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TIME-OPTIMAL TRAJECTORIES FOR STEERED AGENT WITH CONSTRAINTS ON SPEED AND TURNING RATE

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ABSTRACT

Through application of Pontryagin's maximum principle we derive minimum-time optimal trajectories for a "steered particle" agent with constraints on speed and turning rate to reach a point on the plane with free terminal heading. We also present a formulation of the optimal trajectories in the form of a state-feedback control law that is applicable to real time motion planning on a robotic system with these motion constraints.

1 Introduction

Minimum time problems have long been a source of fascination for the mathematics community, reaching as far back as the *brachistochrone* problem that arguably led to the creation of optimal control theory and the calculus of variations [1]. The ongoing progress in the availability and capability of mobile robotic systems have seen an increase in interest in robotic minimum time motion planning problems.

Much work has been done to characterize time-optimal trajectories for car-like systems with limited turning radius, notably the "Dubins vehicle," which has a constant forward speed, and the Reeds-Shepp vehicle, which also allows for reverse motion. See [2] for a review, and [3] for detailed derivation of optimal paths with fixed terminal headings.

In this paper, we solve the problem of reaching a desired point on the plane in minimum time for a "steered particle" agent with independent constraints on angular turning rate and forward speed. Our model differs from both the Dubins and Reeds-Shepp models in that it allows for the agent to rotate in place with zero

forward speed. This type of motion can be achieved in a single-wheeled vehicle with gyroscopic turning, as in the "Gyrover" [4] and "Gyrobot" [5].

We are motivated by the desire to create a movement model with meaningful parameters that is analytically tractable for a single agent. We aim to apply our results for individual optimal trajectories as building blocks for improving models of collective motion, such as the model of group evasion from a pursuer presented in [6] that does not include limits on turning rate.

Our analysis is based closely on the geometric methods of Balkcom and Mason that were applied to differential drive vehicles with limited wheel speeds in [7], and to extremal trajectories for more general constraints in [8]. These methods have also been applied to minimum time trajectories for omni-directional robots [9], and minimum wheel-rotation for the differential drive [10].

2 Steered particle system dynamics

Consider an agent on the plane at position $(x, y) \in \mathbb{R}^2$ and heading angle $\theta \in \mathbb{S}^1$ that moves under the following equations of motion, often known as a "steered particle" model:

$$\begin{aligned}\dot{x} &= v \cos \theta \\ \dot{y} &= v \sin \theta \\ \dot{\theta} &= \omega.\end{aligned}\tag{1}$$

The control inputs are the forward speed v and the turning rate ω . We impose the following constraints on the control inputs:

1. Forward motion: Speed must satisfy $v \geq 0$ for all time, such that the agent never moves in reverse.
2. Limited speed: Let $\bar{v} > 0$ be the maximum speed. The speed control must satisfy $v \leq \bar{v}$ for all time.
3. Limited turning rate: Let $\bar{\omega}$ be the maximum turning rate. Then the turning control must satisfy $|\omega| \leq \bar{\omega}$ for all time.

Let Ω be the set of all admissible inputs $(v, \omega)^T$ satisfying the above constraints.

We consider the problem of reaching a desired point $(x_f, y_f)^T$ in minimum time T starting from the origin $x = 0, y = 0, \theta = 0$, with final heading $\theta(T)$ left unconstrained.

3 Extremal trajectories from Pontryagin's maximum principle

We follow closely the method of [7] to set up the problem and derive the set of extremal controls. In the interest of space we refer the reader to [2] for proofs on existence for optimal trajectories under these dynamics with a convex space of feasible inputs.

The single agent minimum time problem to reach a destination point $(x_f, y_f)^T$ with free terminal heading is described by

$$T = \min_{(v(t), \omega(t)) \in \Omega} \int_0^T 1 dt \quad \text{s.t.} \quad \dot{q}(t) = f(q, v, \omega), \psi(t_f) = 0, \quad (2)$$

where $q = (x, y, \theta)^T$, $f(q, v, \omega) = (v \cos \theta, v \sin \theta, \omega)^T$, and $\psi(t) = (x(t) - x_f)^2 + (y(t) - y_f)^2$ for initial condition $q(0) = (0, 0, 0)^T$.

