

Orientation Control of Multiple Underwater Vehicles with Symmetry-Breaking Potentials

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Abstract

We present a control strategy for stable orientation alignment of autonomous vehicles traveling together as a coordinated group in three-dimensional space. The control law derives from an artificial potential that depends only on the relative orientation of pairs of vehicles. The result is a controlled system of coupled rigid bodies with partially broken rotational symmetry. Semidirect product reduction theory is used to study the closed-loop dynamics, and the energy-Casimir method is applied to the reduced dynamics to prove stability of an alignment of vehicles translating in parallel along the same body axis. For clarity, the theory is described in detail for the case of two underwater vehicles, and the extension to an arbitrary number of underwater vehicles is summarized.

1 Introduction

The study of control laws for groups of autonomous vehicles has emerged as a challenging new research topic in recent years. There are currently few examples of, and yet many possible applications for, groups of highly autonomous agents that exhibit complex collective behavior in the man-made world. Researchers in this area are finding much inspiration from biological examples. Animal aggregations, such as schools of

fish, are believed to use simple, local traffic rules at the individual level but exhibit remarkable capabilities at the group level. These include rapid maneuverability, the ability to quickly process data and a markedly intelligent decision-making ability [2].

Numerous researchers have attempted to model animal aggregation with each agent considered as a point mass (see, for example, [1, 9]). Likewise in engineering applications, group control problems are often formulated with point-mass vehicle models (e.g. [8, 11]). Here we consider a group of underwater vehicles, which are modeled as rigid bodies from the outset. We note that the closest biological analogue to this system, that of a school of fish, is believed to maintain group cohesion through each individual's desire to match the speed and direction of nearby fish, and to be surrounded by a certain amount of open space (see [10]). Our focus in this paper is purely on the orientation matching part of the problem. The control law we derive is generated through judicious selection of an artificial potential function designed to break system symmetry. This work complements the development in [5] in which potential functions are used to address group geometry and inter-vehicle spacing.

The use of symmetry-breaking potentials here leads to the method of semidirect product reduction, for which our primary reference is [7]. Useful examples of this technique are contained in [3, 4]. In [4] stabilization of an arbitrary translation of a single underwater vehicle was achieved using symmetry-breaking potentials. The potentials used in that work inspire our choice of potentials here for orientation alignment of multiple underwater vehicles. As in [4], we prove stability by constructing a Lyapunov function on the reduced phase space by means of the energy-Casimir

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method (see [6]).

In §2 we present our model for a pair of uncontrolled underwater vehicles and describe the symmetry group and the reduced dynamics (see also [3]). In §3 we review semidirect production reduction theory [7] and in §4 we apply this to the dynamics of the pair of underwater vehicles with orientation alignment control derived from a symmetry-breaking potential. In §5 we prove stability of two vehicles moving in alignment and in §6 we describe extension of the development to N vehicles. Final remarks are given in §7.

2 Model and Reduction

We consider two underwater vehicles, which we shall label with the letters A and B . We assume they are identical and model each as an ellipsoidal body of mass m , as in [3]. For each vehicle, let the matrix J be the sum of the body inertia and the *added inertia* from the potential flow model of the fluid. Similarly, let M denote the sum of the body mass m multiplied by the identity matrix and the *added mass matrix*. We assume that m is also the mass of the displaced fluid so that each vehicle is neutrally buoyant. If each ellipsoid has uniformly distributed mass, the center of buoyancy is coincident with the center of gravity and both J and M are diagonal in a coordinate system defined by the ellipsoid's principal axes. In these body coordinates, vehicle A moves through the fluid with translational velocity $v^A = (v_1^A, v_2^A, v_3^A)^T$ and angular velocity $\Omega^A = (\Omega_1^A, \Omega_2^A, \Omega_3^A)^T$, with the analogous quantities for vehicle B defined similarly.

The length of the i th principal axis of each ellipsoid is denoted L_i and we assume that $L_1 > L_2 \geq L_3$. Let the diagonal elements of M be (m_1, m_2, m_3) and the diagonal elements of J be (J_1, J_2, J_3) .

