \theta^a_j$ for all $i, j \in \{1, \ldots, n\}$. So we get that the dissipation control law given by (6.7) yields asymptotic convergence to synchronized motion on a constant momentum surface.

THEOREM 6.1 (ASSM). *Consider a network of n identical SMC systems that each satisfy* **AS1** *and* **AS2***. Suppose for each individual system that the origin is an equilibrium and that the original potential energy is maximum at the origin. Consider the kinetic energy shaping defined in section* 4 *and potential energy coupling $\tilde{V}_\epsilon$ defined in section* 5*, where the terms in $\tilde{V}_\epsilon$ are quadratic in $y^a_i - y^a_j$ and the corresponding interconnection graph is connected. The closed-loop dynamics* (6.2) *derive from the Lagrangian $\tilde{L}_c$ given by* (5.2)*, and the potential energy V'_ϵ is maximized at the relative equilibrium* (5.11)*. The control input takes the form* (6.1)*, where $u^{\mathrm{cons}}_{a,i}$ is given by* (5.15) *and $\rho_i = \rho$, $\kappa_i = \kappa$. The dissipative control term given by* (6.7) *asymptotically stabilizes the solution in which all the vehicles have synchronized dynamics such that $\theta^a_i = \theta^a_j$ and $x^\alpha_i = x^\alpha_j$ for all i and j, and each has the same constant momentum in the θ^a_i direction. The system stays on the constant momentum surface determined by the initial conditions.*

Remark 6.2. Consider a Case II in which we choose $\tilde{u}_{a,n}^{\text{diss}} = -\lambda(J_a - \mu_a)$ and $u_{a,i}$ for $i = 1, \dots, n-1$ as in Case I. Then $J_a = (J_a(0) - \mu_a)\exp(-\lambda t) + \mu_a$ and we can rewrite the reduced system in $(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r)$ coordinates as follows:

$$\mathcal{E}_{\boldsymbol{x}_r}(R^\mu) = \begin{pmatrix} 0 \\ \frac{1}{n}\tilde{\mathbf{u}}^{\text{diss}} \end{pmatrix} + \lambda \bar{M}_{12}\bar{M}^{22}(\mathbf{J}(0) - \boldsymbol{\mu})\exp(-\lambda t). \tag{6.17}$$

Here, $\tilde{\mathbf{u}}^{\text{diss}} = (\tilde{u}_{a,1}^{\text{diss}}, \dots, \tilde{u}_{a,n-1}^{\text{diss}})$ is an rn-dimensional vector, and $\mathbf{J}$ and $\boldsymbol{\mu}$ are r-dimensional vectors with components J_a and μ_b, respectively. When $\lambda = 0$, we get Case I. When $\lambda \neq 0$, the momentum J_a is no longer a conserved quantity. This case needs to be analyzed more carefully since we are pumping energy into the system now to drive it to a particular momentum value. Equation (6.17) can be considered to be a parameter dependent differential equation with the parameter being λ. When $\lambda = 0$, we already know the solution from Case I. From the continuity of dependence of solutions upon parameters, we get that when $0 < \lambda < \delta$, the solution stays within an ϵ-tube of the solution in Case I for time $t \in [0, t_1]$ for some t_1 if the initial conditions are in a δ-neighborhood. Our simulations for pendulum/cart systems suggest that this holds true for the infinite time interval. We plan to investigate this case further in our future work.

Remark 6.3. The simplifying requirement for Theorem 6.1 that all systems be identical is a weakness of the result and motivates the question of robustness to uncertainty in system parameters. Simulations suggest that the stability of Theorem 6.1 is robust to model parameter uncertainty, but a formal robustness analysis is warranted.

In section 8 we illustrate the result of Theorem 6.1 and the dynamics of (6.16) in more detail in the case of a network of inverted pendulum/cart systems. Solutions for this example correspond to synchronized balanced pendula on synchronized moving carts, where the motion of the carts is the sum of a constant velocity plus an oscillation and the motion of the pendula is oscillatory with the same frequency as the carts.