The control Hamiltonian for the system dynamics is given by

$$H(\lambda, q, u) = \lambda \cdot f(q, u) = \lambda_x v \cos \theta + \lambda_y v \sin \theta + \lambda_\theta \omega. \quad (3)$$

The adjoint equations of motion are $\dot{\lambda} = -\frac{\partial H}{\partial q}$, with

$$\begin{aligned} \dot{\lambda}_x &= 0 \\ \dot{\lambda}_y &= 0 \\ \dot{\lambda}_\theta &= \lambda_x v \sin \theta - \lambda_y v \cos \theta. \end{aligned} \quad (4)$$

From this we see that λ_x and λ_y are constant over time. Also λ_θ can be directly integrated:

$$\begin{aligned} \lambda_\theta &= \lambda_x \dot{y} - \lambda_y \dot{x} \\ \lambda_\theta &= \lambda_x y - \lambda_y x - \rho, \end{aligned} \quad (5)$$

for some constant of integration ρ .

Since the terminal heading $\theta(T)$ is free, we must impose the terminal condition that $\lambda_\theta(T) = 0$, where

$$\lambda_\theta(T) = \lambda_x y_f - \lambda_y x_f - \rho. \quad (6)$$

For a nontrivial optimal trajectory, the adjoint vector must be nonzero at all times, implying that λ_x and λ_y cannot both be zero. Without loss of generality, assume $\lambda_x^2 + \lambda_y^2 = 1$, and let $\lambda_x = \sin \gamma$ and $\lambda_y = -\cos \gamma$ for some unknown angle γ , with ρ constrained by Eqn. (6).

Following the derivation of [7], define the following functions:

$$\eta(x, y) = x \cos \gamma + y \sin \gamma - \rho, \quad (7)$$

which describes a line in the x - y plane, and

$$\beta = \theta - (\gamma + \pi/2), \quad (8)$$

which is the the agent's heading relative to the direction tangent to the η line.

Now we can write the Hamiltonian as a dot product between the control vector $u = (v, \omega)^T$ and a vector function of the state $g = (-\cos \beta(\theta), \eta(x, y))^T$,

$$H = -v \cos \beta + \omega \eta = u \cdot g, \quad (9)$$

making it straightforward to apply Pontryagin's maximum principle to find extremal controls as a function of the state.

3.1 Switching functions and generic controls

We define two switching functions that can be used to determine which control input will minimize the Hamiltonian for a given state while satisfying the input constraints:

$$\begin{aligned} \phi_1(q) &= \cos \beta \\ \phi_2(q) &= -\eta(q). \end{aligned} \quad (10)$$

Then the extremal controls take the form $v = \bar{v} \text{sign}(\phi_1)$ and $\omega = \bar{\omega} \text{sign}(\phi_2)$.

The extremal controls for generic (nonsingular) intervals fall into two categories based on the signs of the switching functions:

1. Rotation: When $\phi_1 < 0$, the agent rotates in place, with extremal control given by $v = 0$ and $\omega = \bar{\omega}$ for $\phi_2 > 0$ (left) or $\omega = -\bar{\omega}$ for $\phi_2 < 0$ (right).

2. Turn: When $\phi_1 > 0$, the agent moves forward with full speed while turning at the maximum rate, with $v = \bar{v}$ and $\omega = \bar{\omega}$ for $\phi_2 > 0$ (left) or $\omega = -\bar{\omega}$ for $\phi_2 < 0$ (right). The agent moves on a circular arc with radius $R = \bar{v}/\bar{\omega}$.

3.2 Singular controls

At the boundaries between generic intervals where some $\phi_i = 0$, there exist multiple inputs that minimize the Hamiltonian. When the state arrives at such a switching surface, the control may have an instantaneous switching if the state traverses the switching surface, or an interval of singular control where the state remains on the switching surface for some time interval. We must examine each switching surface separately.