The configuration space of the two vehicle system is $\text{SE}(3) \times \text{SE}(3)$, where $\text{SE}(3)$ is the Euclidean group which globally describes rigid body positions and orientations in three-dimensional space. An element in $\text{SE}(3) \times \text{SE}(3)$ is given by (R^A, b^A, R^B, b^B) , where $R^k \in \text{SO}(3)$ is a rotation matrix that maps body coordinates into inertial coordinates, and describes the orientation of vehicle k . Here b^k is the vector from the origin of the inertial coordinate frame to the origin of the body frame of vehicle k and describes the position of this vehicle.

We now find the equations of motion for the uncontrolled, two-body system by means of reduction, as was done for the single underwater vehicle in [3]. With the above definitions and the crucial assumption that the vehicles are *dynamically decoupled*, the Lagrangian of this system, $T : T\text{SE}(3) \times T\text{SE}(3) \rightarrow \mathbb{R}$, is given by

the kinetic energy

$$T(q^A, q^B) = \frac{1}{2} ((\Omega^A)^T J \Omega^A + (v^A)^T M v^A) + \frac{1}{2} ((\Omega^B)^T J \Omega^B + (v^B)^T M v^B) \quad (2.1)$$

where, for $k = A, B$, we use the shorthand $q^k = (R^k, b^k, R^k \hat{\Omega}^k, R^k v^k) \in T\text{SE}(3)$. The mapping $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$, where $\mathfrak{so}(3)$ is the space of 3×3 skew symmetric matrices, is defined such that for $x, y \in \mathbb{R}^3$, $\hat{x}y = x \times y$.

Under the lift of the left action of $\text{SE}(3) \times \text{SE}(3)$ on itself, by means of left translation $\mathcal{L}_{(\bar{R}^A, \bar{b}^A, \bar{R}^B, \bar{b}^B)} : \text{SE}(3) \times \text{SE}(3) \rightarrow \text{SE}(3) \times \text{SE}(3)$, the Lagrangian (2.1) becomes

$$\begin{aligned} & T(T\mathcal{L}_{(\bar{R}^A, \bar{b}^A, \bar{R}^B, \bar{b}^B)})(q^A, q^B) \\ &= T(\bar{R}^A R^A, \bar{R}^A b^A + \bar{b}^A, \bar{R}^A R^A \hat{\Omega}^A, \bar{R}^A R^A v^A, \\ & \quad \bar{R}^B R^B, \bar{R}^B b^B + \bar{b}^B, \bar{R}^B R^B \hat{\Omega}^B, \bar{R}^B R^B v^B) \\ &= T(q^A, q^B). \end{aligned}$$

This shows the Lagrangian (and therefore the Hamiltonian and thus the equations of motion) to be invariant under translations and rotations of the inertial frame, so we say the system has full $\text{SE}(3) \times \text{SE}(3)$ symmetry. Accordingly, Lie–Poisson reduction (see [6]) of the dynamics by the action of $\text{SE}(3) \times \text{SE}(3)$ induces a Lie–Poisson system on $(\mathfrak{se}(3) \times \mathfrak{se}(3))^* = \mathfrak{se}(3)^* \times \mathfrak{se}(3)^*$, where $\mathfrak{se}(3)$ is the Lie algebra of $\text{SE}(3)$ and $\mathfrak{se}(3)^*$ is the dual of $\mathfrak{se}(3)$. That is, because of symmetry, the equations of motion can be reduced from those on the 24-dimensional phase space $T^*\text{SE}(3) \times T^*\text{SE}(3)$ to the 12-dimensional space $\mathfrak{se}(3)^* \times \mathfrak{se}(3)^*$. We denote an element in the reduced phase space by $\mu = (\Pi^A, P^A, \Pi^B, P^B)$ where for $k = A, B$

$$\Pi^k = J\Omega^k, \quad P^k = Mv^k. \quad (2.2)$$

Written in terms of $\mu \in \mathfrak{se}(3)^* \times \mathfrak{se}(3)^*$ the reduced Hamiltonian is

$$\begin{aligned} H(\mu) &= \frac{1}{2} ((\Pi^A)^T J^{-1} \Pi^A + (P^A)^T M^{-1} P^A \\ & \quad + (\Pi^B)^T J^{-1} \Pi^B + (P^B)^T M^{-1} P^B). \end{aligned}$$

