



## Brief paper

Autonomous rigid body attitude synchronization<sup>☆</sup>Alain Sarlette<sup>a,\*</sup>, Rodolphe Sepulchre<sup>a</sup>, Naomi Ehrich Leonard<sup>b</sup><sup>a</sup> Department of Electrical Engineering and Computer Science, University of Liège, 4000 Liège, Belgium<sup>b</sup> Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08544, USA

## ARTICLE INFO

## Article history:

Received 19 February 2008

Received in revised form

27 August 2008

Accepted 4 September 2008

Available online 23 December 2008

## Keywords:

Attitude control

Autonomous swarms

Consensus

Cooperative control

Energy shaping

## ABSTRACT

Control laws to synchronize attitudes in a swarm of fully actuated rigid bodies, in the absence of a common reference attitude or hierarchy in the swarm, are proposed in [Smith, T. R., Hanssmann, H., & Leonard, N.E. (2001). Orientation control of multiple underwater vehicles with symmetry-breaking potentials. In *Proc. 40th IEEE conf. decision and control* (pp. 4598–4603); Nair, S., Leonard, N. E. (2007). Stable synchronization of rigid body networks. *Networks and Heterogeneous Media*, 2(4), 595–624]. The present paper studies two separate extensions with the same energy shaping approach: (i) locally synchronizing the rigid bodies' attitudes, but without restricting their final motion and (ii) relaxing the communication topology from undirected, fixed and connected to directed, varying and uniformly connected. The specific strategies that must be developed for these extensions illustrate the limitations of attitude control with reduced information.

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## 1. Introduction

Coordination is a common requirement in applications involving robotic swarms or formations (Cortes, Martinez, & Bullo, 2004; Fax & Murray, 2004; Justh & Krishnaprasad, 2004; Leonard et al., 2007; Scardovi, Sepulchre, & Leonard, 2007; Sepulchre, Paley, & Leonard, 2008); understanding basic coordination mechanisms is the subject of ongoing research (Fax & Murray, 2004; Reynolds, 1987; Strogatz, 2003). Controlling a swarm of three-dimensional rigid bodies such that their orientations become asymptotically equal is called *attitude synchronization*. Its main use is in satellite formations (Bondhus, Pettersen, & Grasdahl, 2005; Izzo & Pettazzi, 2005; Krogstad & Grasdahl, 2006; Lawton & Beard, 2002; McInnes, 1996; Ren, 2006; VanDyke & Hall, 2006), e.g. for the Darwin mission, a space interferometer project under study by NASA and ESA, or for on-orbit assembly (Izzo & Pettazzi, 2005; McInnes, 1996). Operational requirements focus on *accuracy* near equilibrium (Beugnon, Buvat, Kersten, & Boulade, 2005). The present paper focuses on *convergence* from arbitrary initial orientations with

*limited information*. This is probably most relevant for deployment or recovery. Each rigid body is assumed fully actuated and called an *agent*.

A first main constraint limits communication links among agents. Under this constraint, (Lawton & Beard, 2002; VanDyke & Hall, 2006) consider attitude synchronization coupled to the tracking of a common external reference. Alternatively, (Bondhus et al., 2005; Krogstad & Grasdahl, 2006) synchronize attitudes in a leader–follower approach.

The qualifier “autonomous” refers to a second main constraint in the present paper: there is no hierarchy in the swarm and no external reference tracking. Autonomous operation in multi-agent systems is well motivated. It can increase robustness, since synchronization does not rely on permanent communication of a common reference, nor on the health of a potential leader. Also, it can stabilize the formation without interfering with its absolute motion. When orientation in inertial frame is not relevant (e.g. in assembly), this additional freedom may lower costs; in other cases, it builds a swarm that behaves more like a single body than a set of individual agents. The autonomous setting relies on the global Lie group structure of orientation manifold  $SO(3)$ . Therefore the popular unitary quaternion representation, containing two elements for each point of  $SO(3)$ , cannot be used.

Control laws for autonomous attitude synchronization are designed in Nair and Leonard (2007); Smith, Hanssmann, and Leonard (2001) with energy shaping. The present paper extends them in two separate ways.

1. In Nair and Leonard (2007); Smith et al. (2001), the dissipative control term, based on angular velocities, imposes the final

<sup>☆</sup> The material in this paper was partially presented at the 17th IFAC Automatic Control in Aerospace Symposium, Toulouse, 2007 and at the 46th IEEE Conference on Decision and Control, New Orleans, 2007. This paper was recommended for publication in revised form by Associate Editor Andrey V. Savkin under the direction of Editor Ian R. Petersen.

\* Corresponding author. Tel.: +32 4 366 2972; fax: +32 4 366 2989.

