Proceedings of the IEEE Conf. on Decision and Control, Los Angeles, CA, 2014

Dynamics of Pursuit and Evasion in a Heterogeneous Herd

William Scott and Naomi Ehrich Leonard

***Abstract*— We propose and analyze a dynamic model of pursuit and evasion on the plane with a single pursuer and a heterogeneous group of evaders. Heterogeneity in the group of evaders is expressed as heterogeneity in the individual maximum speeds. The goal of the pursuer is to capture a single evader in minimum time. The goal of each individual evader is to avoid capture or else to delay capture for as long as possible. Two cases of sensing among agents are considered: global (all-to-all) sensing, and local (radius-limited) sensing. We present pursuer strategies for optimal target selection that achieve bounded capture time. We propose evasion strategies and prove conditions under which they guarantee capture avoidance. In the case of local sensing, our strategy of evader risk reduction leads to aggregation of the evaders where the slowest evader in a group is the only member with a risk of capture. Our results provide insight into the dynamics of aggregation.**

I. INTRODUCTION

We propose and analyze a dynamic model of pursuit and evasion on the plane with a single pursuer and a heterogeneous group of evaders. Our model is inspired by predation in animal herds where individuals within the herd may differ in size and age, and this may lead to inter-herd competition for safety.

Predator avoidance has long been considered a key factor in animal group formation, first studied mathematically for identical evaders on the plane in the "selfish herd" model of Hamilton [1]. Our present investigation into the dynamics of a self-interested group of evaders draws on the spirit of a selfish herd, but is differentiated through the use of continuous-time dynamics for both pursuer and evaders, and the inclusion of heterogeneity in the evaders. Our investigation is also motivated by the problem of designing dynamics for group formation in engineered multi-agent systems.

Hamilton's model has been extended to include evolutionary dynamics, which lead to formation of large groups [2], [3]. Cooperative evader strategies have been studied as differential games in systems where all evaders are captured [4], [5], and in systems where evaders have defensive capabilities [6], [7]. Generalized Voronoi diagrams have been used to analyze systems where evader strategies are known by the pursuers [8]. Several numerical studies have examined properties of group motion in multiple evader systems where biologically inspired strategies are chosen a priori: on the plane [9], [10], [11], in discrete space [12], in three dimensions [13], with multiple pursuers [14], and with strategies based on observations of crabs and shorebirds [15]. Non-spatially explicit game theoretic models of multiple evader systems have been posed for both homogeneous evaders [16], and heterogeneous evaders [17], [18].

Contrary to this literature, the continuous-time dynamic model that we propose is spatially explicit with a heterogeneous group of evaders, and the results we prove focus on capture avoidance for individual evaders when faced with an intelligent pursuer. In our model the goal of the pursuer is to capture (reach the position of) a single evader in minimum time. The goal of each individual evader is to avoid capture or else to delay capture for as long as possible. Every agent is modeled as a particle with a strategy that defines its velocity subject to its maximum speed. Heterogeneity in the group of evaders is expressed as heterogeneity in the individual maximum speeds. The pursuer's maximum speed is assumed to be higher than the maximum speed of every evader, allowing for capture in finite time. Two cases of sensing among agents are considered: global (all-to-all) sensing, and local (radius-limited) sensing.

In both the global and local sensing cases, we propose a pursuer strategy for optimal target selection among the evaders that guarantees bounded capture time of a single evader. In the global case, given a pursuer using this strategy, we show there exists a family of evasion strategies that guarantees that an evader will not become the pursuer's target in the future if it is not currently the target. Under local sensing, we show that evaders that do not sense the location of the pursuer can mitigate the risk of becoming the pursuer's target by approaching a neighboring evader with a lower maximum speed; when such an evader comes sufficiently close to its slower neighbor, it achieves the same guarantee against becoming the target as in the global sensing case. This strategy of evader risk reduction leads to aggregation of the evaders where the slowest evader in a group is the only member with a risk of capture. Our results provide possible insight into the dynamics of herd formation in nature and the initiation of aggregation in design of multi-agent systems.

In Section II we introduce the equations of motion for the model. Section III discusses the optimal target selection strategy for the pursuer in the case of global sensing. In Section IV we present strategies for evasion in the global sensing case and prove conditions for target avoidance. Section V considers pursuit and evasion strategies in the local sensing case. We conclude in Section VI.