7. Asymptotic stabilization of relative equilibria. In the previous section, we proved asymptotic stability of the coordinated network in the case when the network asymptotically converges to the momentum surface $J_a = \mu_a$. This can lead to nontrivial and interesting synchronized group dynamics, as is discussed in section 8. Stabilization was proved using E_μ as a Lyapunov function on the reduced space. The dynamics after adding a dissipative control term are given by $\theta_i^a = \theta_j^a$ and $x_i^\alpha = x_j^\alpha$ for all $i, j = 1, \dots, n$. The dissipative terms are chosen such that the momentum is preserved.

In this section, we demonstrate how to isolate and asymptotically stabilize the particular synchronized and constant momentum solutions corresponding to the relative equilibria given by (5.11). The value of the momentum μ_a can be chosen arbitrarily. We use a different Lyapunov function from that used in section 6. We note that in the example of a network of inverted pendulum/cart systems, the relative equilibrium corresponds to the synchronized motion of all carts moving in unison at a steady speed with all pendula at rest in the upright position; i.e., it is the special case of the motion proved in Theorem 6.1 without the oscillation.

Consider the following function:

$$E_{RE} = \frac{1}{2}(\dot{\boldsymbol{x}}_c - \mathbf{v}_{RE})^T \tilde{M}_c (\dot{\boldsymbol{x}}_c - \mathbf{v}_{RE}) + V'_\epsilon, \tag{7.1}$$

where $\mathbf{v}_{RE}$ is defined by (5.11). E_{RE} is a Lyapunov function in directions transverse to the group orbit of the relative equilibrium, i.e., $E_{RE} > 0$ in a neighborhood of the Euler–Lagrange solution given by $(\boldsymbol{x}_r, \mathbf{z}_n, \dot{\boldsymbol{x}}_r, \dot{\mathbf{z}}_n)$, where $\boldsymbol{x}_r = \mathbf{0}$, $\dot{\boldsymbol{x}}_r = \mathbf{0}$, $z_n^d = \zeta^d t$, $\dot{z}_n^d = \zeta^d$, and ζ^d corresponds to (n times) the constant velocity of the center of mass of the network.

The time derivative of E_{RE} along the flow given by (6.2) can be computed to be

$$\frac{d}{dt}E_{RE} = \frac{1}{n}(\dot{\boldsymbol{x}}_c - \mathbf{v}_{RE}) \cdot \begin{pmatrix} 0 \\ \tilde{\mathbf{u}}^{\text{diss}} \end{pmatrix}.$$

See [3] for the steps involved in proving this identity. Choose

$$\tilde{u}_{a,i}^{\text{diss}} = \begin{cases} n\sigma_i \dot{z}_i^a & \text{for } i = 1, \dots, n-1, \\ n\sigma_n(\dot{z}_n^a - \zeta^a) & \text{for } i = n, \end{cases} \tag{7.2}$$

where control parameters σ_i are positive constants. Then,

$$\frac{d}{dt}E_{RE} = \sum_{j=1}^{n-1} \sigma_i(\dot{z}_j^a)^2 + \sigma_n(\dot{z}_n^b - \zeta^b)^2 \ \geq \ 0.$$

We note here that, unlike the case of asymptotic stabilization in the previous section, where a complete interconnection was required to realize the dissipative control term (6.7), the dissipative control term (7.2) requires only a connected interconnection graph.

Let $\Omega_c^{RE} = \{(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r, \dot{z}_n^a) | E_{RE} \geq c\}$ for $c > 0$. Ω_c^{RE} is a compact set, i.e., E_{RE} is a proper Lyapunov function. Assume that the Euler–Lagrange system (6.2) satisfies the following controllability condition.

AS3. *The system* (6.2) *is linearly controllable at each point in a neighborhood of the relative equilibrium solution manifold.*

Note that checking this condition does not require extensive computation since we already know the expression for the closed-loop Lagrangian.