E-mail addresses: [alain.sarlette@ulg.ac.be](mailto:alain.sarlette@ulg.ac.be), [sarlette@montefiore.ulg.ac.be](mailto:sarlette@montefiore.ulg.ac.be) (A. Sarlette), [r.sepulchre@ulg.ac.be](mailto:r.sepulchre@ulg.ac.be) (R. Sepulchre), [naomi@princeton.edu](mailto:naomi@princeton.edu) (N.E. Leonard).

motion of the swarm. By only using *relative* angular velocities between agents, the present paper obtains a swarm behaving like a single rigid body: any synchronized free rigid body motion is a solution for the controlled swarm.

2. The results in Nair and Leonard (2007); Smith et al. (2001) are valid for fixed, undirected communication topologies. Inspired by consensus strategies on compact Lie groups (Sarlette & Sepulchre, in press; Scardovi, Sarlette, & Sepulchre, 2007), the present paper uses auxiliary variables to allow directed and time-varying communication topologies.

Both extensions illustrate difficulties encountered when reducing available information. The limitations of “relative dissipation” show the difficulty to control a system whose dynamics just have configuration symmetry with a controller that has configuration and velocity symmetry; this illustrates the relevance of formal *reduction techniques* (Hansmann, Leonard, & Smith, 2006). With limited communication, it is well known that for more than local synchronization, the non-convexity of compact Lie groups requires new strategies as compared to simple vector space algorithms (Moreau, 2004).

The paper is organized as follows. Section 2 formalizes the problem and reviews the main result of Nair and Leonard (2007); Smith et al. (2001). Section 3 presents extension 1 and Section 4 presents extension 2.

## 2. Problem setting and previous results

Orientations of  $N$  rigid bodies with respect to an inertial frame are represented by rotation matrices  $Q_k \in SO(3)$ ,  $k = 1 \dots N$ . Their motion follows Euler’s equations

$$Q_k^T \frac{d}{dt} Q_k = [\omega_k]^\wedge \quad (1)$$

$$J_k \frac{d}{dt} \omega_k = [J_k \omega_k]^\wedge \omega_k + \tau_k \quad (2)$$

where  $J_k = \text{diag}(J_{k1}, J_{k2}, J_{k3})$  is the moment of inertia matrix of agent  $k$ ,  $\omega_k \in \mathbb{R}^3$  its angular velocity and  $\tau_k \in \mathbb{R}^3$  its control torque; these are all expressed in body frame and  $J_{k1} \geq J_{k2} \geq J_{k3}$  without loss of generality. Torque and velocity in inertial frame are  $Q_k \tau_k$  and  $Q_k \omega_k$ . Matrix transpose is denoted  $\cdot^T$ , and  $[\cdot]^\wedge$  is defined by

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 \leftrightarrow [a]^\wedge = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

such that  $a \times b = [a]^\wedge b$ ,  $\forall a, b \in \mathbb{R}^3$  with  $\times$  the vector product. The inverse of  $[\cdot]^\wedge$  is denoted  $[\cdot]^\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ . The objective is to design  $\tau_k$ ,  $k = 1 \dots N$  such that  $Q_1 = \dots = Q_N$  asymptotically, under two main constraints.

1. (communication) Communication links among agents are restricted to the edges of a *communication graph*  $G$ ; “ $j$  sends information to  $k$ ” is denoted  $j \rightsquigarrow k$ .
2. (autonomy) Agents use no external reference; available information is expressed relative to body frame.

References (Nair & Leonard, 2007; Smith et al., 2001) solve this problem with *energy shaping*: “shape” the potential and kinetic energy of a system to make the desired state a stable equilibrium; control torques implement the “shaped” energy. Early work uses artificial potentials for robotic navigation and obstacle avoidance (Khatib, 1986; Rimon & Koditschek, 1992). Spacecraft control uses potential (McInnes, 1995) and kinetic (Bloch, Krishnaprasad, & Marsden, 1992) energy shaping. Potential shaping is used in Leonard (1997) for stabilization of rigid bodies in  $SE(3)$ . Energy shaping is used for synchronization of mechanical

system networks in Nair and Leonard (2008) and applied to networks on  $SO(3)$  and  $SE(3)$  in Hansmann et al. (2006); Nair and Leonard (2007); Smith et al. (2001). Kinetic energy shaping can transform any principal axis into the short axis (Nair & Leonard, 2007; Smith et al., 2001); this part is ignored here for simplification. Scalar  $d_{jk} := \sqrt{3 - \text{trace}(Q_k^T Q_j)}$  characterizes the distance between  $Q_k$  and  $Q_j$ , and  $(d_{jk})^2$  is smooth. Therefore (Nair & Leonard, 2007; Smith et al., 2001) use artificial potential