3.2.1 Forward motion When $\phi_2 = 0$ and $\phi_1 > 0$, at some time t' the agent is on the switching surface between left turn and right turn controls. Geometrically, this places the agent on the η line with $\cos \beta > 0$, and the Hamiltonian is minimized by $v = \bar{v}$ with ω taking any value in $[-\bar{\omega}, \bar{\omega}]$. For $\cos \beta \neq 1$, the agent is not aligned with the η line, so the positive speed will bring it off of the line at the next moment. This would manifest as an instantaneous switching from left turn to right turn, or vice-versa, without an extended singular control segment. In the case that $\cos \beta = 1$, the agent is arriving on the η line at a tangent, and forward motion with $\omega = 0$ would keep it on the line for further time $t > t'$. This represents an interval of singular control, and we show that it will be part of many time optimal trajectories.

Consider the case that the destination lies on the positive x -axis. It is clear that the time optimal trajectory to reach that point is forward motion at maximum speed. This illustrates that solutions to the minimum time problem can contain singular intervals of forward motion in this system.

3.2.2 Rotation and slow turn When $\phi_1 = 0$ at some time t' , the Hamiltonian is minimized by $\omega = \pm \bar{\omega}$ (positive for $\phi_2 > 0$, negative for $\phi_2 < 0$) with v taking any value in the range $[0, \bar{v}]$. Taking the derivative of ϕ_1 with respect to time, we can see that it is always nonzero for any of the maximizing controls on the switching surface, so there can be no singular interval for the switching from rotation to slow turn:

$$\begin{aligned} \dot{\phi}_1|_{\beta=\pm\pi/2} &= -\sin(\beta) \frac{\partial \beta}{\partial \theta} \dot{\theta} \Big|_{\beta=\pm\pi/2} \\ &= -\sin(\beta) \omega|_{\beta=\pm\pi/2} = \pm \bar{\omega} \neq 0. \end{aligned} \quad (11)$$

3.3 Doubly singular controls

If both ϕ_1 and ϕ_2 are zero for a given state, then the Hamiltonian is zero. One of the conditions for the application of the maximum principle is that H is constant. Any nonzero control inputs

here would cause H to change value, so doubly singular interval consists of the trivial trajectory of a agent sitting motionless—only an optimal trajectory if the destination lies at the starting configuration.

4 Families of optimal trajectories

Now that we have enumerated the types of extremal trajectory segments, the task is to show which combination of extremal segments make up the time-optimal trajectory to a given destination point. We examine the possible terminal conditions and integrate backwards to find switching conditions compatible with the terminal constraints. In the following section we show that any point in the plane can be reached by one of these “nominal trajectories.”

Theorem 1. *Time-optimal trajectories must end in either a turn or forward motion segment.*

Proof. The terminal condition states that $\phi_2 = 0$ at the final time, and turning and forward motion are the only two controls that can bring the trajectory to (or remain on) that surface.

Theorem 2. *If the heading of the agent is aligned with the baseline vector from the agent to its destination at time t_1 , the time optimal trajectory for $t > t_1$ consists of forward motion at maximum speed.*

Proof. The shortest distance between two points on the plane is a straight line. The agent can move at its maximum speed \bar{v} while traveling in a straight line in the direction of its heading. Thus the minimum time trajectory to a point on the heading tangent line in front of the agent is forward motion at maximum speed.

4.1 Trajectories ending in forward motion

The family of trajectories ending in forward motion includes all trajectories of the following types (for both left and right turns):

1. F : Forward motion only.
2. TF : Turning motion followed by some forward motion.
3. RTF : Rotation, followed by a turn of maximum duration $\bar{\tau}_t$ (defined below), followed by some forward motion.

The rest of this section derives conditions on the lengths of the segments of these trajectories.

For a trajectory to end with a forward motion segment, it must have $\eta = \beta = 0$ at the terminal time. That condition will hold for the duration of the forward segment, no matter how long it lasts. Suppose that at some time the control switches from forward motion to a turn. We can integrate backwards from that point to find the switching time for rotation. Let $\tau_f \geq 0$ be the time interval spent in forward motion for a given trajectory.

Starting at $\beta = 0$, $\eta = 0$ and integrating backwards in time under turning control $v = \bar{v}$, $\omega = \pm\bar{\omega}$, the agent can turn up to $\theta_t = \pi/2$ before reaching the $\phi_1 = \cos\beta = 0$ switching surface. Define $\bar{\tau}_t = \pi/(2\bar{\omega})$ to be the maximum time interval for a turn segment in trajectories ending with forward motion.