We now use a result from nonlinear control theory, which is stated in [3] as Lemma 2.1 and the remark following it, to conclude that the system (6.2) with dissipative control terms given by (7.2) converges exponentially to the set

$$\mathcal{E}_{RE} = \{(\boldsymbol{x}_r, \dot{\boldsymbol{x}}_r, \dot{z}_n^a) \,| E_{RE} = 0\}.$$

On this set, the solution is given by (5.11). Thus, we have shown that the solutions of the controlled system will exponentially converge to $(x_i^\alpha, \theta_i^a, \dot{x}_i^\beta, \dot{\theta}_i^b) = (0, \frac{1}{n}\zeta^a t + \gamma^a, 0, \frac{1}{n}\zeta^b)$, with γ^a constant.

THEOREM 7.1 (ASSRE). *Consider a network of n (not necessarily identical) individual SMC systems that each satisfy assumption* **AS1**. *Suppose for each individual system that the origin is an equilibrium and that the original potential energy is maximum at the origin. Consider the kinetic energy shaping defined in section* 4 *and potential energy coupling $\tilde{V}_\epsilon$ defined in section* 5, *where the terms in $\tilde{V}_\epsilon$ are quadratic in $y_i^a - y_j^a$ and the corresponding interconnection graph is connected. The closed-loop dynamics* (6.2) *derive from the Lagrangian $\tilde{L}_c$ given by* (5.2) *and the potential energy V'_ϵ is maximized at the relative equilibrium* (5.11). *The control input takes the form* (6.1), *where $u_{a,i}^{\text{cons}}$ is given by* (5.15) *and $\rho_i = \rho$. If* (6.2) *satisfies* **AS3**, *then the dissipative control term given by* (7.2) *exponentially stabilizes the relative equilibrium given by* (5.11) *in which $x_i^\alpha = \dot{x}_i^\alpha = 0$ for all $i = 1, \dots, n$ and $\theta_i^a = \theta_j^a$ and $\dot{\theta}_i^a = \dot{\theta}_j^a = \frac{1}{n}\zeta^a$ for all i and j.*

8. Coordination of multiple inverted pendulum/cart systems. As an illustration, we now consider the coordination of n identical planar inverted pendulum/cart systems. For the ith system, the pendulum angle relative to the vertical is x_i and the position of the cart is θ_i. Let the Lagrangian for each system shown in Figure 8.1 be

$$L_i = \frac{1}{2}\alpha\dot{x}_i^2 + \beta\cos(x_i)\dot{x}_i\dot{\theta}_i + \frac{1}{2}\gamma\dot{\theta}_i^2 + D\cos(x_i); \quad i = 1,\dots,n,$$

where l, m, M are the pendulum length, pendulum bob mass, and cart mass, respectively. g is the acceleration due to gravity. The quantities α, β, γ, and D are expressed in terms of l, m, M, g as follows:

$$\alpha = ml^2, \quad \beta = ml, \quad \gamma = m + M, \quad D = -mgl.$$

The equations of motion for the ith system are

$$\mathcal{E}_{x_i}(L_i) = 0,$$
$$\mathcal{E}_{\theta_i}(L_i) = u_i,$$

where u_i is the control force applied to the ith cart.

One can see that θ_i is a symmetry variable. Further, it can be easily verified that each pendulum/cart system satisfies the simplified matching conditions [1, 2]. The n inverted planar pendulum/cart systems lie on n parallel tracks corresponding to the θ_i directions. The coordination problem is to prescribe control forces u_i, $i = 1,\dots,n$, that asymptotically stabilize the solution where each pendulum is in the vertical upright position (in the case of ASSRE) or moving synchronously (in the case of ASSM) and the carts are moving at the same position along their respective tracks with the same common velocity. The relative equilibrium $\mathbf{v}_{RE}$ (5.11) corresponds to $x_i = \dot{x}_i = 0$ for all i, $\theta_i = \theta_j$ for all $i \neq j$, and $\dot{\theta}_i = \frac{1}{n}\zeta$ for some constant scalar velocity ζ.

