Effective Sensing Regions and Connectivity of Agents Undergoing Periodic Relative Motions

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***Abstract*— Time-varying graphs are widely used to model communication and sensing in multi-agent systems such as mobile sensor networks and dynamic animal groups. Connectivity is often determined by the presence of neighbors in a sensing region defined by relative position and/or bearing. We present a method for calculating the effective sensing region that defines the connectivity between agents undergoing periodic relative motions. This method replaces time-varying calculations with time-invariant calculations which greatly simplifies studies of connectivity and convergence of consensus algorithms. We apply the technique to the case of agents moving in a common fixed direction with sinusoidal speed oscillations and fixed relative phases. For agents moving in a straight line, we show analytically how to select dynamics for fast convergence of consensus. Further numerical results suggest graph-level connectivity may be achieved with a sensing radius lower than that predicted by percolation theory for agents with fixed relative positions.**

I. INTRODUCTION

Cooperative dynamics of multi-agent systems depend on information passing among agents; individuals use the feedback from others to modify their own dynamics [1], [2], [3]. In mobile networks, the passing of information can be implicit or explicit. For example, studies of animal aggregations typically assume an implicit communication network generated by individual sensing capabilities [4]. In the design of robotic systems, predefined rules for agent interaction might be in place to determine an explicit communication network [5]. However, in both the natural and engineered settings, it is common for the flow of information from one individual to another to be limited by their pairwise relative position and/or bearing. In a mobile sensor network, power consumption may limit the range of communication. In a school of fish, forward-looking vision may limit the ability of a fish to sense those to its rear.

In these sensory networks, spatial dynamics of agents couple directly with information passing: the spatial dynamics determine the communication network, i.e., which agents can sense which other agents, as a function of time, and the resulting time-varying communication network influences the dynamics of the motion of the agents through feedback. It is thus of great interest to explore this coupling in order to understand what kind of relative motion dynamics leads to high performance cooperative group behavior. In the engineered setting we seek to design individual feedback dynamics that lead to high performance and in the natural setting we seek to understand how animals organize their relative motion and why.

In this paper we consider agents that undergo fixed periodic relative motion with common period T and have a finite sensing region. We define a method to compute effective sensing regions. These effective sensing regions determine a single static graph into which we are able to map calculations of the original time-varying graph's properties.

We compute explicitly the effective sensing region for agents undergoing straight-line motion with sinusoidally time-varying speed. These dynamics are motivated by experimental observations of fish schooling with periodic speed profiles exhibiting phase coordination [6] and a subsequent study that suggests benefits of these oscillations to information passing and spatial motion pattern generation [7].

To illustrate the utility of this tool, we analyze two specific scenarios involving this type of relative motion. First, we determine connectivity conditions and analyze the convergence rate for a consensus algorithm executed over evenly distributed agents. We show that the convergence rate of such a system is maximized when nearest neighbors are antisynchronized. Furthermore, we show that for fixed parameters there is a particular frequency of oscillation that optimizes the convergence rate. We then utilize the effective sensing regions to make manageable a numerical study of spatially randomly distributed agents. We then consider the framework of graph percolation [8], [9], which is used to study connectivity in sensor networks for randomly placed fixed sensors with fixed sensing radius [10], [11]. Our study indicates that connectivity is improved greatly for agents with periodic relative motion.

The network model is described in Section II. In Section III we present our method for computing effective sensing regions for agents with periodic relative motion. These are computed for straight-line motion in Section IV. In Section V we apply the method to study connectivity and performance.

II. MODEL

A. Particle Model

We consider a group of N individual agents and model each as a particle with unit mass. In this paper we restrict motion to the plane and identify $\mathbb{R}^2$ with $\mathbb{C}$. For $k = 1, \ldots, N$, let $r_k \in \mathbb{C}$ denote the position of particle k and $f_k \in \mathbb{C}$ the total external force on particle k. Let $\alpha_k = |\dot{r}_k|$ be the speed of particle k and for $\alpha_k \neq 0$, let $\theta_k \in S^1$ be the direction of motion of particle k relative to an inertial

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frame (also referred to as its orientation). For particle k, we express the velocity as

$$\dot{r}_k = \alpha_k e^{i\theta_k} \tag{1}$$

and hence the dynamics are

$$\ddot{r}_k = \left(\dot{\alpha}_k + \alpha_k \dot{\theta}_k i\right) e^{i\theta_k} \tag{2}$$

for $k = 1, \ldots, N$. The first term on the right of (2) is the component of force in the direction of motion and the second term is the component of force normal to velocity.

