Pursuit, herding and evasion: A three-agent model of caribou predation

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Abstract— **Motivated by problems of pursuit and evasion in coordinated multi-agent systems, we present a model of pursuit, herding and evasion for three agents: a single pursuer, e.g. a bear, chooses a target point along the line connecting two evaders, and the two evaders, e.g. a mother caribou and her calf, each choose a strategy that trades off evasion and herding. The model is based on feedback control of constant speed steered particles on the plane. Dynamics over a reduced set of shape variables are defined. Parallel-motion shape equilibria are studied, with stability analysis and analytic solutions provided for special cases in the parameter space. Simulation results are also presented that suggest existence of optimal strategies for the bear and the caribou in a game theoretic sense.**

I. INTRODUCTION

The mathematical study of pursuit and evasion has a rich history stretching back to the 18th century with Bouguer's famous pirate ship pursuit problem [1]. Although the basic concept of pursuit and evasion is not new, it still remains an active area of study due to the multitude of possible applications and variations. The study of pursuit and evasion for multi-agent systems is motivated by problems in wildlife management, where sensitivity of animal group behavior to environmental change needs to be addressed, and by problems in distributed control of mobile robotic networks, where coordination can be advantageous in maneuvers that require approaching or avoiding a directed signal.

Feedback laws can be used to great effect in describing the interactions among animals, both in herding and pursuit behaviors. One-on-one pursuit and evasion has been studied extensively for steered-particle systems with feedback control laws: recent studies look at "motion camouflage" strategies employed by dragonflies and bats [2], cyclic pursuit [3], and pursuit and evasion strategy selection as an evolutionary game [4], [5]. Voronoi diagram approaches have been useful for systems with multiple pursuers [6], and in strategies for trapping an evader in a limited environment [7]. Systems with two evaders have been studied under the framework of differential game theory, in the "successive pursuit" of [8], and more recently the "cooperative defense" of [9], [10].

As a step towards our goal of developing a framework to examine the role of herding among evaders in pursuit-evasion dynamics, we focus in the present paper on studying how a heterogeneous system with three agents behaves under a combination of feedback control laws inspired by the cohesion/repulsion (herding) feedback rules of [11], and pursuit and evasion feedback rules of [4] and [5].

An example from nature of a herding pair of evaders and a single pursuer is a mother caribou and her calf fleeing from a predator. For the woodland caribou (*Rangifer tarandus caribou*) of Northeastern Canada, population growth is heavily influenced by calf mortality due to predation by bear, wolf, coyote, and lynx. In the wild, female caribou usually produce one calf per year, in the early summer. Calves typically stay close to the mother and are most vulnerable to predation in the first month of life as they struggle to keep pace with the adults [12], [13].

The aim of the present work is to define a mathematical model for predation in which (1) a predator (e.g. a bear) pursues one of the two evaders (e.g., a mother or a calf caribou) or a point along the line that connects the two, and (2) each evader chooses a strategy that is a convex combination of evasion and herding. In Section II of this paper we discuss a steered-particle model for the bear and caribou that features pursuit, evasion, and herding control laws. In Section III we introduce a simplified first-order model that captures much of the same behavior. Reduced-order shape dynamics are presented, and classes of equilibria are defined. In Section IV we study the case of the mother using a pure evasion strategy and prove conditions such that the bear cannot come between the mother and the calf. For special parameter values, we provide a stability analysis and analytic solutions. In Section V we present numerical results that suggest existence of optimal strategies and a way in which the system could be viewed as a differential game.

II. STEERED-PARTICLE MODEL

Our model is motivated by the interactions among a bear, a mother caribou, and her calf. The bear is a pursuer in the classical sense, choosing its target along the line between the two caribou. Each caribou is an evader and a herder; its strategy is a convex combination of classical evasion of the bear and herding with the other caribou.