Here, Π^k and P^k are the angular and linear momentum vectors, respectively, for the k th vehicle. Given two differentiable functions F, Q on $\mathfrak{se}(3)^* \times \mathfrak{se}(3)^*$, the Poisson bracket on $\mathfrak{se}(3)^* \times \mathfrak{se}(3)^*$ which makes $\mathfrak{se}(3)^* \times \mathfrak{se}(3)^*$ a Poisson manifold is

$$\{F, Q\}(\mu) = \nabla F^T \Lambda(\mu) \nabla Q, \quad (2.3)$$

$$\Lambda(\mu) = \text{diag}(\Lambda^A, \Lambda^B); \quad \Lambda^k = \begin{pmatrix} \hat{\Pi}^k & \hat{P}^k \\ \hat{P}^k & 0 \end{pmatrix}.$$

The Lie–Poisson equations of motion are given by

$$\dot{\mu}_i = \{\mu_i, H\}(\mu), \quad (2.4)$$

equivalently, by

$$\begin{bmatrix} \dot{\Pi}^A \\ \dot{P}^A \\ \dot{\Pi}^B \\ \dot{P}^B \end{bmatrix} = \begin{bmatrix} \Pi^A \times \Omega^A + P^A \times v^A \\ P^A \times \Omega^A \\ \Pi^B \times \Omega^B + P^B \times v^B \\ P^B \times \Omega^B \end{bmatrix}, \quad (2.5)$$

which are simply Kirchoff's Equations (see [3] and references therein).

3 Semidirect Product Reduction : Theory

In the case when the Hamiltonian is dependent on a parameter, a , and is invariant under the action of a *subgroup* of the Lie group G , reduction can be achieved by means of semidirect products. Following [7], we denote the space containing a by V^* and define the semidirect product group of G with the vector space V as $S = G \times_{\rho} V$ with multiplication given by

$$(g_1, u_1)(g_2, u_2) = (g_1 g_2, u_1 + \rho(g_1)u_2) \quad (3.1)$$

where ρ is a left representation of G on V , i.e. $\rho : G \rightarrow \text{Aut}(V)$ is a group homomorphism, with $\text{Aut}(V)$ the Lie group of all isomorphisms of V .

Let $\mathfrak{s} = \mathfrak{g} \times_{\rho'} V$ be the Lie algebra of S (where \mathfrak{g} is the Lie algebra of G) with Lie bracket given by

$$[(\xi_1, \nu_1), (\xi_2, \nu_2)] = ([\xi_1, \xi_2], \rho'(\xi_1)\nu_2 - \rho'(\xi_2)\nu_1) \quad (3.2)$$

where $\rho' : \mathfrak{g} \rightarrow \text{End}(V)$ is the induced Lie algebra representation, and $\text{End}(V)$ is the Lie algebra of $\text{Aut}(V)$: the space of all linear maps of V into itself.

The “ \pm ” Poisson bracket of two functions $F, Q : \mathfrak{s}_{\pm}^* \rightarrow \mathbb{R}$ at the point $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times_{\rho'} V^*$ is given by

$$\begin{aligned} \{F, Q\}_{\pm}(\mu, a) &= \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta Q}{\delta \mu} \right] \right\rangle \\ &\pm \left\langle a, \rho' \left(\frac{\delta F}{\delta \mu} \right) \cdot \frac{\delta Q}{\delta a} - \rho' \left(\frac{\delta Q}{\delta \mu} \right) \cdot \frac{\delta F}{\delta a} \right\rangle. \end{aligned} \quad (3.3)$$

The Hamiltonian vector field of $H : \mathfrak{s}_{\pm}^* \rightarrow \mathbb{R}$ is

$$X_H(\mu, a) = \Lambda \nabla H \quad (3.4)$$

where Λ is the matrix representation of the Poisson bracket given in (3.3).

We additionally define \mathcal{L}_g to be the left action on G (i.e. $\mathcal{L}_g : G \rightarrow G, h \mapsto gh$), while $T^*\mathcal{L}_g$ is its cotangent lift to the cotangent bundle T^*G (see [6]).

We are now in a position to state the Semidirect Product Reduction Theorem.