We are particularly interested in motions that result from a periodically time-varying speed profile. We consider control forces that produce a speed profile of the form

$$\alpha_k(t) = 1 + \mu \cos\phi_k(t) \tag{3}$$

where $0 \leq \mu < 1$ is the amplitude of oscillations, we refer to $\phi_k \in S^1$ as the *speed phase*, and, without loss of generality, we have assumed that the mean speed is 1. We assume that, in the absence of external influences, $\dot{\phi}_k = \Omega$. Here we consider only $\dot{\theta}_k = 0$, but in [7] and a forthcoming paper we describe and design control laws for the novel family of trajectories generated when $\dot{\theta}_k \neq 0$ and for a more general family of periodic speed profiles.

B. Communication Model

Here we formally review the terms and notions from graph theory and consensus dynamics that we use throughout this paper, following [5], [2], and we introduce notation for periodically time-varying graphs.

The interaction network between N agents is defined by a time-varying directed graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), A(t)\}$ with node set $\mathcal{V} = \{1, \ldots, N\}$, time-varying edge set $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ and adjacency matrix $A(t)$. We consider each node to represent one agent such that when agent k can sense agent j at time t, there is an edge $(j,k) \in \mathcal{E}(t)$. $A(t)$ is defined such that element $A_{kj}(t) \geq \delta$ for some $\delta > 0$ when $(j,k) \in \mathcal{E}(t)$. The interactions for agent k at time t are calculated over the neighborhood set $\mathcal{N}_k(t) = \{j \in \mathcal{V} : (j,k) \in \mathcal{E}(t)\}$. The Laplacian matrix $L(t)$ associated with the graph $\mathcal{G}(t)$ is defined by its elements as $L_{kj} = \sum_i A_{ki}$, $j = k$ and $L_{kj} = -A_{kj}$, $j \neq k$. When the context is clear, we use $L = L(\mathcal{G})$ and $\mathcal{G}$ interchangeably.

For the case of a time-invariant graph, i.e., $\mathcal{G}(t) = \mathcal{G}$, the graph $\mathcal{G}$ is *strongly connected* (resp. *weakly connected*) if and only if any two distinct nodes can be connected by a path that respects (resp. does not necessarily respect) the edge directions of the graph. An undirected graph $\mathcal{G}$ (every edge bi-directional) that is weakly connected is also strongly connected and so we call it a connected graph.

Now consider a time-interval I and a time-varying graph $\mathcal{G}(t)$. The time-invariant graph $\bar{\mathcal{G}}_I$ corresponding to the graph $\mathcal{G}(t)$ over the interval I is $\bar{\mathcal{G}}_I = \left\{\mathcal{V}, \bar{\mathcal{E}}, \bar{A}\right\}$ where $\bar{\mathcal{E}} = \cup_{t \in I} \mathcal{E}(t)$ and $\bar{A}$ the adjacency matrix corresponding to $\bar{\mathcal{E}}$ with weights proportional to the time duration of edges. We say that $\mathcal{G}(t)$ is *strongly connected over the interval* I if $\bar{\mathcal{G}}_I$ is strongly connected. Further, a node k is said to be connected to node j ($j \neq k$) in the interval I if there is a path from k to j in $\bar{\mathcal{G}}_I$ that respects edge directions. The time-varying graph $\mathcal{G}(t)$ is *uniformly connected* if there exists an index k and a time horizon $T > 0$ such that, for all t, node k is connected to all other nodes in the interval $I = [t, t+T]$. It follows that a graph $\mathcal{G}(t)$ is uniformly connected if the graph $\bar{\mathcal{G}}_I$ is strongly connected for all intervals I of length T.

We say that a graph is periodic with period T if $\mathcal{G}(t+T) = \mathcal{G}(t)$ for all t. If a graph is periodic and strongly connected over any interval of length T, then it is uniformly connected with a horizon T. We write $\bar{\mathcal{G}}_T$ to denote the graph $\bar{\mathcal{G}}_I$ for any $I = [t, t+T]$, $t > 0$.