The equations of motion are based on a steered-particle model: the input u_j controls the angular velocity of agent j. In the case of constant speeds considered here, this is equivalent to choosing the instantaneous curvature of the trajectory. The mother and calf are taken to have unit speed, and the bear has a speed $v \geq 1$. The agents are taken to evolve on the complex plane, so system states are the position vectors $r_j \in \mathbb{C}$ and heading angles θ_j measured counterclockwise from the real axis, as shown in Fig. 1. The equations of motion are given by

$$\dot{r}_b = ve^{i\theta_b}, \qquad \dot{\theta}_b = u_b$$
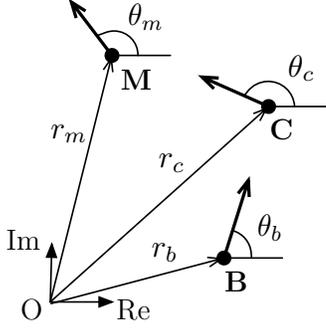$$\dot{r}_m = e^{i\theta_m}, \qquad \dot{\theta}_m = u_m \qquad (1)$$

Fig. 1. System states for steered-particle model of bear (B), mother (M), and calf (C).

$$\dot{r}_c = e^{i\theta_c}, \qquad\qquad \dot{\theta}_c = u_c.$$

The feedback control law for the bear is taken from the "classical pursuit" law of [4]:

$$u_b = -\eta \left\langle \frac{r_p}{|r_p|}, ie^{i\theta_b} \right\rangle, \qquad (2)$$

$$\text{with } r_p = r_b - [w_t r_c + (1 - w_t) r_m],$$

where r_p is the vector from the target point to the bear's position. The target point is a point along the line connecting the calf and the mother parameterized by a target parameter, $w_t \in [0, 1]$; when $w_t = 1$, the bear targets the calf, when $w_t = 0$ it targets the mother, and for intermediate values of w_t the bear targets a point between the two. $\eta > 0$ is a gain which, when high enough, guarantees convergence to a "pursuit manifold" in finite time as discussed in [4].

Feedback laws for the mother and calf are taken to be a convex combination of evasion from the bear and herding with the other caribou. The weights $w_m, w_c \in [0, 1]$ represent the mother's and calf's reliance on herding, respectively:

$$u_m = w_m u_{m,herd} + (1 - w_m) u_{m,evade}$$
$$u_c = w_c u_{c,herd} + (1 - w_c) u_{c,evade}. \qquad (3)$$

The evading rule $u_{j,evade}$ is the "classical evasion" law from [5]:

$$u_{j,evade} = -\eta \left\langle \frac{r_b - r_j}{|r_b - r_j|}, ie^{i\theta_j} \right\rangle. \qquad (4)$$

The herding rule comes from the repulsion-orientation-attraction (ROA) laws for group motion presented in [11], with concentric non-overlapping zones (see also [14]). When the distance between calf and mother is less than repulsion radius r_r, a repulsion rule is used. With only two agents herding, the repulsion rule takes the form of the evasion rule (4). For a distance greater than r_r but less than orientation radius r_o, an orientation rule is used which steers the two agents towards alignment. For a distance greater than r_o but less than attraction radius r_a, an attracting rule is used, which, for only two agents herding, takes the form of the pursuit rule (2). For a distance greater than r_a, no interaction occurs and $u_{j,herd} = 0$. For the calf, then, the herding

feedback rule is

$$u_{c,herd} = \begin{cases} -\eta \left\langle \frac{r_m - r_c}{|r_m - r_c|}, ie^{i\theta_c} \right\rangle & \text{if } |r_c - r_m| < r_r, \\ \left\langle ie^{i\theta_c}, e^{i\theta_m} \right\rangle & \text{if } r_r \le |r_c - r_m| < r_o, \\ \eta \left\langle \frac{r_m - r_c}{|r_m - r_c|}, ie^{i\theta_c} \right\rangle & \text{if } r_o \le |r_c - r_m| < r_a, \\ 0 & \text{if } |r_c - r_m| \ge r_a. \end{cases} \qquad (5)$$

The herding rule for the mother $u_{m,herd}$ is the same with indices c and m switched.

For the remainder of this paper, the bear's velocity is taken to be the same as the caribou ($v = 1$), in order to study steady-state behavior of the system.

III. SIMPLIFIED FIRST-ORDER MODEL

In [4] it is shown that under the classical pursuit steering law (2) with high enough gain, the "pursuit manifold" of the system is reachable within finite time, such that the pursuer will be traveling in a direction directly towards the target. Similarly the classical evasion control law will bring the states towards a corresponding evasion manifold where the evader travels directly away from the pursuer.