Theorem 3.1 (Marsden et al. [7]) *Let $H_a : T^*G \rightarrow \mathbb{R}$ be a Hamiltonian depending smoothly on a parameter $a \in V^*$, and left invariant under the action on T^*G of the stabilizer G_a , defined as:*

$$G_a = \{g \in G \mid \rho^*(g)a = a\} \quad (3.5)$$

where ρ is a left representation of G on the vector space V , and ρ^* is the associated right representation of G on V^* . The family of Hamiltonians $\{H_a \mid a \in V^*\}$ induces a Hamiltonian function H on the space \mathfrak{s}_{\pm}^* , defined by $H((T_c\mathcal{L}_g)^*\alpha_g, \rho^*(g)a) = H_a(\alpha_g)$, thus yielding Lie-Poisson equations on \mathfrak{s}_{\pm}^* .

4 Semidirect Product Reduction : Orientation Control Application

We now revisit our system of two underwater vehicles and introduce an artificial potential to stabilize the equilibrium $R^A = KR^B$, where $K \in \text{SO}(3)$ is a (matrix) parameter we are free to choose. We are particularly interested in the case $K = I_3$, the 3×3 identity matrix, since this corresponds to two vehicles moving with orientations in alignment.

The form of our potential, V_p , is given by

$$V_p = \sigma \text{Tr}(I_3 - (R^A)^T K R^B) \quad (4.1)$$

where Tr is the trace operator for square matrices and σ is a scalar control gain which, if chosen to be strictly positive, ensures that V_p has a global minimum of zero at $R^A = KR^B$. With this potential the Lagrangian becomes

$$L(q^A, q^B) = T(q^A, q^B) - \sigma \text{Tr}(I_3 - (R^A)^T K R^B)$$

with kinetic energy (2.1). Under the left action of $\text{SE}(3) \times \text{SE}(3)$ the Lagrangian becomes

$$\begin{aligned} &L(T_{(\bar{R}^A, \bar{b}^A, \bar{R}^B, \bar{b}^B)}(q^A, q^B)) \\ &= T(q^A, q^B) - \sigma \text{Tr}(I_3 - (R^A)^T (\bar{R}^A)^T K \bar{R}^B R^B). \end{aligned}$$

The Lagrangian (and likewise the Hamiltonian and the equations of motion) will be invariant provided

$$(\bar{R}^A)^T K \bar{R}^B = K. \quad (4.2)$$

The mechanical system is thus invariant under action of the group

$$\begin{aligned} G_K &= \{(R^A, b^A, R^B, b^B) \in \text{SE}(3) \times \text{SE}(3) \\ &\mid (R^A)^T K R^B = K\}. \end{aligned} \quad (4.3)$$

We remark that G_K is (isomorphic to) a semidirect product of $\text{SO}(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$, where the pertinent representation of $\text{SO}(3)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ depends on the choice of K . In case $K = I_3$ this representation is simply the diagonal action $R \mapsto (R, R)$ and G_{I_3} is (isomorphic to) the double semidirect product S considered in Section 2.3.2 of [3].

The potential energy coupling terms have thus broken the full $\text{SE}(3) \times \text{SE}(3)$ symmetry, leaving the (left) symmetry group of the system to be G_K . This means

that the Lagrangian is unchanged by arbitrary translations of bodies A and B and arbitrary rotations R such that $R^B = R$ and $R^A = KRK^T$ (i.e. $R^A = R^B$ for $K = I_3$).

Define $\rho : \text{SE}(3) \times \text{SE}(3) \rightarrow \text{Aut}(\text{SO}(3))$ by $\rho(R^A, b^A, R^B, b^B)K = R^A K (R^B)^T$ where $\text{Aut}(\text{SO}(3))$ is the Lie group of all automorphisms of $\text{SO}(3)$. Given two matrices $Y \in \mathbb{R}^{n_1 \times m_1}$, $Z \in \mathbb{R}^{n_2 \times m_2}$, we define their *Kronecker Product*, denoted by $Y \otimes Z \in \mathbb{R}^{n_1 n_2 \times m_1 m_2}$, as

$$Y \otimes Z = \begin{bmatrix} y_{11}Z & \cdots & y_{1m_1}Z \\ \vdots & \ddots & \vdots \\ y_{n_1 1}Z & \cdots & y_{n_1 m_1}Z \end{bmatrix}. \quad (4.4)$$