Following (5.2), the closed-loop Lagrangian for the total system in the coordinates $\boldsymbol{x}_c = (x_1,\dots,x_n,z_1,\dots,z_n)$, where $z_i = y_1 - y_{i+1}$ for $i = 1,\dots,n-1$, $z_n = y_1 + \dots + y_n$, $y_i = \theta_i + p\sin x_i$, and $p = \frac{\beta}{\gamma}(\kappa + 1 - \frac{1}{\rho})$, is

$$\tilde{L}_c = \frac{1}{2}\dot{\boldsymbol{x}}^T\tilde{M}_c\dot{\boldsymbol{x}} - V'_\epsilon(x_1,\dots,x_n,z_1,\dots,z_{n-1}). \tag{8.1}$$

$\tilde{M}_c$ is as in (5.6) and M_c is as in (4.3),

$$\tilde{g}^i_{\alpha\beta} = \alpha - \left(\kappa + 1 - \frac{1}{\rho}\right)\frac{\beta^2}{\gamma}\cos^2(x_i), \qquad \tilde{g}^i_{\alpha a} = \beta\cos(x_i),$$
$$\tilde{g}^i_{ab} = \rho\gamma, \qquad V'_\epsilon = -D\sum_{i=1}^{n-1}\left(\cos(x_i) - \frac{1}{2}\epsilon\frac{\gamma^2}{\beta^2}z_i^2\right) - D\cos(x_n) \tag{8.2}$$

with $\epsilon > 0$. The control law (6.1) for the ith system is

$$u_i = \frac{\kappa\beta\left(\sin x_i\left(\alpha\dot{x}_i^2 + \cos(x_i)D\right) - B_i\left(\frac{\partial V'_\epsilon}{\partial\theta_i} - u_i^{\text{diss}}\right)\right)}{\alpha - \frac{\beta^2}{\gamma}(1+\kappa)\cos^2(x_i)}, \tag{8.3}$$

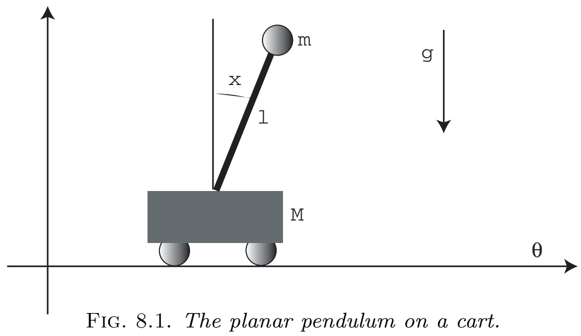

FIG. 8.1. *The planar pendulum on a cart.*

where

$$B_i = \frac{1}{\rho}\left(\alpha - \frac{\beta^2 \cos^2(x_i)}{\gamma}\right).$$

Note that we have chosen $\rho_i = \rho$ and $\kappa_i = \kappa$. In the case $u_i^{\mathrm{diss}} = 0$, by Theorem 5.2, we get stability of the relative equilibrium $\mathbf{v}_{RE}$ (SSRE) if we choose $\rho < 0$, $\epsilon > 0$, and κ such that $m_\kappa := \alpha - (\kappa+1)\frac{\beta^2}{\gamma} < 0$. The choice of u_i^{diss} depends upon what kind of asymptotic stability we want, i.e., convergence to a synchronized constant momentum solution or to a relative equilibrium.

The dependence of V'_ϵ on z_i^2 in (8.2) implies that coupling between the pendulum/cart systems introduced by the control is a function of terms $y_i - y_j$ rather than $\theta_i - \theta_j$. That is, our approach to simultaneous stabilization and synchronization of a network of planar pendulum/cart systems yields coupling not simply as a function of relative cart positions but, rather more subtly, as a function of the horizontal component of relative positions of pendulum bobs (where pendulum length is scaled by p). Numerical simulations show that naively coupling the positions of the carts for the purpose of synchronization in fact destabilizes the network. This particular example illustrates the need to integrate synchronization and stabilization tasks.

8.1. Asymptotic stability on a constant momentum surface (ASSM). Following (6.7), we let u_1^{diss} be

$$u_1^{\mathrm{diss}} = d_1\left(\sum_{k=1}^{n-1}(\dot{z}_k)\right)$$

and u_i^{diss} for $i = 2, \dots, n$ be

$$u_i^{\mathrm{diss}} = d_i\left(-(n-1)\dot{z}_{i-1} + \sum_{k=1, k\neq i-1}^{n-1} \dot{z}_k\right),$$

where coefficients d_i are constant positive scalars.