If we limit the caribou herding interaction to just the attraction mode, the caribou control law becomes a linear interpolation between classical evasion of the bear and classical pursuit of the other caribou. With high gains on the control inputs, the heading dynamics will quickly settle on the desired directions. We may simplify the model by eliminating the heading dynamics, instead taking the instantaneous desired direction of each agent as the control input.

In this way, the parameters w_t, w_c, and w_m serve to interpolate between two different direction vectors. For convenience, the relative vector between agent positions is written as $r_{jk} = r_j - r_k$. The bear's parameter w_t defines its direction of travel as a convex combination of the direction towards the calf (r_{cb}) and the direction towards the mother (r_{mb}). The calf's parameter w_c defines its direction of travel as a convex combination of the direction towards the mother (r_{mc}) and the direction away from the bear (r_{cb}). Similarly, the mother's parameter w_m defines its direction of travel as a convex combination of the direction towards the calf (r_{cm}) and the direction away from the bear (r_{mb}). The equations of motion for the three agents thus become

$$\dot{r}_m = \hat{u}_m,$$
$$\dot{r}_b = \hat{u}_b, \qquad (6)$$
$$\dot{r}_c = \hat{u}_c,$$

with unit-length direction inputs

$$\hat{u}_m = e^{i[w_m \angle r_{cm} + (1 - w_m) \angle r_{mb}]},$$
$$\hat{u}_b = e^{i[w_t \angle r_{cb} + (1 - w_t) \angle r_{mb}]}, \qquad (7)$$
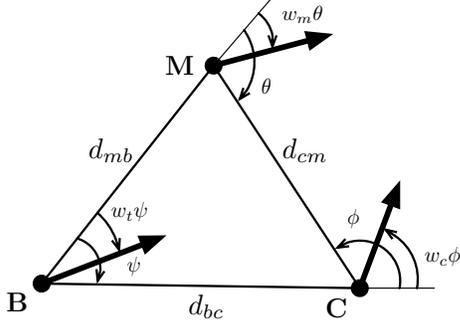$$\hat{u}_c = e^{i[w_c \angle r_{mc} + (1 - w_c) \angle r_{cb}]}.$$

Fig. 2. Shape variables $\psi, \phi, \theta, d_{bc}, d_{cm}, d_{mb}$ describing relative positions of the bear (B), the mother caribou (M) and the caribou calf (C), along with directions of motion as determined by parameters w_t, w_c, and w_m.

It is important to note that the control laws become undefined when any two agents are coincident, due to their dependence on the heading angles of the relative vectors r_{jk}.

Since the control laws rely solely on the relative vectors between the three agents, the relative motion of the agents is independent of the positions of the agents with respect to a global coordinate frame. Thus it is possible to reduce the order of the system further and directly study the dynamics of "shape variables," which describe the triangle formed by the agents, as shown in Fig. 2.

By the geometry of the triangle, the three angles ψ, ϕ, θ are related by

$$\pi + \psi = \phi + \theta, \tag{8}$$

and the side lengths d_{bc}, d_{cm}, d_{mb} can be related to the angles by the law of sines:

$$\frac{d_{bc}}{\sin(\theta)} = \frac{d_{cm}}{\sin(\psi)} = \frac{d_{mb}}{\sin(\phi)}. \tag{9}$$

A. Length dynamics

By projecting the motion of the agents along each of the sides of the triangle, we can find how the side lengths change over time:

$$
\begin{aligned}
\dot{d}_{bc} &= \cos(w_c\,\phi) - \cos((1 - w_t)\psi), \\
\dot{d}_{cm} &= -\cos((1 - w_m)\theta) - \cos((1 - w_c)\phi), \\
\dot{d}_{mb} &= \cos(w_m\,\theta) - \cos(w_t\,\psi).
\end{aligned} \tag{10}
$$