With this definition it is easy to check that there is a direct correspondence between the elements of $R^A K (R^B)^T \in \mathbb{R}^{3 \times 3}$ and those of $(R^A \otimes R^B)\tilde{K} \in \mathbb{R}^9$, where $\tilde{K} = (e_1^T K, e_2^T K, e_3^T K)^T$. We can thus equivalently define $\rho : \text{SE}(3) \times \text{SE}(3) \rightarrow \text{Aut}(\mathbb{R}^9)$ by $\rho(R^A, b^A, R^B, b^B)\tilde{K} = (R^A \otimes R^B)\tilde{K}$, where $\text{Aut}(\mathbb{R}^9)$ denotes the set of invertible linear mappings on \mathbb{R}^9 .

An element in the Lie group $S = (\text{SE}(3) \times \text{SE}(3)) \times_{\rho} \mathbb{R}^9$ can be represented by an 18×18 matrix of the form

$$\begin{bmatrix} R^A & b^A & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & R^B & b^B & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & R^A \otimes R^B & \tilde{K} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the group action is given simply by matrix multiplication.

Let ρ^* be the associated right representation of $\text{SE}(3) \times \text{SE}(3)$ on \mathbb{R}^9 , given by

$$\rho^*(R^A, b^A, R^B, b^B)\tilde{K} = ((R^A)^T \otimes (R^B)^T)\tilde{K}. \quad (4.5)$$

Then G_K as defined in (4.3) is the stabilizer of $\tilde{K} \in \mathbb{R}^9$ under ρ^* , i.e.

$$G_K = \{(R^A, b^A, R^B, b^B) \in \text{SE}(3) \times \text{SE}(3) \mid \rho^*(R^A, b^A, R^B, b^B)\tilde{K} = \tilde{K}\}.$$

By Theorem 3.1 the Hamiltonian dynamics on $T^*(\text{SE}(3) \times \text{SE}(3))$ can be reduced to a Lie-Poisson system on the 21-dimensional space \mathfrak{s}^* , the dual of the Lie algebra \mathfrak{s} of S . In fact, $\mathfrak{s} = (\mathfrak{se}(3) \times \mathfrak{se}(3)) \times_{\rho'} \mathbb{R}^{3 \times 3}$ where ρ' is the induced Lie algebra representation, given by

$$\rho'(\hat{\alpha}, \nu^A, \hat{\beta}, \nu^B)\kappa = \hat{\alpha}\kappa - \kappa\hat{\beta}, \quad (4.6)$$

where $(\hat{\alpha}, \nu^A), (\hat{\beta}, \nu^B) \in \mathfrak{se}(3)$, $\kappa \in \mathbb{R}^{3 \times 3}$.

From (3.2) we can compute the Lie bracket of two elements of the Lie algebra \mathfrak{s} . Elements in the Lie algebra \mathfrak{s} can be represented by matrices of the form

$$\begin{bmatrix} \hat{\alpha} & \nu^A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{\beta} & \nu^B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{\alpha} \otimes I + I \otimes \hat{\beta} & \tilde{\kappa} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{\kappa} = (e_1^T \kappa, e_2^T \kappa, e_3^T \kappa)^T$. The Lie bracket of two Lie algebra elements in this form is given simply by the matrix commutator.

Recalling the definition $\mu = (\Pi^A, P^A, \Pi^B, P^B)$, where the elements of this vector are defined in (2.2), we note that $\mu_s = (\mu, X)$ is an element in \mathfrak{s}^* where $X \in \text{SO}(3) \subset \mathbb{R}^{3 \times 3}$ is defined by

$$X = \rho^*(R^A, b^A, R^B, b^B)K = (R^A)^T K R^B. \quad (4.7)$$

For future notational convenience we write

$$X^T = (\Delta, \Sigma, \Gamma) \quad \text{where } \Delta, \Sigma, \Gamma \in \mathbb{R}^3.$$

Written in terms of these new variables the reduced Hamiltonian is

$$\begin{aligned} H_s(\mu_s) &= \frac{1}{2} ((\Pi^A)^T J^{-1} \Pi^A + (P^A)^T M^{-1} P^A \\ &\quad + (\Pi^B)^T J^{-1} \Pi^B + (P^B)^T M^{-1} P^B) \\ &\quad + \sigma(\Delta \cdot e_1 + \Sigma \cdot e_2 + \Gamma \cdot e_3). \end{aligned} \quad (4.8)$$