II. MODEL FORMULATION

We consider a system with a single pursuer agent P and a heterogeneous group of n evader agents E_i. Each evader

This research is supported in part by NSF grant ECCS-1135724, ONR grants N00014-09-1-1074, N00014-14-1-0635, and ARO grant W911NG-11-1-0385. The authors are with the Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08540, USA wlscott@princeton.edu, naomi@princeton.edu

has an individual maximum speed v_i, position on the plane $\mathbf{r}_i(t) \in \mathbb{R}^2$ at time t, and velocity at time t given by control input $\mathbf{u}_i(t) \in \mathbb{R}^2$ with $\|\mathbf{u}_i\| \leq v_i$. Similarly the pursuer P has maximum speed $v_p > v_i$ for $i = 1, 2, ..., n$, position $\mathbf{r}_p(t) \in \mathbb{R}^2$, and velocity control input $\mathbf{u}_p(t) \in \mathbb{R}^2$ with $\|\mathbf{u}_p\| < v_p$. The system evolves as

$$\begin{aligned} \dot{\mathbf{r}}_i &= \mathbf{u}_i, \text{ for } i = 1, 2, ..., n, \\ \dot{\mathbf{r}}_p &= \mathbf{u}_p. \end{aligned} \tag{1}$$

For convenience, we define the baseline vector from P to E_i as $\mathbf{r}_{ip} = \mathbf{r}_i - \mathbf{r}_p$, the length of that vector as $d_{ip} = \|\mathbf{r}_{ip}\|$, and the unit vector in that direction as $\hat{\mathbf{r}}_{ip} = \mathbf{r}_{ip}/\|\mathbf{r}_{ip}\|$. Similarly for the baseline vector from E_j to E_i we define $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $d_{ij} = \|\mathbf{r}_{ij}\|$, and $\hat{\mathbf{r}}_{ij} = \mathbf{r}_{ij}/\|\mathbf{r}_{ij}\|$.

In this system, we take the pursuer's goal to be the capture of a single evader in minimum time. The goal of each evader is to avoid capture altogether, or else delay capture as long as possible. We consider two sensing regimes, global and local. The global case assumes all-to-all sensing, such that each agent has knowledge of the position and maximum speed of all other agents. In the local case, agents can only sense the locations and maximum speeds of other agents within a radius limited neighborhood.

III. Global pursuer strategy: optimal target selection

In a system of a single pursuer and a single evader on the plane, it has been shown that the time optimal strategy for the pursuer is to move at maximum speed in the direction of the evader (classical pursuit), and for the evader to move at maximum speed away from the pursuer (classical evasion) [19]. Consider a pursuer and an evader with maximum speeds $v_p > 0$ and $v_e > 0$ and positions $\mathbf{r}_p(t) \in \mathbb{R}^2$ and $\mathbf{r}_e(t) \in \mathbb{R}^2$, respectively. We define *capture* to be the coincidence of the pursuer and evader positions. Under the strategies of classical pursuit and classical evasion, the trajectories of the two agents follow a straight line path; the *time to capture from time t* is

$$T_{cap}(t) = \frac{\|\mathbf{r}_p(t) - \mathbf{r}_e(t)\|}{v_p - v_e}, \tag{2}$$

for $v_p > v_e$. For a pursuer using the classical pursuit strategy, $T_{cap}(0)$ is an upper bound on the total time to capture from $t = 0$, since any deviation from the classical evasion strategy on the part of the evader will allow capture in shorter time.

In the full multiple evader system with global (all-to-all) sensing, a pursuer strategy of classical pursuit of evader E_i will guarantee capture from time t within $T_{cap,i}(t)$, defined by (2) with i replacing e. Thus, the pursuer will have the shortest guaranteed time to capture by using a strategy of classical pursuit targeting evader E_g, with control input

$$\mathbf{u}_p = v_p \, \hat{\mathbf{r}}_{gp}, \text{ where } g = \underset{j}{\operatorname{argmin}} \, T_{cap,j}. \tag{3}$$

We refer to $E_g(t)$ as the *optimal target at time t* and $T_{cap,g}(t)$ as the *time to capture the target from time t*.