B. Angle dynamics

By a similar method, we can project the motion of each agent along the axis normal to a side to find how that side of the triangle rotates in time. Then the dynamics of the individual angles are given by the difference between the rotation of the sides constituting each angle:

$$
\begin{aligned}
\dot{\psi} = {}& [\sin(w_t\,\psi) - \sin(w_m\,\theta)]/d_{mb} \\
& - [\sin(w_c\,\phi) - \sin((1 - w_t)\psi)]/d_{bc},
\end{aligned}
$$

$$
\begin{aligned}
\dot{\phi} = {}& [\sin((1 - w_c)\phi) - \sin((1 - w_m)\theta)]/d_{cm} \\
& - [\sin(w_c\,\phi) - \sin((1 - w_t)\psi)]/d_{bc},
\end{aligned}
$$

$$
\begin{aligned}
\dot{\theta} = {}& [\sin(w_t\,\psi) - \sin(w_m\,\theta)]/d_{mb} \\
& - [\sin((1 - w_c)\phi) - \sin((1 - w_m)\theta)]/d_{cm}. \tag{11}
\end{aligned}
$$

One must note, however, that a triangle can be defined by three side lengths, two angles and a side length, or two side lengths and an angle, but not by three angles alone. So to study the behavior of this system one should choose three appropriate variables (e.g. d_{bc}, ψ, and ϕ), and eliminate the others using the constraint equations (10)-(11) to be left with three first-order equations in three variables.

C. Parallel motion equilibria

Several classes of "shape equilibria" corresponding to parallel motion of all agents can be found at fixed points of the length dynamics ($\dot{d}_{bc} = \dot{d}_{cm} = \dot{d}_{mb} = 0$). These occur when the following three equations are satisfied:

$$
\begin{aligned}
w_c\,\phi &= (1 - w_t)\psi \\
w_m\,\theta &= w_t\,\psi \tag{12} \\
(1 - w_c)\phi + (1 - w_m)\theta &= \pi.
\end{aligned}
$$

Since the side lengths are not involved in these equations, any shape configuration that is a similar triangle to an equilibrium configuration will also be an equilibrium configuration.

Adding these three equations yields the angle constraint (8): using that constraint to solve for one angle we are left with two linear equations in two angles and three parameters. For any pair of angles, (12) gives a one-parameter family of equilibrium solutions for w_m, w_c, and w_t. Special cases where the three agents are collinear are discussed below:

1) B—C—M: When the calf is in between the bear and mother, $\psi = \phi = 0$ and $\theta = \pi$. This configuration is only an equilibrium when $w_m = 0$, such that the mother caribou's strategy is pure evasion of the bear.

2) B—M—C: When the mother is in between the bear and calf, $\psi = \theta = 0$ and $\phi = \pi$. In symmetry with the previous case, this configuration is only an equilibrium when $w_c = 0$, such that the calf's strategy is pure evasion.

3) M—B—C: When the bear is between the two caribou, $\psi = \phi = \theta = \pi$. This is a parallel motion equilibrium when $w_t = w_m = 1 - w_c$. In this case, the direction of travel is not necessarily along the line formed by the agents (as in the previous two cases), but at an angle of $w_t\pi$ from the line.

The next sections present analysis of the behavior of the dynamics in the case $w_m = 0$.

IV. MOTHER CARIBOU IN PURE EVASION

It has been observed that in the heat of a predation event, a mother caribou may make her own safety her priority and focus on evasion, only changing her course to go back for her calf once the threat has passed [15]. The calf is expected to follow, but it does not always do so and may become separated from its mother, which makes the calf very vulnerable. This situation can be modeled with (7) by setting the mother's parameter to be $w_m = 0$ (pure evasion).

In this case, the length dynamics simplify to

$$\dot{d}_{bc} = \cos(w_c\,\phi) - \cos((1-w_t)\psi),$$
$$\dot{d}_{cm} = \cos(\phi-\psi) - \cos((1-w_c)\phi), \qquad (13)$$
$$\dot{d}_{mb} = 1 - \cos(w_t\,\psi),$$

and we can note that $\dot{d}_{mb} > 0$ for all $\psi, w_t \neq 0$.

A. Avoiding mother-calf separation

We prove conditions on w_c and w_t for the model (13) where $w_m = 0$ such that the bear can never come directly between the calf and mother (M–B–C configuration) from initial conditions satisfying $d_{cm} < d_{mb}$. For the same set of initial conditions in the special case that the calf 'ignores' the bear completely and uses pure herding ($w_c = 1$), the bear can never come directly between the calf and mother.