Since we have performed reduction by means of the left action, the reduced phase space is \mathfrak{s}_*^* (with lower signs in (3.3)). The first term in the expression for the Poisson bracket (3.3) is given by the expression in (2.3). Using this, and defining the pairing between matrices in $\mathbb{R}^{3 \times 3}$ and $\mathbb{R}^{3 \times 3}$ to be $\langle Y, Z \rangle = \text{Tr}(Y^T Z)$ yields the following expression for the Poisson bracket:

$$\begin{aligned} &\{F, Q\}(\Pi^A, P^A, \Pi^B, P^B, X) \\ &= -\Pi^A \cdot (\nabla_{\Pi^A} F \times \nabla_{\Pi^A} Q) - \Pi^B \cdot (\nabla_{\Pi^B} F \times \nabla_{\Pi^B} Q) \\ &\quad - P^A \cdot (\nabla_{P^A} F \times \nabla_{P^A} Q) - P^B \cdot (\nabla_{P^B} F \times \nabla_{P^B} Q) - \\ &\quad \text{Tr} \left[X^T \left[\frac{\delta F}{\delta \Pi^A} \frac{\delta Q}{\delta X} - \frac{\delta Q}{\delta X} \frac{\delta F}{\delta \Pi^B} - \frac{\delta Q}{\delta \Pi^A} \frac{\delta F}{\delta X} + \frac{\delta F}{\delta X} \frac{\delta Q}{\delta \Pi^B} \right] \right]. \end{aligned}$$

We summarize our results in the following Proposition.

Proposition 4.1 *The dynamics of two rigid underwater vehicles coupled by means of a control law that realizes the artificial potential (4.1) describes a system that is Hamiltonian on $T^*\text{SE}(3) \times T^*\text{SE}(3)$ with symmetry group G_K . Reduction by this symmetry group yields Lie-Poisson dynamics on \mathfrak{s}_*^* , the dual of the Lie algebra of the semidirect product $S = (\text{SE}(3) \times \text{SE}(3)) \times_{\rho} \mathbb{R}^{3 \times 3}$. In the coordinates $\mu_s =$*

$(\Pi^A, P^A, \Pi^B, P^B, \Delta, \Sigma, \Gamma)$ of the reduced space \mathfrak{s}_-^* the differential equations governing the reduced dynamics are

$$\begin{aligned}\dot{\Pi}^A &= \Pi^A \times \Omega^A + P^A \times v^A - u \\ \dot{\Pi}^B &= \Pi^B \times \Omega^B + P^B \times v^B + u \\ \dot{P}^A &= P^A \times \Omega^A \\ \dot{P}^B &= P^B \times \Omega^B \\ \dot{\Delta} &= [0 \quad -\Gamma \quad \Sigma] \Omega^A + \Delta \times \Omega^B \\ \dot{\Sigma} &= [\Gamma \quad 0 \quad -\Delta] \Omega^A + \Sigma \times \Omega^B \\ \dot{\Gamma} &= [-\Sigma \quad \Delta \quad 0] \Omega^A + \Gamma \times \Omega^B\end{aligned}\quad (4.9)$$

where the control torque u on vehicles A and B is given by

$$u = \sigma (\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3).$$

These dynamics correspond to $\dot{\mu}_s = \Lambda_s(\mu_s) \nabla H_s(\mu_s)$ where H_s is given in (4.8) and

$$\Lambda_s(\mu_s) = \begin{bmatrix} \hat{\Pi}^A & \hat{P}^A & 0 & 0 & a & b & c \\ \hat{P}^A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{\Pi}^B & \hat{P}^B & \hat{\Delta} & \hat{\Sigma} & \hat{\Gamma} \\ 0 & 0 & \hat{P}^B & 0 & 0 & 0 & 0 \\ -a^T & 0 & \hat{\Delta} & 0 & 0 & 0 & 0 \\ -b^T & 0 & \hat{\Sigma} & 0 & 0 & 0 & 0 \\ -c^T & 0 & \hat{\Gamma} & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.10)$$