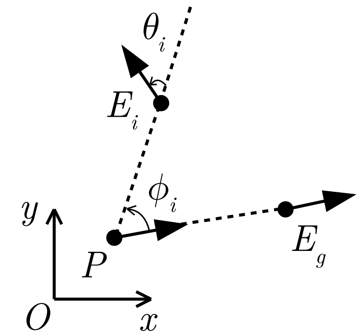

Fig. 1. Relative angles between pursuer P, targeted evader E_g, and untargeted evader E_i. Arrows denote direction of travel, with P in classical pursuit of E_g, and E_g in classical evasion.

IV. Global evader strategies: target-avoidance

The aim of each evader is to avoid capture if possible. Since only the targeted evader will be captured, it is in the best interest of each evader to remain untargeted. For the case of global sensing, we show that for any initially untargeted evader, there exists a family of control inputs that guarantee that the evader will remain untargeted against a pursuer using classical evasion with optimal target selection (3).

Theorem 4.1: In the case of global sensing for pursuer and evaders, where the pursuer uses strategy (3), there exists a family of control inputs that guarantees an initially untargeted evader will avoid becoming the target for all time. When such a strategy is used by all evaders in the system, the initially targeted evader will be captured by the pursuer at the initial value of $T_{cap,g}(0)$.

Proof: Consider a system under global sensing where the pursuer uses the optimal target selection strategy (3). Against that pursuer strategy, the condition for an untargeted evader E_i to remain untargeted is that $T_{cap,i}(t) > T_{cap,g}(t)$ for all time $t \geq 0$. If we can show that there always exists a control input for E_i such that $\frac{d}{dt}T_{cap,i} > \frac{d}{dt}T_{cap,g}$, then by continuity E_i can use that input to remain untargeted.

During classical pursuit of P towards its targeted evader E_g, the "relative dynamics" of the baseline vector from P to a given evader E_i are described by

$$\begin{aligned} \dot{d}_{ip} &= \|\mathbf{u}_i\| \cos\theta_i - v_p \cos\phi_i, \\ \dot{\phi}_i &= \frac{1}{d_i}(\|\mathbf{u}_i\| \sin\theta_i + v_p \sin\phi_i), \end{aligned} \tag{4}$$

where ϕ_i represents the angle from the target baseline vector $\mathbf{r}_{gp}$ to the baseline vector $\mathbf{r}_{ip}$, and θ_i is the angle from $\mathbf{r}_{ip}$ to the direction of motion of E_i as shown in Figure 1. The rate of change of $T_{cap,i}$ for each evader is thus

$$\frac{d}{dt}T_{cap,i} = \frac{\dot{d}_{ip}}{v_p - v_i} = \frac{\|\mathbf{u}_i\| \cos\theta_i - v_p \cos\phi_i}{v_p - v_i}. \tag{5}$$

As in the one-on-one case, the optimal strategy for the targeted evader to avoid capture for as long as possible is classical evasion, $\mathbf{u}_g = v_g \, \hat{\mathbf{r}}_{gp}$. With P moving towards E_g at a rate of v_p and E_g moving away from P at a rate of v_g,

$$\frac{d}{dt}T_{cap,g} = \frac{v_g \cos(0) - v_p \cos(0)}{v_p - v_g} = -1, \tag{6}$$

i.e., the bound on capture time goes down by one second per second.

By continuity, any other evader E_i with an initially higher value of $T_{cap,i}(0)$ can then avoid becoming the target if it is always able to choose a control input that satisfies $\frac{d}{dt}T_{cap,i} > -1$. This condition is given by

$$\|\mathbf{u}_i\| \cos\theta_i \geq v_i - v_p(1 - \cos\phi_i). \tag{7}$$

The right hand side is less than or equal to v_i for all values of ϕ_i, so it is always possible to choose $\mathbf{u}_i$ which satisfies the inequality (7). In the limiting case that $\phi_i = 0$ (P is targeting E_i), the only admissible choice of evasive control is $\theta_i = 0$, $\|\mathbf{u}_i\| = v_i$, which is equivalent to classical evasion.