Continuing backward from that switching time, the agent performs a rotation segment $v = 0$, $\omega = \pm\bar{\omega}$, in the same direction as the turn.

Theorem 3. *A minimum-time optimal trajectory ending in a forward motion segment of duration τ_f preceded by a turning segment of maximum duration $\bar{\tau}_t$ can include a rotation motion segment of duration up to*

$$\bar{\tau}_r = (\pi - \text{atan2}(R + \tau_f/\bar{v}, R))/\bar{\omega}. \quad (12)$$

Proof. For an agent starting at the origin with its heading along the positive x -axis at time $t = 0$, an *RTF* trajectory with rotation, full-length turn, and forward segments of length τ_r , $\bar{\tau}_t$, and τ_f respectively will reach its destination at the point

$$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} R \cos(\pm\bar{\omega}\tau_r) \mp (R + \bar{v}\tau_f) \sin(\pm\bar{\omega}\tau_r) \\ R \sin(\pm\bar{\omega}\tau_r) \pm (R + \bar{v}\tau_f) \cos(\pm\bar{\omega}\tau_r) \end{pmatrix}, \quad (13)$$

with $+$ for left turning trajectories, and $-$ for right turning trajectories.

If we substitute $\tau_r = \bar{\tau}_r$ from Eqn. (12) into Eqn. (13), we find that $y_f = 0$ and $x_f = -\sqrt{R^2 + (R + \bar{v}\tau_f)^2} < 0$ for both left and right turning trajectories. Due to the symmetry of the control limits for left and right turns, destinations lying directly behind the agent's initial position can be reached in equal time through either a left or right turning trajectory. It can be verified with Eqn. (13) that a left turning trajectory with $\tau_r = \bar{\tau}_r + \varepsilon$ reaches the same destination point as a right turning trajectory with $\tau_r = \bar{\tau}_r - \varepsilon$, and vice-versa, for any $\varepsilon \in [0, \bar{\tau}_r]$. Thus any trajectory of *RTF* type including a rotation segment of duration longer than $\bar{\tau}_r$ can be replaced by an *RTF* trajectory turning the opposite direction to reach the same destination point in less time.

4.2 Trajectories ending in a turn

The family of trajectories ending in a turn includes all trajectories of the following types (for both left and right turns):

1. *T*: Turn only.
2. *RT*: Rotation, followed by a turning segment.

For trajectories with $\eta(T) = 0$ and $\beta(T) \neq 0$, we can again integrate backwards to find families of optimal trajectories, but in this case the switching angle for the turning segment will be a function of the terminal value of β .

Let β' be the value of β at the terminal time. Integrating backwards with turning motion, we find that the maximum duration of a turn segment is $\bar{\tau}_t(\beta') = \theta_t(\beta')/\omega_c$ with

$$\theta_t(\beta') = \pi/2 - \beta'. \quad (14)$$

As above with the forward-ending trajectories, we can calculate the maximum amount of rotation for a given turning duration τ_t such that the destination lies directly behind the agent's initial location. Here the maximum rotation duration is given by $\bar{\tau}_r(\tau_t) = \pi/\bar{\omega} - \tau_t/2$.

5 The Optimal Trajectory

Here we present the explicit form of the time optimal trajectories. There are five types of trajectory (in each direction) corresponding to the different possible combinations of rotation, turn, and forward trajectory segments. We will first describe the partition of the plane into regions for the different trajectory types, and then present the explicit form of the optimal trajectory for each type individually.

5.1 Trajectory parameterized by switching times

Given the initial position $P_0 = (x_0, y_0)^T$ and heading θ_0 at time t_0 and a set of time durations spent in each control mode, we can write the trajectory explicitly as a sequence of translations and rotations about different points. Let τ_r , τ_t , and τ_f represent the time spent in rotation, turn, and forward modes, respectively. Switching times and the final time are then given simply by

$$\begin{aligned} t_t &= t_0 + \tau_r \\ t_f &= t_t + \tau_t \\ T &= t_f + \tau_f. \end{aligned} \quad (15)$$