The 3×3 matrices a, b and c are given by

$$a = \begin{bmatrix} 0 \\ \Gamma^T \\ -\Sigma^T \end{bmatrix}, b = \begin{bmatrix} -\Gamma^T \\ 0 \\ \Delta^T \end{bmatrix}, c = \begin{bmatrix} \Sigma^T \\ -\Delta^T \\ 0 \end{bmatrix}.$$

The maximal rank of the Poisson bracket (4.10) on the whole 21-dimensional space \mathfrak{s}_-^* being 12, there may be up to nine independent *Casimirs*, i.e. functions that Poisson commute with any other function and as a result are constant along system trajectories for any Hamiltonian. We found the following six Casimir functions:

$$\begin{aligned}C_1 &= \|P^A\|^2 \\ C_2 &= \|P^B\|^2 \\ C_3 &= (P^B)^T (\Delta, \Sigma, \Gamma) P^A \\ C_4 &= \|\Delta\|^2 + \|\Sigma\|^2 + \|\Gamma\|^2 \\ C_5 &= \|\Delta \times \Sigma\|^2 + \|\Sigma \times \Gamma\|^2 + \|\Gamma \times \Delta\|^2 \\ C_6 &= (\Delta \times \Sigma) \cdot \Gamma.\end{aligned}$$

The functions $\|\Delta\|^2, \|\Sigma\|^2, \|\Gamma\|^2, \Delta \cdot \Sigma, \Sigma \cdot \Gamma, \Delta \cdot \Gamma$ are not Casimirs on the whole space \mathfrak{s}_-^* ; they only become constant along system trajectories once we restrict to the subset $\mathfrak{t}_-^* = (\mathfrak{se}(3) \times \mathfrak{se}(3)) \times_{\rho'} SO(3)$ of \mathfrak{s}_-^* . On this 15-dimensional submanifold the Casimirs C_4, C_5, C_6 assume the fixed values $C_4 \equiv 3, C_5 \equiv 3, C_6 \equiv 1$.

Proposition 4.2 On the subset $\mathfrak{t}_-^* \subseteq \mathfrak{s}_-^*$ given by $\|\Delta\|^2 = \|\Sigma\|^2 = \|\Gamma\|^2 = 1, \Delta \cdot \Sigma = \Sigma \cdot \Gamma = \Delta \cdot \Gamma = 0$ the 12-dimensional symplectic leaves are obtained by fixing the values of the three functionally independent Casimir functions C_1, C_2 and C_3 .

5 Stability of the Reduced System

We examine dynamic stability at equilibria of the reduced system of the form

$$\begin{aligned}(\Pi^A, P^A, \Pi^B, P^B, \Delta, \Sigma, \Gamma) \\ = (0, P_0^A e_3, 0, P_0^B e_3, e_1, e_2, e_3).\end{aligned}\quad (5.1)$$

with P_0^A, P_0^B both different from zero. Physically, such equilibria correspond to underwater vehicles A and B being aligned, non-rotating and each translating along its shortest axis. We note that motion of an underwater vehicle along any but its shortest axis is unstable unless it is spinning, or stabilized by a controlled moment ([3, 4, 12]). The control laws derived in this paper provide no such moment and thus, without additional control terms, stability along the shortest axis is the best that can be hoped for.

To find a Lyapunov function, we make the ansatz

$$H_\Phi = H_s + \Phi(C_1, C_2, C_3, C_4, C_5, C_6).$$

Let $\Phi^{(l)}$ be the derivative of Φ with respect to C_l at the equilibrium (5.1). The first variation of H_Φ will be zero at this equilibrium provided

$$\begin{aligned}\frac{P_0^A}{m_3} + 2P_0^A \Phi^{(1)} + P_0^B \Phi^{(3)} &= 0 \\ \frac{P_0^B}{m_3} + 2P_0^B \Phi^{(2)} + P_0^A \Phi^{(3)} &= 0 \\ -\sigma + 2\Phi^{(4)} + 4\Phi^{(5)} + \Phi^{(6)} &= 0 \\ -\sigma + P_0^A P_0^B \Phi^{(3)} + 2\Phi^{(4)} + 4\Phi^{(5)} + \Phi^{(6)} &= 0.\end{aligned}$$