Theorem 1: Consider the system (13) corresponding to $w_m = 0$, and suppose that $0 < d_{cm} < d_{mb}$ at time $t = t_0$. If w_c, w_t are such that the inequality

$$-\cos(2\psi) - \cos((1-w_c)(\pi-\psi)) + \cos(w_t\,\psi) < 1 \quad (14)$$

is satisfied for all $\psi \in [0, \pi/2]$, then the system will never reach an M–B–C configuration at any future time $t > t_0$. An M–B–C configuration is defined such that the distances satisfy $d_{cm} = d_{mb} + d_{bc}$ with $d_{mb} > 0$ and $d_{bc} > 0$. In case $d_{bc} = 0$, $d_{cm} = 0$, or $d_{mb} = 0$, the dynamics are assumed to terminate.

In the special case that $w_c = 1$ (the calf ignores the bear) for the same initial conditions, the system will never reach an M–B–C configuration at any time $t > t_0$ for any $w_t \in [0, 1]$.

Proof: By definition, at an M–B–C configuration, $d_{cm} > d_{mb} > 0$. Hence, if we can show that $d_{cm} \leq d_{mb}$ for all time, then we have shown that the system can never reach an M–B–C configuration.

Consider system configurations in which $d_{cm} = d_{mb}$; these form a surface δS that separates the space of length configurations into $S^+ = \{d_{cm} > d_{mb}\}$ and $S^- = \{d_{cm} < d_{mb}\}$. Thus, we are done if we can show that solutions stay in $S^- \cup \delta S$.

On δS, if $d_{bc} = 0$, $d_{cm} = 0$, or $d_{mb} = 0$, then the dynamics are terminated. Where $d_{bc} \neq 0$, $d_{cm} \neq 0$, and $d_{mb} \neq 0$ on this surface, the agents form an isosceles triangle with $\phi = \pi - \psi$ and $\psi \in (0, \pi/2)$ and a line (B-M-C configuration) when $\phi = \pi - \psi$ and $\psi = 0$. Substituting $\phi = \pi - \psi$ in (13) gives

$$\dot{d}_{cm} = -\cos(2\psi) - \cos((1-w_c)(\pi-\psi)),$$
$$\dot{d}_{mb} = 1 - \cos(w_t\,\psi). \qquad (15)$$

The inequality condition (14) is simply the condition that $\dot{d}_{cm} < \dot{d}_{mb}$ whenever $d_{cm} = d_{mb} \neq 0$ and $d_{bc} \neq 0$.

If $d_{cm} < d_{mb}$ at time $t = t_0$, then the state is in S^-, and by continuity the system cannot reach S^+ without first passing through δS. If the system reaches δS and $d_{cm} = d_{mb} = 0$ or $d_{cb} = 0$, then the dynamics terminate. If the system reaches δS and $d_{cm} = d_{mb} \neq 0$ and $d_{bc} \neq 0$ and the inequality condition (14) holds, then $\dot{d}_{cm} < \dot{d}_{mb}$ and the dynamics
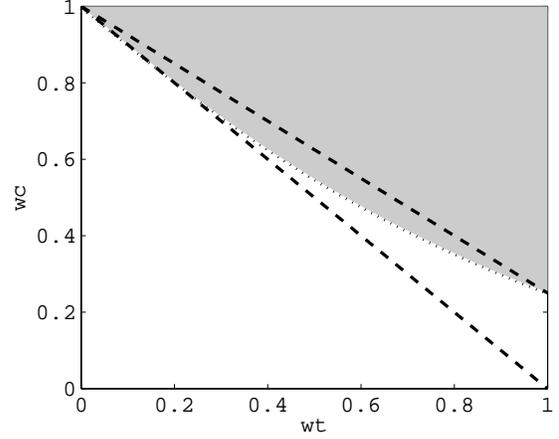

Fig. 3. Range of parameters w_c and w_t such that the mother and calf caribou cannot be separated by the bear when starting with $d_{cm} < d_{mb}$. The shaded area is the safe range computed from (14). Dashed lines $w_c = 1 - w_t$ and $w_c = 1 - \frac{3}{4}w_t$ are shown for reference

must remain in $S^- \cup \delta S$. Thus, the system can never reach an M–B–C configuration when starting with $d_{cm} < d_{mb}$.