The heading angle, for initial heading θ_0 , is given by the following expression, with $+$ for left turn, $-$ for right:

$$\theta(t) = \begin{cases} \theta_0 \pm \bar{\omega}(t - t_0), & \text{for } t_0 \leq t < t_f \\ \theta_0 \pm \bar{\omega}(\tau_r + \tau_t), & \text{for } t_f \leq t \leq T. \end{cases} \quad (16)$$

We can write the agent's position at a given time as the sum of vectors for each segment. Define the following "turning vector," which represents the relative translation due to a left/right turn with change in heading $\Delta\theta$, as

$$\begin{aligned} T(\Delta\theta) &= R \begin{pmatrix} \sin(|\Delta\theta|) \\ \text{sign}(\Delta\theta)(1 - \cos(\Delta\theta)) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \text{sign}(\Delta\theta)R \end{pmatrix} - B(\Delta\theta) \begin{pmatrix} 0 \\ \text{sign}(\Delta\theta)R \end{pmatrix}. \end{aligned} \quad (17)$$

Also define the “forward vector” $F(d)$ for a given distance d as

$$F(d) = \begin{pmatrix} d \\ 0 \end{pmatrix}, \quad (18)$$

and let $B(\theta)$ represent the standard rotation matrix,

$$B(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (19)$$

The position of the agent $P = (x, y)^T$ at time t is given by the following expression (with $+/-$ for left/right turns respectively):

$$P(t) = \begin{cases} P_0, & \text{for } t_0 \leq t < t_l \\ P_0 + B(\theta(t_l))T_s(\pm\bar{\omega}(t-t_l)), & \text{for } t_l \leq t < t_f \\ P_0 + B(\theta(t_l))T_s(\pm\bar{\omega}\tau_t) + B(\theta(t_f))F(\bar{v}(t-t_f)), & \text{for } t_f \leq t < T. \end{cases} \quad (20)$$

6 Trajectory-type partition

In this section, we describe the locus curves of destination points that can be reached by multiple trajectory types, and show that this collection of curves implies a partition of the space of destinations into regions such that any point in a particular region can be reached in minimum time by a trajectory of that region’s type. See Fig. 1 for an illustration of the partition of the plane into trajectory type regions.

6.1 Boundary curves

6.1.1 Left *TF*–Right *TF* boundary The boundary curve separating left turn-forward and right turn-forward trajectories is the set of points reachable by a forward-only trajectory, which is the positive x -axis.

6.1.2 *TF*–*RT* boundary For this and the remaining boundary types, we assume a left turning trajectory. The boundary curve separating turn-forward from rotate-turn trajectories is the set of points reachable by a turn-only trajectory:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R \sin \theta \\ R(1 - \cos \theta) \end{pmatrix}, \quad \theta \in [0, \pi/2]. \quad (21)$$

6.1.3 *TF*–*RTF* boundary All *RTF* trajectories include a turning segment of full duration $\tau_t = \pi/(2\bar{\omega})$, and all *TF* trajectories have no rotation segment, $\tau_r = 0$. So, the set of points reachable by both *RTF* and *TF* trajectories are the end-points of trajectories with no rotation, a full turn segment, and

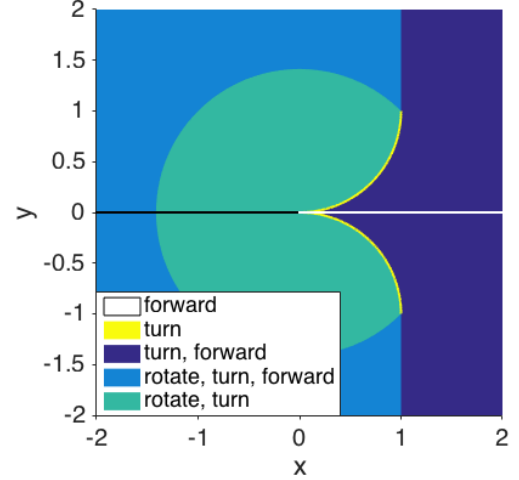


FIGURE 1. Regions show the sequence of control modes for different destination regions for time optimal of trajectories of an agent starting at the origin with heading in the direction of the positive x -axis with $\bar{v} = 1$, $\bar{\omega} = 1$. Destinations with $y > 0$ use left turn controls, and those with $y < 0$ use right turns. For those destinations along the positive x axis, no turning is necessary. For those on the negative x axis, left and right turning trajectories take the same amount of time.