$$V = \frac{\sigma}{2} \sum_k \sum_{j \rightsquigarrow k} \text{trace}(Q_k^T Q_j), \quad \sigma < 0 \quad (3)$$

whose global minimum is attitude synchronization.  $V$  can have local minima when  $G$  is not a tree or complete graph (Sarlette & Sepulchre, in press). For  $G$  undirected (i.e.  $j \rightsquigarrow k \Leftrightarrow k \rightsquigarrow j$ ), the conservative control torques only depend on the relative orientations  $Q_k^T Q_j$  of agents  $j \rightsquigarrow k$  with respect to  $k$ :

$$\tau_k^{(P)} = -[\text{grad}_{Q_k}(V)]^\vee = -\sigma \sum_{j \rightsquigarrow k} [Q_k^T Q_j - Q_j^T Q_k]^\vee. \quad (4)$$

Write  $\tau_k = \tau_k^{(P)} + \tau_k^{(D)}$ , energy  $H = T + V$  with kinetic energy  $T = \sum_k T_k = \sum_k \frac{1}{2} \omega_k^T J_k \omega_k$ , and angular momentum  $M = \sum_k Q_k J_k \omega_k$ . Then  $\frac{d}{dt} H = \sum_k \omega_k^T \tau_k^{(D)}$  and  $\frac{d}{dt} M = \sum_k Q_k \tau_k$ . When  $\tau_k^{(D)} = 0$ , attitude synchronization with rotation around the short axis is (Lyapunov) stable. *Asymptotic* stability requires  $\tau_k^{(D)}$  to decrease  $H$ . Nair and Leonard (2007) exponentially stabilize this rotation when  $V$  contains an additional term *aligning the short axis with a specific direction in inertial space*; this does not satisfy autonomy. An alternative torque satisfying autonomy is

$$\text{with } \tau_k^{(P)} \text{ from (4), } \tau_k^{(D)} = -\gamma \omega_k, \quad \gamma > 0 \quad (5)$$

as in Ren (2006), for which  $H$  decreases till  $\omega_k = 0 \forall k$ . This asymptotically stabilizes attitude synchronization with zero velocity.

## 3. Extension 1: Relative angular velocities

Dissipation (5), although it preserves symmetry with respect to *orientation* of the agents (autonomy constraint), imposes their *motion*. A dissipative term preserving motion symmetry should drive agent velocities *towards each other* instead of towards 0, comparing  $Q_k \omega_k$  to the  $Q_j \omega_j$  (inertial frame), or equivalently  $\omega_k$  to the  $Q_k^T Q_j \omega_j$  (body frame). The corresponding torque is

$$\tau_k^{(D)} = \gamma \sum_{j \rightsquigarrow k} (Q_k^T Q_j \omega_j - \omega_k), \quad \gamma > 0. \quad (6)$$

Attitude synchronization is more difficult with (6) than with (5) because the  $\tau_k$  only influence *relative* velocities, while the rigid body dynamics still depend on  $\omega_k$ ; indeed, the nonlinearity of (2) with respect to  $\omega_k$  cannot be reduced. The study of mechanical systems that are invariant with respect to configuration symmetries, but still depend on the associated velocities, is the subject of *reduction techniques*—see (Hansmann et al., 2006) for a discussion about  $SO(3)$  and  $SE(3)$ . The following result illustrates typical difficulties caused by the remaining velocity in the dynamics. The proof also illustrates the difficulty to obtain results without resorting to formal reduction techniques.

**Theorem 1.** Consider  $G$  fixed, undirected and connected and control  $\tau_k = \tau_k^{(P)} + \tau_k^{(D)}$  with  $\tau_k^{(D)}$  defined by (6) and  $\tau_k^{(P)} = -[\text{grad}_{Q_k}(V)]^\vee$ , where  $V$  is a bounded potential.

- (a) Regardless of  $V$ , for any initial conditions, velocities in inertial frame  $Q_k \omega_k$  asymptotically synchronize.

(b) For identical rigid bodies ( $J_k = J\forall k$ ) and  $V$  defined by (3), angular momentum  $M$  is conserved. Given  $M_{\max}$ , there exists  $\sigma^* < 0$  (depending on  $N, J, G$  and  $M_{\max}$ ) such that for  $|\sigma| > |\sigma^*|$ , the set of free rigid body motions with synchronized attitudes  $Q_k(t) = Q_j(t)\forall j, k$  and  $\|M\| \in (0, M_{\max})$  is locally asymptotically stable.