In the special case with $w_c = 1$ where the calf uses a pure herding strategy and ignores the bear, the inequality condition (14) simplifies to

$$\cos(w_t\,\psi) - \cos(2\psi) < 2, \qquad (16)$$

which always holds for $\psi \in [0, \pi/2]$. Thus for $w_c = 1$, the system can never reach an M–B–C configuration when starting with $d_{cm} < d_{mb}$, regardless of the value of the bear's parameter w_t. ∎

Fig. 3 shows numerical calculations of the range of parameters w_c and w_t for which the inequality (14) holds. Note that when $w_c \leq 1/4$ there is no guarantee, for any $w_t \in [0, 1]$, that an M-B-C configuration will be avoided.

B. Disadvantage of pure evasion by the calf

In the case that both mother and calf use a pure evasion strategy, it can be shown that the calf will always become separated from the mother unless starting from a configuration with $\psi = 0$. When $w_m = w_c = 0$, the dynamics of the system simplify greatly, and can be described in terms of lengths d_{bc}, d_{mb}, and angle ψ by

$$\dot{d}_{bc} = 1 - \cos((1-w_t)\psi)$$
$$\dot{d}_{mb} = 1 - \cos(w_t\,\psi)$$
$$\dot{\psi} = \frac{\sin(w_t\,\psi)}{d_{mb}} + \frac{\sin((1-w_t)\psi)}{d_{bc}}. \qquad (17)$$

When starting with $\psi = 0$, the system can be in either B–C–M, or B–M–C collinear equilibria. Otherwise for all initial conditions with d_{bc}, d_{mb}, $\psi \neq 0$, the three variables will increase monotonically, with ψ eventually approaching π, taking the system towards an M–B–C configuration with the caribou separated from each other by the bear.

C. When the bear ignores the calf

Fig. 3 suggests an advantage for the bear to target the mother rather than the calf in order to separate the calf from its mother. Here, we study the dynamics in the case that $w_m = w_t = 0$, i.e., the bear purely pursues the mother and the mother purely evades the bear. The length dynamics for d_{mb} simplify to $\dot{d}_{mb} = 0$, so the distance between the mother and the bear remains constant throughout the trajectory. This reduces the shape dynamics to two dimensions, which can be described in terms of ϕ, and ψ with

$$\dot{\phi} = \frac{\sin(\phi)}{d_{mb}} \left[\frac{\sin(\psi) - \sin(w_c\,\phi)}{\sin(\phi - \psi)} \right.$$
$$\left. + \frac{\sin((1 - w_c)\phi) - \sin(\phi - \psi))}{\sin(\psi)} \right]$$
$$\dot{\psi} = \frac{\sin(\phi)}{d_{mb}} \left[\frac{\sin(\psi) - \sin(w_c\,\phi)}{\sin(\phi - \psi)} \right]. \tag{18}$$

A line of equilibria exists where $\psi = w_c\phi$, which corresponds to parallel motion of all three agents, with the bear directly following the mother and the calf off to the side, or between them.

We compute the Jacobian for points along the equilibria line $\psi = w_c\phi$ by substituting for ψ. Because this is a line of equilibria, one eigenvalue must be zero with its corresponding eigenvector pointing along the line of equilibria. Since the Jacobian is a 2×2 matrix, the other eigenvalue must be given by the trace of the Jacobian, which simplifies to

$$\lambda = \frac{\sin(\phi)\big[(1 - w_c)\sin(2w_c\,\phi) - w_c\sin(2(1 - w_c)\phi)\big]}{2d_{mb}\sin(w_c\,\phi)\sin((1 - w_c)\phi)}. \tag{19}$$