some forward segment at the end. This takes the form of a vertical line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R \\ R+d \end{pmatrix}, \quad d \geq 0. \quad (22)$$

6.1.4 *RT*–*RTF* boundary All *RTF* trajectories include a turning segment of full duration $\tau_t = \pi/(2\bar{\omega})$, and all *RT* trajectories have no forward segment, $\tau_f = 0$. So, the boundary between the two trajectory types is the set of point reachable by a trajectory with no forward segment, a full turning segment, and a rotation segment of up to the maximum duration $\tau_r \in [0, (3\pi/4)/\bar{\omega}]$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R(\cos \theta_r - \sin \theta_r) \\ R(\sin \theta_r + \cos \theta_r) \end{pmatrix}, \quad \theta_r \in [0, 3\pi/4]. \quad (23)$$

6.2 Optimal switching times for each compound trajectory type

Here we derive the optimal switching times for trajectories in each of the regions defined above. For all, we assume the agent starts at the origin with $\theta_0 = 0$, and that the destination lies in the upper half-plane so that left turns are used. See Fig. 2 for examples of optimal trajectories, and Fig. 3 for a contour plot of the minimum time to reach different destinations.

6.2.1 TF trajectory This trajectory type is parameterized by the turn angle $\theta_t = \bar{\omega}\tau_t$ and the forward distance $d = \bar{v}\tau_f$. We start by writing the expression for the destination point in terms of the parameters:

$$P(T) = T(\theta_t) + B(\theta_t)F(d)$$

$$P(T) = \begin{pmatrix} 0 \\ R \end{pmatrix} + B(\theta_t) \left(F(d) - \begin{pmatrix} 0 \\ R \end{pmatrix} \right). \quad (24)$$

Rearranging and taking the norm of both sides, we are able to solve for the forward distance

$$d = \sqrt{x^2 - R^2 + (y - R)^2}, \quad (25)$$

and using that we can find the turn angle

$$\theta_t = \text{atan2}(y - R, x) - \text{atan2}(-R, d). \quad (26)$$

6.2.2 RT trajectory This trajectory type is parameterized by two angles, $\theta_r = \bar{\omega}\tau_r$ for rotation and $\theta_t = \bar{\omega}\tau_t$ for turning. We use the formula for the length of a circle chord to find θ_t , since the distance to the destination does not depend on θ_r :

$$x^2 + y^2 = 2R \sin(\theta_t/2), \quad (27)$$

$$\theta_t = 2 \sin^{-1} \left(\frac{\sqrt{x^2 + y^2}}{2R} \right). \quad (28)$$

The rotation angle is then computed as the difference between the actual heading to the destination and the heading of a turn of θ_t with no rotation:

$$\theta_r = \text{atan2}(y, x) - \theta_t/2. \quad (29)$$

6.2.3 RTF trajectory Since it includes both rotation and forward segments, the turn segment takes on its full possible duration of $\tau_t = \pi/(2\bar{\omega})$. Thus we only need to calculate the rotation angle $\theta_r = \tau_r/\bar{\omega}$ and the distance along the forward segment $d = \bar{v}\tau_d$. The location of the destination can be described as a function of those two quantities:

$$P = \begin{pmatrix} x \\ y \end{pmatrix} = B(\theta_r) \begin{pmatrix} R \\ R + d \end{pmatrix}. \quad (30)$$

By taking the norm of both sides we can solve for d ,

$$d = -R + \sqrt{x^2 + (y - R)^2 - R^2}. \quad (31)$$

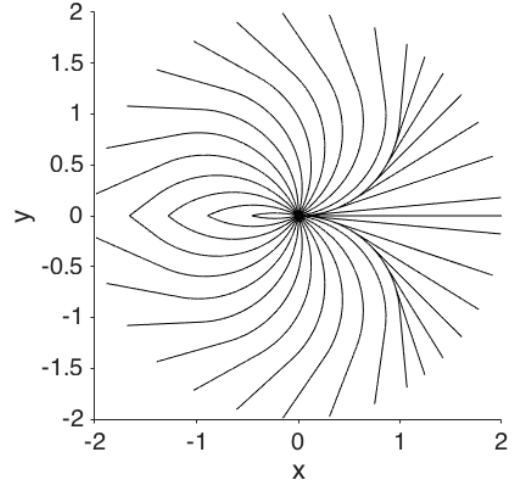


FIGURE 2. Minimum-time optimal trajectories for an agent starting at the origin with heading in the direction of the positive x -axis with $\bar{v} = 1$ and $\bar{\omega} = 1$.