**Proof.** (a)

$$\begin{aligned} \frac{d}{dt}H &= \sum_k \omega_k^T \tau_k^{(D)} \\ &= \gamma \sum_k \sum_{j \rightsquigarrow k} \omega_k^T (Q_k^T Q_j \omega_j - \omega_k) \\ &= \gamma \sum_k \sum_{j \rightsquigarrow k} (Q_k \omega_k)^T (Q_j \omega_j - Q_k \omega_k) \\ &= -\gamma (\Omega^a)^T (L \otimes I_3) \Omega^a \end{aligned}$$

where  $\Omega^a$  is the  $3N$ -vector containing all  $Q_k \omega_k$  and  $L \otimes I_3$  is the Kronecker product of the Laplacian<sup>1</sup>  $L$  of  $G$  with the  $3 \times 3$  identity matrix. For undirected graphs,  $L$  is positive semidefinite; its kernel reduces to  $x = (c \ c \ c \cdots \ c)^T$ ,  $c \in \mathbb{R}$ , if and only if  $G$  is connected. Thus  $H$  decreases unless all  $Q_k \omega_k$  are equal. Orientations evolve in a compact set where  $V$  is bounded, and  $H$  is radially unbounded in the non-compact dimension corresponding to velocities. Therefore a LaSalle argument proves that the swarm converges to an invariant set where  $Q_k \omega_k = Q_j \omega_j \forall j, k$ , under the dynamics (1) and (2) with  $\tau_k = \tau_k^{(P)}$ .

(b) Conservation of  $M$  is equivalent to  $\sum_k Q_k \tau_k = 0$  which is easy to verify. For synchronization of the  $Q_k$ , the proof is in two steps. First, show that given a neighborhood  $W \ni (Q_1 \cdots Q_N, \omega_1 \cdots \omega_N)$  of the set  $S_{M^*}$  of free rigid body motions with synchronized attitudes  $Q_k(t) = Q_j(t)\forall j, k$  and total angular momentum  $\|M^*\| < M_{\max}$ , there exist  $|\sigma_1|$  and a neighborhood  $U$  of  $S_{M^*}$  such that starting in  $U$  implies staying in  $W$  if  $|\sigma| > |\sigma_1|$ . Then show that there exist  $|\sigma_2|$  and a neighborhood  $W_1$  of  $S_{M^*}$  such that for  $|\sigma| > |\sigma_2|$ , solutions of (1) and (2) with identical  $Q_k \omega_k$  and  $\tau_k = \tau_k^{(P)}$  as in (4) that stay in  $W_1$  are necessarily in  $S_{M_0}$ , where  $M_0$  is the initial angular momentum of the system. Then taking  $W = W_1$  and  $|\sigma^*| > \max(|\sigma_1|, |\sigma_2|)$  concludes the proof.

For the first part, recalling  $(d_{jk})^2 = 3 - \text{trace}(Q_k^T Q_j)$ , let  $W = \{(Q_1 \cdots Q_N, \omega_1 \cdots \omega_N) : (d_{jk})^2 < \varepsilon \forall k, j \text{ and } \|M - M^*\| < \delta\}$ . If  $E$  is the number of edges in  $G$ ,

$$\frac{1}{2} \sum_k \sum_{j \rightsquigarrow k} (3 - \text{trace}(Q_k^T Q_j)) = 3E - V(t)/\sigma < \varepsilon$$

(the factor  $\frac{1}{2}$  comes from counting each distance twice) is sufficient for a solution starting with  $\|M - M^*\| < \delta$  to be in  $W$  at time  $t$ . Since  $H$  decreases, for  $t \geq 0$ ,  $T(t) + V(t) \leq T(0) + V(0)$  so  $V(t) - V(0) \leq T(0) - T(t) \leq T(0)$ . Hence if  $|\sigma| > |\sigma_1|$ , then  $(V(0) - V(t))/\sigma \leq T(0)/|\sigma_1|$  and so

$$3E - V(t)/\sigma \leq (3E - V(0)/\sigma) + T(0)/|\sigma_1|.$$

Choose a neighborhood  $U_1 \subseteq W$  of  $S_{M^*}$  such that  $\max_k \|Q_k \omega_k - \frac{M^*}{N}\| < \frac{\beta}{N} \|M^*\|$ , for some  $\beta > 0$ . Initial conditions in  $U_1$  imply  $T(0) < \frac{\|M^*\|^2}{2J_3 N} (1 + \beta)^2$ . Then taking (assuming actual  $M^*$  unknown)  $|\sigma_1| > \frac{M_{\max}^2}{\varepsilon J_3 N} (1 + \beta)^2$  ensures  $T(0)/|\sigma_1| < \frac{\varepsilon}{2}$ . Also define  $U_2$  such that  $3E - V(0)/\sigma < \varepsilon/2$ . Then with initial conditions in  $U = U_1 \cap U_2$ , the system stays in  $W$  for  $t \geq 0$ .