When all initially untargeted evaders successfully use such a target-avoidance strategy, the initially targeted evader must remain targeted, thus there is no possibility for the targeted evader to "lure" the pursuer towards another target. Being unable to avoid capture, the goal of the targeted evader is to delay capture as long as possible, so optimally it must use classical evasion. Thus $\frac{d}{dt}T_{cap,g} = -1$ for the duration of the pursuit, and the final capture time is equal to the original value of $T_{cap,g}(0)$. ■

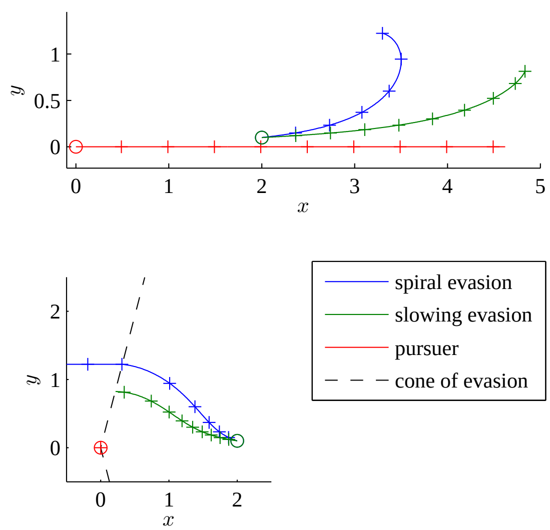

Fig. 2. Comparison of the "slowing" and "spiral" reactive evasion strategies for pursuer with speed $v_p = 1$ and evader with speed $v_e = 0.75$. The pursuer starts at the origin $\mathbf{r}_p = (0, 0)$ and travels along the x-axis, and the evader starts at $\mathbf{r}_e = (2, 0.1)$. Circles indicate initial positions and crosses show positions at 0.5 second intervals, continuing until the slowing evader leaves the cone of evasion. Top: trajectories of the agents in the inertial frame. Bottom: the same trajectories in a frame relative to the pursuer. Spiral evasion leaves the cone of evasion at $t = 3.0$ s, and slowing evasion at $t = 4.6$ s. Note that the slowing evasion strategy leaves the cone of evasion at a smaller distance from the pursuer.

An untargeted evader E_i does not need to take action until $T_{cap,i}$ is close to $T_{cap,g}$. We define *reactive evasion* to be the class of strategies that satisfy

$$\|\mathbf{u}_i\| \cos\theta_i = v_i - v_p(1 - \cos\phi_i) \tag{8}$$

for $T_{cap,i} - T_{cap,g} \leq \epsilon$, and $\mathbf{u}_i = 0$ for $T_{cap,i} - T_{cap,g} > \epsilon$, where $\epsilon > 0$ is a constant. The evader E_i must begin evading before $T_{cap,i}$ is equal to $T_{cap,g}$, since in that situation the predator could freely choose either evader for the same time to capture. For a physical interpretation, ϵ can be thought of as a buffer to account for the evader's reaction time, how long it will take to start moving, or a buffer to account for inexact estimates on the values of T_{cap}.

Note that the left hand side of (7) represents the projection of $\mathbf{u}_i$ onto $\hat{\mathbf{r}}_{ip}$, so (7) is a condition on the component of the evader's velocity in the direction away from the predator. For $\cos\phi_i = (v_p - v_i)/v_p$, the right hand side becomes zero, and for greater magnitudes of ϕ_i it becomes negative. This implies that the evader does not need to engage in evasive action in those cases, as the relative motion of the pursuer causes the inequality to be satisfied even for $\mathbf{u}_i = 0$. Thus a reactive evasion strategy need only be defined for an evader E_i inside the *cone of evasion*, $\phi_i \in (-\bar{\phi}_i, \bar{\phi}_i)$ where $\bar{\phi}_i = \cos^{-1}((v_p - v_i)/v_p)$.

At each instant during reactive evasion, the angle and speed of E_i can be chosen arbitrarily so long as they satisfy (8). Setting the maximum speed, $\|\mathbf{u}_i\| = v_i$, we can solve for the maximum angle,

$$\bar{\theta}_i(\phi_i) = \pm\cos^{-1}\left(1 - \frac{v_p}{v_i}(1 - \cos\phi_i)\right). \tag{9}$$

The sign $\pm$ is set so that the evader turns away from the direction of the pursuer's motion, towards the edge of the cone of evasion. We can parameterize a one-dimensional family of input vectors for reactive evasion with parameter $\alpha \in [0, 1]$ by taking $\theta_i = \alpha\,\bar{\theta}_i$, and

$$\|\mathbf{u}_i\| = \frac{v_i - v_p(1 - \cos\phi_i)}{\cos(\alpha\,\bar{\theta}_i)}. \tag{10}$$

The following sections analyze strategies for the two boundary cases: *slowing evasion* for $\alpha = 0$, and *spiral evasion* for $\alpha = 1$. Trajectories for the two strategies are shown in Figure 2.