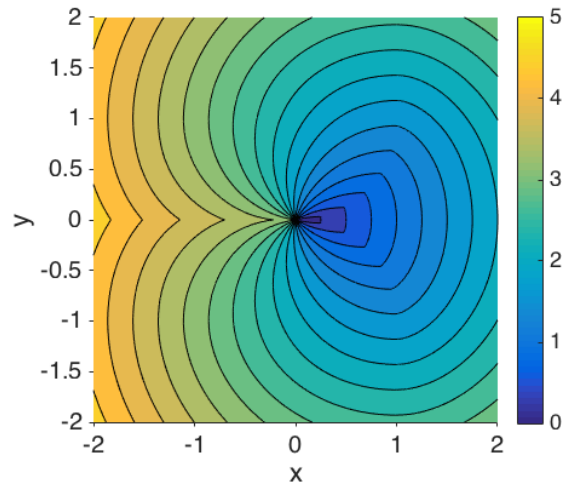


FIGURE 3. Minimum time to reach contours for an agent starting at the origin with heading in the direction of the positive x -axis with $\bar{v} = 1$ and $\bar{\omega} = 1$. Contours are drawn at 0.25 second intervals.

Then the rotation angle is given by the difference

$$\theta_r = \text{atan2}(y, x) - \text{atan2}(R + d, R). \quad (32)$$

7 State-feedback formulation of optimal control law

The optimal minimum-time trajectories can be described in terms of a state-feedback law where we consider the state to be

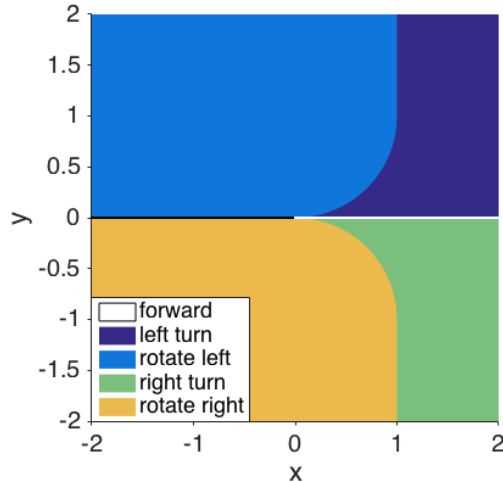


FIGURE 4. Control switching regions for the destination point in a body-fixed frame that moves with the agent, and has its x -axis aligned with the agent's heading direction, for an agent with $\bar{v} = 1$ and $\bar{\omega} = 1$.

the location of the destination point in a body-fixed frame that moves with the agent, and has its x -axis aligned with the agent's heading direction. The optimal control consists of the following rules:

1. If destination is on the positive x -axis, go forward.
2. Else if destination is in a trajectory-type region with turn as the initial segment, use the turn control in the appropriate direction.
3. Else, rotate in the appropriate direction.

See Fig. 4 for a diagram of the control regions.

This follows from Bellman's principle of optimality that for any point in time along an optimal trajectory, the remainder of the optimal trajectory is the same as if it was computed starting at that point.

8 Conclusions

We have presented explicit solutions to the problem of reaching a destination point in minimum time for an agent with limits on turning rate and forward speed. The solution to this minimum time problem can serve as a useful building block to motion planning in a multi-agent system. Since all trajectories and the time-to-reach surface are explicit and analytic, trajectories can be computed on a robot with limited computational capacity that can sense the distance and relative heading to its desired destination.

In future work, adding further refinements to the model in the form of constraints on acceleration will allow us to apply it to describe terrestrial animal motion. A strong understanding of

individual time-optimal trajectories gives insight into possibilities for collective motion in heterogeneous groups.

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