We now analyze the dynamics on the LaSalle surface. On this surface, we have $\dot{y}_i = \dot{y}_j$ for all $i, j \in \{1, \dots, n\}$ and $J = \mu$, where momentum μ is determined by the initial conditions. From the calculations made in section 6, we also get $y_i = y_j$ and $\cos(x_i)\dot{x}_i = \cos(x_j)\dot{x}_j$. The x_i dynamics are given by (6.15) with

$$L^\mu = \sum_{i=1}^{n}\left(\frac{1}{2}\left(\alpha - (\kappa+1)\frac{\beta^2}{\gamma}\cos^2(x_i)\right)\dot{x}_i^2 + D\cos(x_i)\right). \tag{8.4}$$

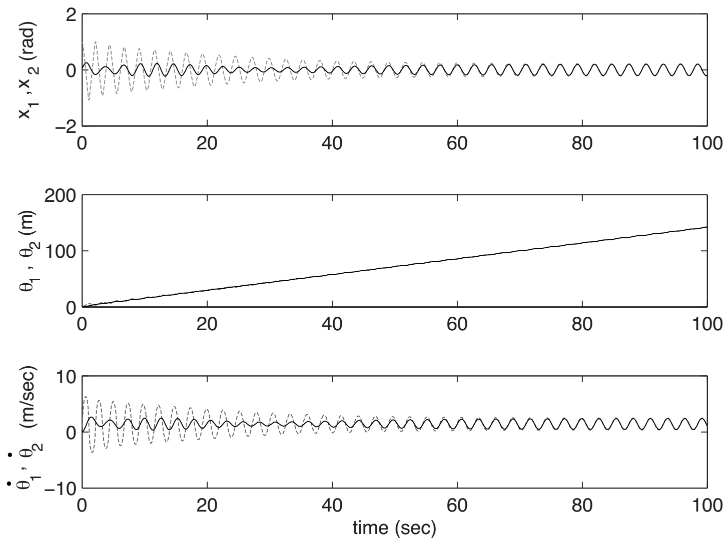

Fig. 8.2. *Simulation of a controlled network of pendulum/cart systems with dissipation designed for ASSM. The pendulum angle, cart position, and cart velocity are plotted as a function of time for each of the two pendulum/cart systems in the network.*

To verify **AS2** we need to check that if $\cos(x_i)\dot{x}_i = \cos(x_j)\dot{x}_j$ about the origin for a system corresponding to the Lagrangian L^μ, then $x_i = x_j$ identically. This condition can also be written as $\sin(x_i) = \sin(x_j) + c$, where c is a constant. Note that if $x_i(t)$ is an Euler–Lagrange solution corresponding to L^μ for the ith vehicle, then $-x_i(t)$ is also a solution. Since we have a stable pendulum oscillation about the upright position, $x_i(t)$ and therefore $|\sin(x_i(t))|$ oscillates with mean zero for all i. This can also be concluded from the fact that the solution curves are closed level curves in the $(x_i, \dot{x}_i)$ plane of L^μ given by (8.4) and L^μ is invariant under the sign change $(x_i, \dot{x}_i) \mapsto -(x_i, \dot{x}_i)$. Since $|\sin(x_i)|$ oscillates with zero mean for all i, the constant c must be zero. Hence, $x_i(t) = x_j(t)$ for all i, j identically and **AS2** is verified. Thus, by Theorem 6.1 the pendulum network asymptotically goes to an ASSM.