The second part involves more calculations, which will not all be detailed. Denote the final common velocity by  $Q_k \omega_k = \Omega(t)\forall k$ ;

note that  $\|\Omega\| \leq \frac{\|M\|}{NJ_3}$ . The time derivative of  $\Omega = Q_k \omega_k$  along solutions of the closed-loop system is (note that  $\frac{d}{dt}(Q_k^T Q_j) = 0$  when  $Q_k \omega_k = Q_j \omega_j$ )

$$\begin{aligned} \frac{d}{dt}\Omega &= -\sigma Q_k J^{-1} \sum_{l \rightsquigarrow k} [Q_k^T Q_l - Q_l^T Q_k]^\vee \\ &\quad + Q_k J^{-1} Q_k^T [Q_k J Q_k^T \Omega]^\wedge \Omega \end{aligned} \quad (7)$$

which must hold  $\forall k$ . Denoting the first and second terms on the right side of (7) by  $(7a)_k$  and  $(7b)_k$  respectively,

$$\|(7a)_k - (7a)_j\|^2 = \|(7b)_k - (7b)_j\|^2 \quad (8)$$

$\forall k, j$ . The right side of (8) is bounded by (calculations)

$$\|(7b)_k - (7b)_j\|^2 \leq \frac{16J_1^2}{J_3^2} \|\Omega\|^4 (d_{jk})^2.$$

Thus the same bound must hold for the left side of (8),

$$\|(7a)_k - (7a)_j\|^2 \leq \frac{16J_1^2}{J_3^2} \|\Omega\|^4 (d_{jk})^2.$$

Summing the last condition over all  $k, j$ , using conservation of  $M$  and linearizing leads to (calculations)

$$\frac{2\sigma^2 \lambda_2^3}{J_1^2} (d_{\max}^2 + \mathcal{O}(d_{\max}^4)) \leq \frac{16J_1^2 E \|M^*\|^4}{J_3^6 N^2} d_{\max}^2 + \mathcal{O}(d_{\max}^3) \quad (9)$$

where  $\lambda_2 > 0$  is the second-smallest eigenvalue of the Laplacian  $L$  of  $G$  and  $d_{\max}^2$  denotes the maximal value of  $(d_{jk})^2$  among all pairs of connected agents. Choosing  $W_1$  such that the higher-order terms represent less than  $\gamma_1 < 1$  and  $\gamma_2 < 1$  respectively on the left and right side of (9), the condition becomes

$$d_{\max}^2 \leq \frac{8J_1^4 E}{(1 - \gamma_1)(1 - \gamma_2) J_3^6 \lambda_2^3} \frac{\|M^*\|^4}{\sigma^2} d_{\max}^2. \quad (10)$$

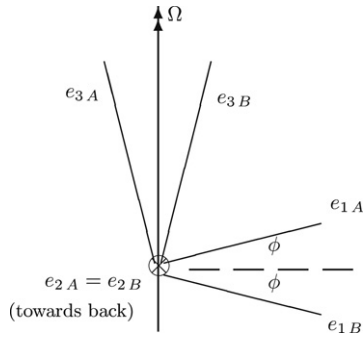
Taking  $\sigma^2 > (\sigma_2)^2 := \frac{8J_1^4 E M_{\max}^4}{(1 - \gamma_1)(1 - \gamma_2) J_3^6 \lambda_2^3}$ , (10) can only be satisfied if  $d_{\max}^2 = 0 \Leftrightarrow Q_k = Q_j \forall k, j$ .  $\square$

Several comments are in order about [Theorem 1](#).

- [Theorem 1\(a\)](#) still holds for time-varying (uniformly connected) and directed, but balanced graphs<sup>2</sup> because  $x^T L x$  is still non-negative in this case.
- When the swarm is synchronized, all control torques  $\tau_k$  vanish. Hence, the (unimposed) motion of the synchronized swarm can be any free rigid body motion.
- [Theorem 1\(b\)](#) is a local result. However, simulations indicate a large basin of attraction  $U$ . The proof contains three conditions for  $U$ . Conditions with  $\delta$  and  $\beta$  basically impose  $\|M\|$  lower and upper bounded and  $\|\omega_k\|$  upper bounded; for “infinite  $\sigma$ ”, this still allows almost any initial condition. The critical constraint is with  $\varepsilon$  chosen to bound high-order terms in (9). For “infinite  $\sigma$ ” relevant high-order terms are on the left; they are like  $\sin^2(\theta) - \theta^2$  around  $\theta = 0$ . Then it is sufficient that all  $Q_k$  are inside a geodesic ball of radius  $\pi/2$ , also the maximal convex set of  $SO(3)$ . This is consistent with synchronization being the only minimum of  $V$  for all  $Q_k$  in a convex set.