A. Slowing evasion

For the strategy (10), $\alpha = 0 =$ constant, so $\theta_i = 0$ and the velocity of the evader E_i is aligned with the baseline $\mathbf{r}_{ip}$. The speed of the evader is given by

$$\|\mathbf{u}_i\| = v_i + v_p(\cos\phi_i - 1), \tag{11}$$

which is maximum at v_i when $\phi_i = 0$, and smoothly decreases to zero at the edge of the cone of evasion.

Integrating (4) for initial conditions $d_{ip}(0) = d_0$, $\phi_i(0) = \phi_0$, we have the analytic trajectory

$$\begin{aligned} d_{ip}(t) &= d_0 - (v_p - v_i)t \\ \phi_i(t) &= 2\tan^{-1}\left(\tan(\frac{\phi_0}{2})\left(1 - \frac{v_p - v_i}{d_0}t\right)^{\frac{-v_p}{v_p - v_i}}\right). \end{aligned} \tag{12}$$

Of all possible reactive evasion strategies, an evader using slowing evasion will take the longest time to leave the cone of evasion. This is because with $\theta_i = 0$, the evader's velocity has no component normal to the baseline $\mathbf{r}_{ip}$, so ϕ_i increases only due to the motion of the pursuer. The distance to

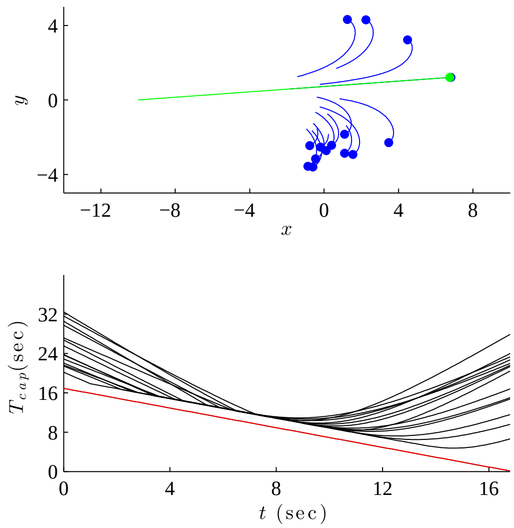

Fig. 3. Simulation results for global pursuit and evasion strategies, with $n = 20$ evaders using the "spiral evasion" strategy with $\epsilon = 2$ sec. Pursuer has maximum speed $v_p = 1$ and evader speeds are chosen uniformly in the range $v_i \in [0.5, 0.7]$. Top: trajectories on the plane, with evaders shown in blue, and pursuer in green. Circles denote final position at capture time $t = 16.9$ sec. Bottom: Plot of $T_{cap,i}$ over time for each evader during the pursuit, with the targeted evader shown in red.

the pursuer d_{ip} decreases at a constant rate during reactive evasion. By taking the longest to leave the cone of evasion, the slowing evasion strategy will lead to the minimum value of d_{ip}. However, in a physical system this reactive evasion strategy may not be the least efficient in terms of energy expenditure.

B. Spiral evasion

This strategy corresponds to $\alpha = 1 =$ constant. Here, the speed is maximum ($\|\mathbf{u}_i\| = v_i$) and the angle is maximum ($\theta_i = \bar{\theta}_i(\phi_i)$) at all times during evasion, so the component of velocity normal to the baseline is at its maximum. Consequently an evader using the strategy of spiral evasion leaves the cone of evasion in minimum time, and with maximum final distance to the pursuer. Unfortunately it does not afford an analytic solution for trajectories as with slowing evasion. Though the spiral evasion strategy minimizes time spent evading, it may not be the most efficient in terms of energy use for an evader. Figure 3 shows simulation results for the global pursuer strategy against evaders using spiral evasion.

Although slowing evasion and spiral evasion both use a constant value of α, that need not be the case in general. One example of a strategy with varying α would be to set the angle to be $\theta_i = \bar{\theta}_i(t_0) =$ constant, and vary the speed to satisfy (8). This type of "constant bearing" strategy has been observed in fish [20], where evaders aim to keep the pursuer at a constant position in the visual field.