From (8.4), it can be seen that on the LaSalle surface, the dynamics of x_i are decoupled from the dynamics of x_j for all $i \neq j$. For small x_i, the dynamics of each individual term in L^μ corresponds to the stable dynamics of a spring-mass system with a κ-dependent mass $-m_\kappa > 0$ and spring constant $-D > 0$. The mass $-m_\kappa$, which determines the oscillation frequency of the pendulum for each individual cart, can be controlled by the choice of κ. For the nonlinear system also, constant energy curves are closed curves in the $(x_i, \dot{x}_i)$ plane. Hence, we have a periodic orbit for the angle made by each pendulum with the vertical line with a κ-dependent frequency. On the LaSalle surface, $J = \rho\gamma\dot{\theta}_i + (\beta + p\rho\gamma)\cos(x_i)\dot{x}_i =$ constant. Therefore, the velocity of the cart $\dot{\theta}_i$ oscillates about a constant velocity with the same frequency as the pendulum oscillation.

Figure 8.2 shows the results of a MATLAB simulation for the controlled network of pendulum/cart systems using the following values for the system parameters. The pendulum/cart systems have identical pendulum bob masses, lengths, and cart masses. The pendulum bob mass is chosen to be $m = 0.14$ kg, cart mass is $M = 0.44$ kg, and pendulum length is $l = 0.215$ m. The control gains are $\rho = -0.27$, $\kappa = 40$, $d_i = d = 0.2$, and $\epsilon = 0.0005$. We compute $m_\kappa = -0.058$ kgm$^2 < 0$ as required for

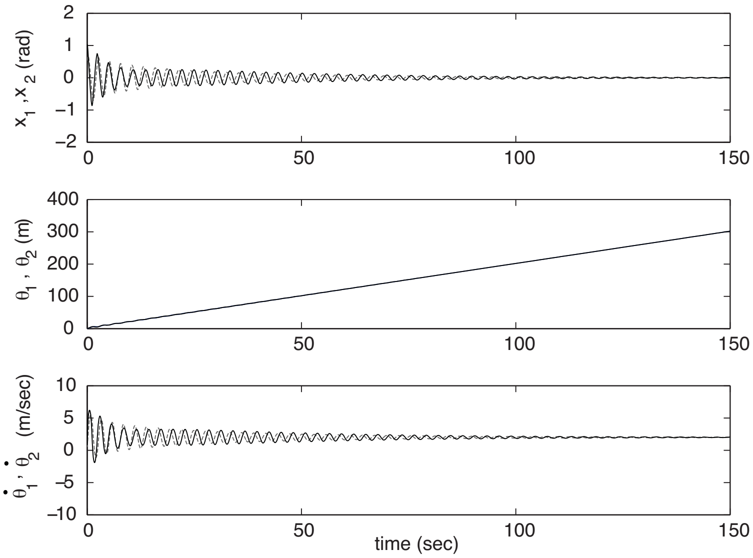

FIG. 8.3. *Simulation of a controlled network of pendulum/cart systems with dissipation designed for ASSRE. The pendulum angle, cart position, and cart velocity are plotted as a function of time for each of the two pendulum/cart systems in the network.*

stability. The initial conditions for the two systems shown are

$$\begin{aligned}&(\ x_1(0) \quad \dot{x}_1(0) \quad \theta_1(0) \quad \dot{\theta}_1(0) \quad x_2(0) \quad \dot{x}_2(0) \quad \theta_2(0) \quad \dot{\theta}_2(0)\ )\\ &= (\ 0.48 \quad 0.99 \quad 0.37 \quad 0.53 \quad 0.18 \quad 0.50 \quad 0.42 \quad 0.66\ ).\end{aligned}$$

Figure 8.2 shows plots of the pendulum angle, cart position, and cart velocity as a function of time for two of the coupled pendulum/cart systems. Convergence to an ASSM is evident. The frequency of oscillation of the pendula can be observed to be the same as the frequency of oscillation in the cart velocities. This frequency of oscillation can be computed as $\omega = \sqrt{D/m_\kappa}$ and the period of oscillation as $T = 2\pi/\omega = 2.8$ s, which is precisely the period of the oscillations observed in Figure 8.2.