<sup>1</sup> The Laplacian of a graph has entries  $l_{kj} = -1$  if  $j \rightsquigarrow k$  and  $l_{kj} = 0$  otherwise for  $j \neq k$ ;  $l_{kk} = -\sum_{j \neq k} l_{kj}$ .

<sup>2</sup> A directed graph is *balanced* when at each node, the number of incoming edges equals the number of outgoing edges.



**Fig. 1.** Two rigid bodies in a situation from which (4) and (6) do not synchronize attitudes for small  $\sigma$ . All vectors lie in the same plane, except  $e_{2A} = e_{2B}$  perpendicular to the page.

- The bound on  $|\sigma|$  reflects that the controller must overcome unknown “perturbations”  $[J\omega_k]^\wedge \omega_k$ . Sliding mode control like  $\text{grad}_k(V)/\|\text{grad}_k(V)\|$  is similar to “infinite  $\sigma$ ” near synchronization. However, the resulting chattering depends on controller parameters, and  $\omega_k$ -dependent bounds will still be required to stay in a neighborhood of synchronization where the “perturbations” do not desynchronize the system. This even gets more difficult since  $\frac{d}{dt}M \neq 0$  and  $\frac{d}{dt}H \neq 0$ .
- With no condition on  $\sigma$ , there are situations arbitrarily close to synchronization but from which it is never reached. Take two agents A and B synchronized and rotating around  $e_3$  with velocity  $\Omega$ , where  $e_1, e_2, e_3$  denote principal axes of  $J_1 > J_2 > J_3$ . Now (see Fig. 1), with respect to this synchronized state, A and B are tilted by  $\phi$  and  $-\phi$  respectively around  $e_2$ , with  $\phi$  arbitrarily small; they still rotate with  $Q_A\omega_A = Q_B\omega_B = \Omega$  around the axis aligned with the initial synchronized  $e_3$ , which now makes an angle  $\phi$  with actual axes  $e_{3A}$  and  $e_{3B}$ . Then  $[J\omega_k]^\wedge \omega_k$  pulls A and B further apart, while  $\tau_k^{(P)}$  pulls them together. For a particular ratio  $\|\Omega\|^2/\sigma$ , both effects exactly cancel.
- In Jalnapurkar and Marsden (1998), general forms for relative dissipation are proposed based on reduction techniques; they require that consecutive Poisson brackets of allowed torques restore full rank, which is not the case here.

#### 4. Extension 2: Directed and varying graphs

The previous algorithms require fixed, undirected graphs. “Consensus algorithms”, see (Moreau, 2004; Olfati-Saber, Fax, & Murray, 2007; Tsitsiklis & Athans, 1984), globally synchronize variables in vector spaces with directed and time-varying graphs. Therefore, (Sarlette & Sepulchre, in press) embeds  $SO(3)$  in  $\mathbb{R}^{3 \times 3}$  and builds a consensus algorithm for auxiliary variables  $Y_k \in \mathbb{R}^{3 \times 3}$ ;  $Q_k$  tracks the projection of  $Y_k$  on  $SO(3)$  such that synchronizing  $Y_k$  implies synchronizing  $Q_k$ . Here (Sarlette & Sepulchre, in press) is extended to mechanical system (1) and (2).

The consensus algorithm is, in inertial frame,

$$\frac{d}{dt}Y_k = \beta \sum_{j \sim k} (Y_j - Y_k), \quad \beta > 0 \quad (11)$$

or equivalently in body frames with  $X_k = Q_k^T Y_k$

$$\frac{d}{dt}X_k = \beta \sum_{j \sim k} (Q_k^T Q_j X_j - X_k) - [\omega_k]^\wedge X_k \quad (12)$$

for  $k = 1 \dots N$ . Projection and tracking are implemented with artificial potential

$$V = \sigma \sum_k \text{trace}(Q_k^T Y_k), \quad \sigma < 0. \quad (13)$$

Indeed, the distance from  $Y_k$  to  $Q \in SO(3)$  in  $\mathbb{R}^{3 \times 3}$  is

$$\begin{aligned} \|Y_k - Q\|^2 &= \text{trace}((Y_k - Q)^T(Y_k - Q)) \\ &= 3 + \text{trace}(Y_k^T Y_k) - 2\text{trace}(Q_k^T Y_k). \end{aligned}$$

The resulting torque is

$$\tau_k = -\sigma [X_k - X_k^T]^\vee + \tau_k^{(D)}, \quad k = 1 \dots N. \quad (14)$$

Controller (12) and (14) only involves variables in body frame (autonomy) but, unlike extension 1, contains  $\omega_k$ . Cost is added to store, update and exchange the  $X_k$ .