V. Local (sensing-limited) pursuit and evasion

Up to this point we have assumed all-to-all sensing, so that each agent has full information about all other agent positions and speeds. This is unrealistic for systems with limited range on sensing or those that suffer from occlusion. In this section, limits on sensing are defined and the pursuit and evasion strategies introduced in the previous section are adapted to local strategies that address the uncertainty imposed by limited sensing.

In the local (sensing-limited) system, we define d_{sense} as the sensing radius for all agents. An agent's local sensing neighborhood consists of the set of agents within the sensing radius. The local neighborhood of the pursuer is defined as

$$\mathcal{N}(P) = \{E_i | d_{ip} \leq d_{sense}\}\,, \qquad (13)$$

and the neighborhood of evader E_i is defined as

$$\mathcal{N}(E_i) = \{E_j | d_{ij} \leq d_{sense}\} \cup \{P | d_{ip} \leq d_{sense}\}\,. \qquad (14)$$

A. Local target selection

Under local sensing, the pursuer must choose a control law based only on information about evaders within its neighborhood, $\mathcal{N}(P)$. The best possible strategy then is the same as the global strategy, targeting the evader within the sensing neighborhood that can be captured in the minimum time,

$$\mathbf{u}_p = v_p\,\hat{\mathbf{r}}_{gp}, \text{ where } g = \operatorname*{argmin}_{E_j \in \mathcal{N}(P)} T_{cap,j}. \qquad (15)$$

During pursuit, if a new evader E_i comes into view with $T_{cap,i}$ less than $T_{cap,g}$, the pursuer will switch to targeting the new evader, leading to a decrease in the remaining bound on time to capture. In this way we can see that total time spent pursuing will necessarily be less than or equal to the minimum local T_{cap} calculated at the start of pursuit. This local strategy will not lead to an under estimate of the total time spent pursuing.

B. Local evasion strategy

To assess local evader strategies we partition the plane into cells: a cell is associated with the evader that has the minimum value of T_{cap} when the pursuer is in that cell. The cell corresponding to evader E_i is its *domain of danger*: this is the set of pursuer positions such that evader E_i is the optimal target for a pursuer under global sensing. Formally we define the domain of danger of evader E_i as

$$\mathcal{D}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \;\middle|\; i = \operatorname*{argmin}_j \frac{\|\mathbf{r}_j - \mathbf{p}\|}{v_p - v_j} \right\}. \qquad (16)$$

This partitioning is equivalent to a multiplicatively-weighted Voronoi diagram [21], where the weight on each evader's distance is given by $(v_p - v_i)$. This notion generalizes the standard Voronoi domain of danger partition of Hamilton's selfish herd model of identical evaders [1].

In a local sensing system, evaders might not have knowledge of the position of the pursuer until late in the course of a pursuit. To address this uncertainty, we propose a local evasion strategy that consists of 1) a *risk reduction phase* which is used when the pursuer is not in sensing range, and 2) a *local reactive evasion phase* which is used when the pursuer comes into view.