This implies $\Phi^{(3)} = 0$, and taking $\Phi^{(5)} = \Phi^{(6)} = 0$ we may choose Φ to have the following simple form:

$$\begin{aligned}\Phi &= (\|P^A\|^2 - P_0^A P_0^A)^2 - \|P^A\|^2 / (2m_3) \\ &+ (\|P^B\|^2 - P_0^B P_0^B)^2 - \|P^B\|^2 / (2m_3) \\ &+ \frac{\sigma}{2} (\|\Delta\|^2 + \|\Sigma\|^2 + \|\Gamma\|^2).\end{aligned}$$

We now check for definiteness of the second variation at the equilibrium point. The Hessian of H_Φ has an associated matrix given by

$$\text{diag}(J^{-1}, \widetilde{M}_A, J^{-1}, \widetilde{M}_B, \sigma I_3, \sigma I_3, \sigma I_3)$$

where $\widetilde{M}_k = \text{diag}(1/m_1 - 1/m_3, 1/m_2 - 1/m_3, 8(P_0^k)^2)$, for $k = A, B$.

The Hessian is thus positive definite provided $\sigma > 0$. Stability follows by the energy-Casimir method [6].

Proposition 5.1 Consider two rigid underwater vehicles with controlled dynamics described by (4.9). Take $\sigma > 0$. Then, the equilibrium (5.1) corresponding to the two vehicles with full (three-dimensional) alignment in orientation and each translating along its shortest axis is stable.

6 The Case of N Vehicles

The case of N underwater vehicles, when considered as $N(N - 1)/2$ possible pairwise interactions, can be treated as a natural extension of the two-vehicle problem. We label each of our N vehicles with an index (i) for $i = 1, \dots, N$, and now seek an appropriate artificial potential to stabilize the equilibrium $R^{(j)} = K^{(jk)} R^{(k)}$, for all pairs j, k satisfying

$$1 \leq j < k \leq N \quad (6.1)$$

where $K^{(jk)} \in SO(3)$ is a (matrix) parameter relating the orientation of body (j) to that of body (k). For consistency, we must include the requirement that

$$K^{(ji)} K^{(ik)} = K^{(jk)} \quad \forall j, k \quad \text{and} \quad j < i < k.$$

This requirement will be trivially satisfied in the case $K^{(jk)} = I_3 \quad \forall j, k$ satisfying (6.1). As in §4, this choice corresponds to our desire to stabilize the equilibrium consisting of all N vehicles having the same orientation. Adjusting the definitions of V_p , \bar{K} and ρ appropriately the semidirect product reduction follows through in much the same way as in §4. Further details will be provided in a future publication.

7 Concluding Remarks

We have shown in this paper how to use symmetry-breaking potentials to generate a control law to effect stable orientation control for a pair of underwater vehicles. To prove asymptotic stability we can add a dissipation term to the control law and use the same Lyapunov function H_Φ constructed in §5. In fact, the Lyapunov function can be used like a control Lyapunov function to prescribe the form of the dissipation control term and to study stability in the presence of physical dissipation, i.e., drag (see [12]).

We remark that the various symmetry groups G_K for (non-aligning) relative orientation $K \in SO(3)$ are all isomorphic to G_{I_3} through an inner automorphism of $SE(3) \times SE(3)$. This may allow for an extension of the approach presented here to the case of prescribed time-varying relative orientations $K(t)$.

In a future publication we will present the details of the extension of this work to N vehicles that was introduced in §6. Further, we plan to couple this work with

the theory presented in [4] to allow the stabilization of motion of our underwater vehicles with arbitrary steady translations, and to prevent translational drift of our vehicle system. This will allow for stabilization of the vehicles in alignment and moving along their long axes (long axis motion being the preferred motion for streamlined bodies).

It is also of interest to investigate the addition of control terms which depend on the relative distance between vehicles. These terms will be intended to break translational symmetry, forcing $P_0^A = P_0^B$ in (5.1). For example, relative-distance-dependent factors might be included in the orientation control such that each vehicle tries to align itself only with its nearest neighbors. We will also consider adding to the orientation control the type of control laws featured in [1, 5] such that we produce formations of aligned vehicles with desirable group geometries and inter-vehicle spacing.

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