8.2. Asymptotic stability of relative equilibria (ASSRE). In this case, we want to asymptotically stabilize the relative equilibrium $\mathbf{v}_{RE}$, i.e., $x_i = \dot{x}_i = 0$ for all i, $\theta_i = \theta_j$ for all $i \neq j$, and $\dot{\theta}_i = \frac{1}{n}\zeta$ for all i, and any constant scalar velocity ζ. Recall that this corresponds to each pendulum angle at rest in the upright position and all carts aligned and moving together with the same constant velocity $\frac{1}{n}\zeta$. Following (7.2), we let

$$u_i^{\text{diss}} = n d_i \dot{z}_i$$

for $i = 1, \ldots, n-1$ and

$$u_n^{\text{diss}} = n d_n (\dot{z}_n - \zeta),$$

where the control parameters d_i are positive constants.

Figure 8.3 shows the results of a MATLAB simulation for the controlled network of pendulum/cart systems with this dissipative control. We choose $\zeta = 2n$ m/s, and the remaining system and control parameters are as above in the ASSM case. The initial conditions for the two systems shown are

$$\begin{aligned}&\begin{pmatrix} x_1(0) & \dot{x}_1(0) & \theta_1(0) & \dot{\theta}_1(0) & x_2(0) & \dot{x}_2(0) & \theta_2(0) & \dot{\theta}_2(0) \end{pmatrix}\\ &= \begin{pmatrix} 0.53 & 1.12 & 0.56 & 0.50 & 1.02 & 0.63 & 0.24 & 0.81 \end{pmatrix}.\end{aligned}$$

Figure 8.3 shows convergence to the relative equilibrium; the pendula are stabilized in the upright position, the cart positions become synchronized, and the cart velocities converge to 2 m/s.

9. Final remarks. We have derived control laws to stabilize and stably synchronize a network of mechanical systems with otherwise unstable dynamics. We have proved stability of relative equilibria corresponding to synchronization in all variables and common steady motion in the actuated directions. Using two different choices of a dissipative term in the control law, we prove two different kinds of asymptotic stability. In the first case of dissipation, we show how to drive the network to a synchronized motion on the constant momentum surface determined by the initial conditions. Such a synchronized motion can be interesting when examined in physical space. In our example of a network of planar pendulum/cart systems, we show that the synchronized motion is periodic and the period of the oscillation can be controlled with a control parameter. In the second case of dissipation, we show how to isolate and asymptotically stabilize the relative equilibrium for any choice of constant momentum. We illustrate all of our results for a network of pendulum/cart systems. For this example, our approach yields a subtle choice in the coupling variables: The coupling that leads to stable synchronization is a function of relative positions of pendulum bobs, not simply relative positions of carts. Indeed, coupling as a function of relative cart positions destabilizes the network.

For asymptotic stabilization of the relative equilibrium, we assume that the interconnection graph for the network is connected. However, for asymptotic stabilization of a synchronized motion on the constant momentum surface, we assume that the interconnection graph for the dissipative control is completely connected. It is of interest in future work to determine whether this latter condition can be relaxed.

In Theorem 6.1 we prove asymptotic stabilization of a synchronized motion on the constant momentum surface; however, we cannot select the value of the momentum because it is determined by the initial conditions. In Remark 6.2 we propose a control law to simultaneously drive the momentum to a desired value. This control law appears to work in simulation; however, the stability analysis is more subtle. It raises a number of interesting questions. For example, suppose we have a dynamical system depending upon a parameter λ, i.e., the Lagrangian is given by a function $L(q, \dot{q}, \lambda)$, where q is the state variable. Assume that for each $\lambda \in [0, \epsilon]$, the (controlled) system is Lyapunov stable. If we now let λ evolve in time such that it "slowly" goes to a value $\bar{\epsilon} \in (0, \epsilon)$, can we still conclude that the system is Lyapunov stable in the infinite time domain? See [12] for results in the case when the unperturbed system has a uniformly asymptotically stable equilibrium. We plan to build on these tools to study our parameter dependency problem in future work.

Another future direction is the inclusion of collision avoidance in our framework. For instance, in our example, the carts move on parallel tracks, and hence collision avoidance is not an issue. However, it is interesting to consider the case in which all of the carts are on the same track and the pendulum/cart systems can be controlled without collisions for stable synchronization.

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