**Theorem 2.** For  $G$  uniformly connected<sup>3</sup>, attitude synchronization with  $\omega_k = 0 \forall k$  is almost globally asymptotically stable for (1) and (2) with controller (12), (14) and (5).

**Proof.** The  $Y_k$  evolve under (11) independently of the agents’ motions. Hence classical consensus results (e.g. (Moreau, 2004)) ensure their exponential convergence to a common constant  $Y_\infty$  for  $G$  uniformly connected. The  $Y_\infty$  such that  $v(Q) := \sigma \text{trace}(Q^T Y_\infty)$  has several minimizers on  $SO(3)$ , are non-generic (Sarlette & Sepulchre, in press); the following assumes that  $v(Q)$  has a unique minimizer  $Q^* = \text{Proj}_{SO(3)}(Y_\infty)$ .

From the previous paragraph, (1), (2), (14) and (5) form an asymptotically autonomous system where agents are decoupled; the limiting (autonomous) system is obtained by replacing  $X_k$  with  $Q_k^T Y_\infty$ . Solutions of an asymptotically autonomous system converge to a chain recurrent set of the limiting system (Mischaikow, Smith, & Thieme, 1995). A point  $x$  in state space is chain recurrent if and only if it belongs to the intersection of all locally asymptotically stable sets containing the positive limit set  $L^+(x)$  of  $x$  (Oprocha, 2005). The limiting system for  $k$  is of the “shaped energy” form with  $H$  bounded below and  $\frac{d}{dt}H = -\gamma \|\omega_k\|^2$ . A LaSalle argument on  $H$  as for Theorem 1(a) shows that the positive limit set  $L^+(x)$  for the autonomous system of any point  $x := (Q_k, \omega_k)$  only contains equilibria  $x_e$ , where  $\omega_k = 0$  and  $Q_k$  is at a critical point of  $v(Q_k)$ ; the set of equilibria is denoted by  $\mathcal{E}$ . Any  $x \in \mathcal{E}$  is chain recurrent. For  $x \notin \mathcal{E}$ , define  $\mu := \max_{y \in L^+(x)} (H(y)) < H(x)$  and  $S := \{y : H(y) \leq \mu\}$ .  $S$  contains  $L^+(x)$  but not  $x$ , since  $H(y) < H(x)$  for any  $y \in L^+(x)$  when  $x \notin \mathcal{E}$ . For any  $Y_\infty$ ,  $v(Q_k)$  takes a finite number of values at critical points (Sarlette & Sepulchre, in press). Then there exists  $\nu > \mu$  such that  $H(y) \in (\mu, \nu) \Rightarrow y \notin \mathcal{E}$ . Choose an  $\varepsilon$ -neighborhood  $S_\varepsilon$  of  $S$  and define  $\rho = \min_{y \in (SO(3) \times \mathbb{R}^3) \setminus S_\varepsilon} (H(y)) > \mu$ . Select a  $\delta$ -neighborhood  $S_\delta$  of  $S$  where  $H(y) < \min(\{\nu, \rho\})$ . Starting in  $S_\delta$  ensures staying in  $S_\varepsilon$  so  $S$  is locally stable; every point of  $S_\delta$  must converge to a point of  $\mathcal{E} \cap S_\delta \subseteq \mathcal{E} \cap S$  so  $S$  is locally asymptotically stable. Thus  $S$  is an asymptotically stable set containing  $L^+(x)$  but not  $x$ . Then from (Oprocha, 2005), non-equilibrium points are not chain recurrent: the autonomous system’s chain recurrent set reduces to the critical points of  $v(Q_k)$ . For a generic  $Y_\infty$ , the latter contain the unique minimum  $Q^*$  and three unstable points (Sarlette & Sepulchre, in press). All solutions starting outside these three points and their stable manifolds converge to  $Q_k = Q^*$ .  $\square$

Theorem 2 deserves the following comments.

- Unstable solutions that do not converge to synchronization are (i) situations where  $\text{Proj}_{SO(3)}(Y_\infty)$  is not unique; (ii) a few unstable critical points of  $V$  in (13).
- The choice  $X_k(0) = \alpha_k I_3$  with  $\alpha_k \in \mathbb{R}^+ \forall k$  avoids unnecessary transients when the  $Q_k(0)$  are close.