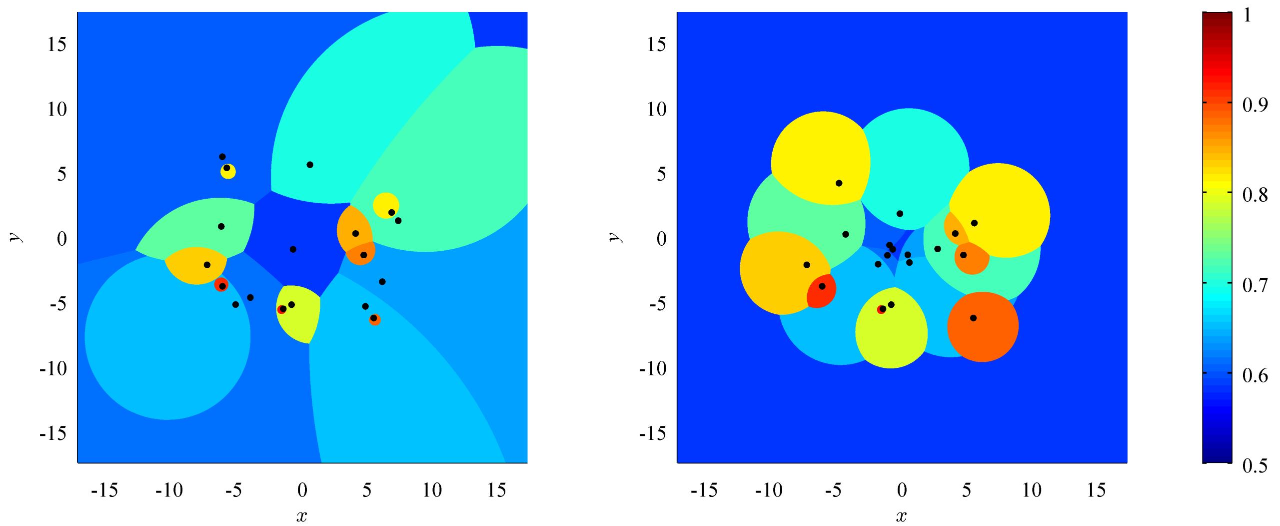

Fig. 4. Weighted-Voronoi domain of danger partition, calculated for a pursuer with maximum speed $v_p = 1$ and position not sensed by evaders. Each black dot denotes the position of an evader and the color of the surrounding cell (domain of danger) indicates the evader's maximum speed. Left: initially with random initial positions. Right: after running the risk reduction strategy for locally sensing evaders with sensing radius $r_{sense} = 10$.

1) Risk reduction phase: During the risk reduction phase, each evader aims to reduce the size of its domain of danger. Let E_f and E_s be two evaders with $v_f > v_s$. E_f's domain of danger is bounded by the Apollonius circle [21] formed by the locus of points where $T_{cap,f} = T_{cap,s}$. The circle is centered at

$$\mathbf{r}_{Apol,fs} = \mathbf{r}_f + \frac{(v_p - v_f)^2}{(v_p - v_s)^2 - (v_p - v_f)^2}\,\mathbf{r}_{fs} \tag{17}$$

with radius

$$R_{Apol,fs} = \frac{(v_p - v_s)(v_p - v_f)}{(v_p - v_s)^2 - (v_p - v_f)^2}\,d_{fs}. \tag{18}$$

All points outside the circle have $T_{cap,s} < T_{cap,f}$, so $T_{cap,f}$ cannot be the minimum there. The maximum distance from E_f to the boundary of the circle is

$$d_{App,fs} = \left(\frac{v_p - v_f}{v_f - v_s}\right) d_{fs}, \tag{19}$$

in the direction $\hat{\mathbf{r}}_{fs}$. Since $d_{App,fs}$ is linear in d_{fs}, E_f may reduce this bound on the size of its domain of danger by approaching E_s. Since $v_f > v_s$, E_f can always choose its velocity such that $\dot{d}_{fs} < 0$.

We define E_f as *risk minimized with respect to* E_s if the following condition is satisfied:

$$d_{fs} + d_{App,fs} < d_{sense} \tag{20}$$

which is equivalent to

$$d_{fs} < \left(\frac{v_f - v_s}{v_p - v_s}\right) d_{sense}. \tag{21}$$

The phase of risk reduction consists of each evader moving towards a chosen slower neighbor until either the inequality is satisfied or the pursuer enters the evader's sensing range and local reactive evasion is triggered. Figure 4 shows how the domains of danger decrease in size during risk reduction.

2) Local reactive evasion phase: When pursuer P enters evader E_i's sensing range, E_i only knows the T_{cap} of its neighbors in $\mathcal{N}(E_i)$, and the pursuer P chooses its target based only on the T_{cap} of its neighbors in $\mathcal{N}(P)$. In this context, an evader must use its best estimate of the pursuer's estimate of the minimum T_{cap} in order to decide when to begin its reactive evasion strategy. Thus, E_i should begin reactive evasion when $T_{cap,i} - \bar{T}_{cap} \leq \epsilon$, where $\bar{T}_{cap}$ is the minimum T_{cap} for the evaders in the set $\mathcal{N}(E_i) \cap \mathcal{N}(P)$. As in the global case, reactive evasion continues until reaching the cone of evasion at $\phi_i = \cos^{-1}((v_p - v_i)/v_p)$, and an evader that begins reactive evasion while untargeted will remain untargeted.

The following theorem states that a risk minimized evader is guaranteed to avoid capture under local sensing.