The stability of the equilibria line is thus determined by the sign of this eigenvalue. Since the leading term and each term in the denominator of (19) are positive for $w_c \in (0, 1)$ and $\phi \in (0, \pi)$, we need only consider the expression in brackets:

$$f(w_c, \phi) = (1 - w_c)\sin(2w_c\,\phi) - w_c\sin(2(1 - w_c)\phi). \tag{20}$$

We show for all $\phi \in (0, \pi)$ that $\lambda > 0$ for $w_c \in (0, 1/2)$ and $\lambda < 0$ for $w_c \in (1/2, 1)$. First note that $f(w_c, 0) = 0$. The partial derivative of $f(w_c, \phi)$ with respect to ϕ is given by

$$\frac{\partial f}{\partial \phi} = 4w_c(1 - w_c)\sin(\phi)\sin((1 - 2w_c)\phi). \tag{21}$$

This derivative is positive for $w_c \in (0, 1/2)$ and negative for $w_c \in (1/2, 1)$ for all $\phi \in (0, \pi)$. $f(w_c, \phi)$ is monotonically increasing in ϕ for $w_c \in (0, 1/2)$, so $f(w_c, \phi) > f(w_c, 0) = 0$, and we can conclude that the eigenvalue λ must be positive in that range. Similarly $f(w_c, \phi)$ is monotonically decreasing in ϕ for $w_c \in (1/2, 1)$, so λ must be negative in that range. Thus the line of parallel-motion equilibria is stable for $w_c \in (1/2, 1)$ and unstable for $w_c \in (0, 1/2)$ for all $\phi \in (0, \pi)$. The change in stability of the parallel equilibria at $w_c = 1/2$ suggests the presence of a local bifurcation.

In the special case of $w_m = w_t = 0$ and $w_c = 1$, the bear and the mother caribou travel in a straight line with
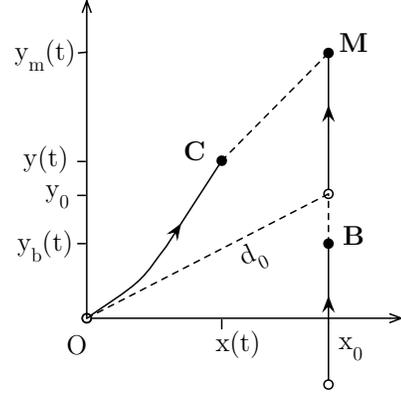

Fig. 4. Coordinate diagram for trajectories with $w_m = w_t = 0$, $w_c = 1$. Open circles show initial positions, and closed circles show positions after time t.

constant d_{mb}, and the calf directly pursues the mother. This configuration is equivalent to the classic Bouguer problem of a pirate ship in classical pursuit of a merchant ship that is traveling in a straight line. Analytic solutions for certain initial conditions are presented in [1], [16], and the following steady-state analysis for the caribou system is based closely on the method of [1].

Consider an inertial frame with origin fixed at the initial position of the calf, with the positive y-axis in the direction of the vector from the bear's initial position to the mother's (see Fig. 4). Let (x_0, y_0) be the initial coordinates of the mother on this frame, and let $d_0 = d_{cm,0}$ be the initial distance from mother to calf at time $t = t_0$.

Mother and bear both travel along the line $x_m = x_0$, with the mother's position at time t given by

$$y_m(t) = y_0 + t, \tag{22}$$

and the bear follows below at a constant distance.

The calf's trajectory traces out a curve $y = f(x)$ such that at each point its tangent will pass through the current position of the mother, since the calf is engaging in classical pursuit. So at each point in time,

$$y' = \frac{dy}{dx} = \frac{y_m - y}{x_0 - x} = \frac{y_0 - y + t}{x_0 - x}. \tag{23}$$

The agents move at unit speed, so the arclength of the calf's curve is simply the elapsed time. By solving (23) for t and setting it equal to the formula for arclength of the calf's curve we arrive at an integro-differential equation, which, when solved at initial condition $y'|_{x=0} = y_0/x_0$, yields

$$y' = \frac{1}{2}\left(\frac{y_0 + d_0}{x_0 - x} + \frac{x_0 - x}{y_0 + d_0} \right). \tag{24}$$

At any given time, the distance from the calf to the mother is

$$d_{cm}^2 = (x_0 - x)^2 + (y_m - y)^2$$
$$= (x_0 - x)^2 \left[1 + \left(\frac{y_m - y}{x_0 - x} \right)^2 \right]. \tag{25}$$