<sup>3</sup> Moreau (2004):  $G$  is uniformly connected if there exist an agent  $k$ , and time periods  $T, \Delta > 0$  such that,  $\forall t$ , there is a directed path from  $k$  to all other agents in the union of all edges appearing at least during  $\Delta$  time units in the interval  $[t, t+T]$ .

- Embedding  $SO(3)$  in  $\mathbb{R}^{3 \times 3}$  uses nine-dimensional  $X_k$ . In fact, the three-dimensional  $SO(3)$  can be embedded in  $\mathbb{R}^{3 \times 2}$  by only retaining the first two columns of  $Q_k$ . It is even possible to embed  $SO(3)$  in  $\mathbb{R}^5$  (Stuelpnagel, 1964).
- The proof of Theorem 2 uses vector space consensus in cascade with projection and tracking for the actual system. Sarlette and Sepulchre (in press) applies this to first-order integrators. Sarlette, Sepulchre, and Leonard (2007) consider (1) and (2) with “consensus tracking”; “energy shaping” control is expected to be more robust.
- Dissipation (5) forces zero final velocities. It is tempting to combine extensions 1 and 2 to obtain global convergence without restricting final motion. Unfortunately this is not easy, independently of  $\tau_k^{(D)}$ , because the final motion is dictated by  $Q^* = \text{Proj}(Y_\infty)$  and (11) leads to a constant  $Y_\infty$ . Different adaptations have been explored in simulation, with no conclusive result.

## 5. Conclusion

This paper presents two extensions of results in Nair and Leonard (2007); Smith et al. (2001) for rigid body attitude synchronization with limited communication links and no external reference.

A first controller avoids imposing the final motion of the swarm. For fixed, undirected, connected communication graphs, angular velocities globally synchronize. Asymptotic attitude synchronization is local and requires adapting the strength of the interaction potential to the initial total angular momentum. These limitations illustrate difficulties, in accordance with insights of formal reduction techniques (Hansmann et al., 2006), encountered when using controllers with configuration and velocity symmetry for mechanical systems that just have configuration symmetry.

A second controller achieves almost-global attitude synchronization for time-varying and directed graphs, at the cost of introducing auxiliary variables that communicating agents exchange; connectedness can be relaxed to the union of all links appearing in a fixed time span.

Combining both extensions remains speculative. Future work could add actuator constraints to the communication constraints. Also, similar proofs can likely be repeated for mechanical systems on other Lie groups.

## Acknowledgements

This paper presents research results of the Belgian network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors. The first author is supported as an FNRS fellow (Belgian Fund for Scientific Research) and achieved part of this work as a Visiting Student Research Collaborator at Princeton University, MAE department, whose hospitality and intellectually inspiring environment are gratefully acknowledged; this was partly supported by the 1st Odissea prize initiated by the Belgian Senate. The third author was supported in part by ONR grants N00014-02-1-0826 and N00014-04-1-0534, by NSF grant CMS-0625259 and by AFOSR grant #FA9550-07-1-0528.

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**Alain Sarlette** received the engineering degree in space physics and technologies from the University of Liege, Liege, Belgium, in 2005. He is currently a Ph.D. student at this university, researching on the topic of “Geometry and Symmetries in Coordination Control”.



**Rodolphe Sepulchre** received the engineering and Ph.D. degrees in applied mathematics from the University of Louvain, Louvain, Belgium, in 1990 and 1994, respectively. During 1994–1996, he was a Postdoctoral Fellow at the University of California, Santa Barbara. From 2002 to 2003, he was with Princeton University, Princeton, NJ. He is currently with the Department of Electrical Engineering and Computer Science, Institut Montefiore, University of Liege, Liege, Belgium, as a Professor. His current research interests include nonlinear dynamics and control, collective dynamics and consensus problems on manifolds, and optimization on manifolds. He is the coauthor of the books “Constructive Nonlinear Control” (Springer-Verlag, 1997) and “Optimization Algorithms on Manifolds” (Princeton University Press, 2007).



**Naomi Ehrich Leonard** received the B.S.E. degree in mechanical engineering from Princeton University, Princeton, NJ, in 1985, and the M.S. and Ph.D. degrees in electrical engineering from the University of Maryland, College Park, in 1991 and 1994, respectively. From 1985 to 1989, she was an Engineer in the Electric Power Industry, MPR Associates, Inc. She is currently a Professor of mechanical and aerospace engineering and an Associated Faculty Member of the Program in Applied and Computational Mathematics, Princeton University. Her current research interests include nonlinear control and dynamics, cooperative control, mobile sensor networks, autonomous underwater vehicles, adaptive ocean sampling, and collective motion in animal aggregations.