Theorem 5.1: For system (1) under local sensing with pursuer P using local pursuit strategy (15), let E_f and E_s be evaders with maximum speeds $v_f > v_s$. If E_f is risk minimized with respect to E_s at the time t_0 when P first enters the sensing radius of E_f, then there exists a control input that guarantees that E_f will avoid capture.

Proof: We must consider two cases separately. In the first case, P does not target E_f at time t_0, instead targeting another evader E_g. In that case, $T_{cap,f} > T_{cap,g}$, so a local reactive evasion strategy may be used by E_f in order to avoid becoming the target.

In the second case, P targets E_f at time t_0. At that time, $d_{fp} = d_{sense}$, since P enters the sensing range of E_f at that moment. As long as P remains outside of the Apollonius circle defined by $T_{cap,f} = T_{cap,s}$ (which is the case at t_0 by the risk minimization condition (20)), it must switch to targeting E_s at the moment that E_s enters the sensing range of P under optimal target selection. If we can guarantee that P will sense E_s before entering the circle, then P must switch to a different target before capturing E_f, and E_f will be able to use reactive evasion at that point to avoid capture.

Under the local pursuit strategy with E_f as the target, $\dot{d}_{fp} < 0$, so P monotonically approaches E_f.

We will show that P cannot enter the circle without first sensing E_s, so long as E_f remains risk minimized. At the point when P senses E_s, it will switch to targeting E_s, and E_f will be able to use local reactive evasion to avoid becoming the target again.

Suppose $d_{fp} = d_{App,fs}$. By the triangle inequality, $d_{sp} \leq d_{fp} + d_{fs}$. Substituting,

$$\begin{aligned} d_{sp} &\leq d_{App,fs} + d_{fs} \\ &\leq \left(\frac{v_p - v_f}{v_f - v_s} + 1 \right) d_{fs} = \left(\frac{v_p - v_s}{v_f - v_s} \right) d_{fs}. \end{aligned} \tag{22}$$

By the risk minimization condition (21), we have $d_{sp} < d_{sense}$. Thus by contradiction P cannot enter the Apollonius circle without first sensing E_s. ■

Consider a graph $\mathcal{G}$ where evaders act as nodes, and an edge e_{ij} from evader E_i to evader E_j is present only if E_i is risk minimized with respect to E_j. This forms a directed graph with edges only going from a faster evader to a slower evader. Due to that hierarchy, any connected component must contain a spanning tree with the slowest evader in the component as the root.

Theorem 5.2: Under the local evasion strategy, an evader can only be captured if it is the slowest evader in a connected component of $\mathcal{G}$.

Proof: Let E_1 be the initial target of pursuer P under local sensing when P enters the sensing range of E_1 at time t_0. If E_1 is risk minimized with respect to another evader E_2 at t_0, then by the definition of $\mathcal{G}$ it is not the slowest evader in its connected component, and by Theorem 5.1 the target of P will eventually switch to another evader. If E_1 is not risk minimized it must be the slowest evader within its connected component, and the other evaders will be able to use reactive evasion to avoid becoming the target, leading to the capture of E_1. ■

VI. Final Remarks

Our analysis of the local sensing system shows that it is beneficial for evaders to approach and remain close to slower neighbors. All evaders within a group benefit from their membership, except for the slowest member, mirroring the result of the non-spatially explicit model of Eshel [17]. A significant difference in our model is that the slowest member will never be able to leave the group, as the faster evaders will always be able to keep up with it. This does not preclude the slowest evader from wandering around in search of an even slower evader—if the slowest evader in each group performs a random walk, we would expect that eventually all evaders would connect into one group, with all of the capture risk concentrated on the globally slowest evader.

In both natural and engineered systems, the careful use of limited energy resources is of paramount importance. We plan to extend our analysis to consider pursuer and evader strategies for energy minimization. One limitation of the current model is that agents must know the maximum speeds of the other agents in order to calculate their strategies. Further work on this subject that addresses uncertainty in the speed measurements is warranted. The risk reduction strategy presented here provides a ready framework for the development of aggregation algorithms in engineered multi-agent systems with local sensing capabilities, and the weighted-Voronoi analysis may be leveraged to study coverage problems where a bounded time to reach a set of points is desired.

Acknowledgments

We thank Dan Rubenstein, Simon Levin, and Philip Holmes for helpful discussions on this work.

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