Recalling (23), the final term in the brackets in (25) is simply $(y')^2$, and thus we arrive at an equation for d_{cm} as a function of x:

$$d_{cm}^2 = x_0^2 \left[\frac{1}{2} \left(1 - \frac{x}{x_0} \right)^2 + \frac{1}{4} \left(\frac{y_0 + d_0}{x_0} \right)^2 \right.$$
$$\left. + \frac{1}{4} \left(\frac{x_0}{y_0 + d_0} \right)^2 \left(1 - \frac{x}{x_0} \right)^4 \right]. \quad (26)$$

By inspection of Fig. 4, it is clear that as t grows large, x approaches x_0, so the steady-state distance is simply

$$d_{cm,ss} = \lim_{t \to \infty} d_{cm}^2 = \lim_{x \to x_0} d_{cm}^2 = \left(\frac{y_0 + d_0}{2} \right)^2, \quad (27)$$

or in terms of the initial distances,

$$d_{cm,ss} = \frac{(d_{cm,0} + d_{mb,0})^2 - d_{bc,0}^2}{4 \, d_{mb,0}}. \quad (28)$$

Then for $d_{cm,ss} < d_{mb,0}$, the system will end up in a B–C–M configuration, and for $d_{cm,ss} > d_{mb,0}$, the system will end up in an M–B–C configuration. Setting $d_{cm,ss} = d_{mb,0}$, we can find the locus of initial conditions where the calf ends up directly on the bear.

V. SIMULATIONS AND ESTIMATED SOLUTION TO A DIFFERENTIAL GAME

Trajectories were computed in Matlab using the forward-Euler method of integration on the first-order model (6)-(7). Leaving the initial distance from bear to calf constant at $d_{bc} = 10$, the initial position of the mother was varied over an equally-spaced grid on the circular area centered at the calf with radius 10, (i.e. only initial conditions where $d_{cm} < d_{bc}$).

Trials were run with $\Delta t = 0.1$ s for 2500 s, or until the agents reached a collinear configuration. "Capture" of the calf was taken to occur when $d_{bc} \leq 1$ or $\psi \geq \pi/2$ at the final time, with the assumption that in the wild a calf separated from its mother would eventually tire and be captured.

Leaving $w_m = 0$ constant (mother using pure evasion), the parameters w_c and w_t were varied over the range [0,1] in increments of 0.05. The percentage of trials ending in capture for each particular w_c, w_t combination gives a measure of the "fitness" of those strategies against one another, which can be seen in Fig. 5.

If we consider a zero-sum game in which the bear aims to choose a value of w_t that maximizes the fraction of captures, and the calf aims to choose a value of w_c that minimizes the fraction of captures, a saddle point in this w_t-w_c strategy space represents a solution to the game: optimal strategies for the two players. For these simulations, we estimate a saddle point at $w_t = 0.3$, $w_c = 0.8$, with a capture fraction of 0.137.

Although this saddle is only the optimum strategy on average for certain initial conditions, its presence suggests that a similar optimal strategy may exist for each particular set of initial conditions. In future work we aim to devise a suitable cost function so that the bear and caribou system may be studied under the "differential game" framework established by Isaacs in [17].
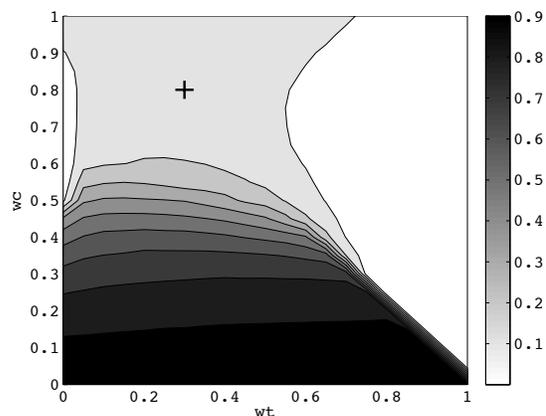

Fig. 5. Fraction of capture events for different combinations of w_t and w_c, with $w_m = 0$ for initial conditions satisfying $d_{cm} < d_{bc}$. The estimated location of the saddle point is denoted by the cross at $w_t = 0.3$, $w_c = 0.8$, which has a capture fraction of 0.137.

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