Uniform connectedness of a directed, time-varying graph is a useful property since it provides a sufficient condition for global convergence of linear consensus dynamics [5]:

$$\dot{x} = -L(t)x \tag{4}$$

where $x = \left[\begin{array}{ccc} x_1 & \ldots & x_N \end{array}\right]^T$ and x_k, $k \in 1, \ldots, N$, is a scalar quantity updated by agent k. Uniform connectedness can also be used to prove global convergence of synchronization problems on nonlinear spaces (e.g., on the N-torus) when auxiliary consensus variable dynamics are included [12].

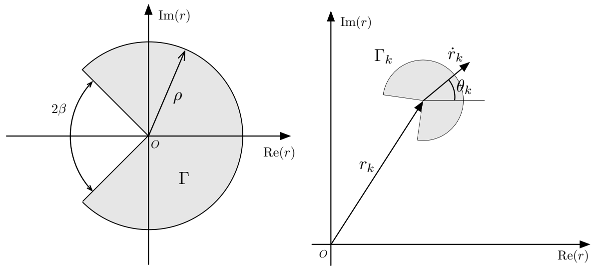
When $\mathcal{G}$ is time-invariant, the convergence rate of the consensus dynamics can be described precisely by the second smallest eigenvalue λ_2 of $L(\mathcal{G})$, which is also called the algebraic connectivity or the Fiedler constant of $\mathcal{G}$ [13], [14]. However, when $\mathcal{G}(t)$ is time-varying, the convergence rate is typically difficult to compute. Different approaches have been followed to estimate the convergence rate by utilizing the fact that $L(t)$ is taken from a compact set of matrices. Notions in matrix theory, such as the joint spectrum radius and scrambling constants, have been used to lower bound the convergence rate [15], [3]. We show below how in the case of agents undergoing periodic relative motion, the effective sensing regions we define and the periodicity of the time-varying graph can be used to compute the convergence rate.

C. Sensing Regions

Sensing regions define the mapping between spatial configuration of agents and their neighbor relationships, i.e. the communication graph. Let $\Gamma \subset \mathbb{C}$ be the *sensing region template* defined as the set of points in $\mathbb{C}$ sensed by a particle fixed at the origin with its direction of motion pointing along the real axis. Fig. 1(a) shows an example with finite sensing radius ρ combined with rear blind-angle 2β. We define the *sensing region of agent k at time t* (called the perceptual zone in [16]) as the set $\Gamma_k(t) \subset \mathbb{C}$ of points sensed by an agent located at $r_k(t)$ with orientation $\theta_k(t)$. Assuming that all agents share a common fixed sensing region template Γ, we have, for $k = 1, \ldots, N$ that

$$\begin{aligned} \Gamma_k(t) &= e^{i\theta_k(t)}\Gamma + r_k(t) \\ &:= \{x \in \mathbb{C} \mid (x - r_k(t))e^{-i\theta_k(t)} \in \Gamma\}. \end{aligned}$$

An agent j is therefore in $\mathcal{N}_k(t)$ if and only if $r_j \in \Gamma_k(t)$.



(a) (b)

Fig. 1. (a) Example sensing region template Γ for a limited sensing radius ρ with rear blindspot with half-angle β. (b) Illustration of $\Gamma_k = e^{i\theta_k}\Gamma + r_k$ for Γ as defined in (a).

III. EFFECTIVE SENSING REGIONS FOR PERIODIC RELATIVE MOTIONS

In this section we show how to compute connectedness properties of a time-varying communication graph associated with periodic relative motion. The approach is to map the system with N time-varying sensing regions $\Gamma_k(t)$ to an equivalent system with $N^2 - N$ time-invariant sensing regions $\bar{\Gamma}_{jk}$, $j \neq k$, where $\bar{\Gamma}_{jk}$ is defined below.

Consider a group of N agents undergoing periodic relative motion with period T. That is, for all t and for each pair j, k we have $r_{jk}(t) = r_{jk}(t+T)$ where $r_{jk}(t) = r_j(t) - r_k(t)$. Define the average relative position $\bar{r}_{jk}$ as

$$\bar{r}_{jk} = \frac{1}{T}\int_0^T r_{jk}(\tau)d\tau. \tag{5}$$

We define the *effective sensing region* such that there exists a $t^* > 0$ for which

$$r_j(t^*) \in \Gamma_k(t^*) \Leftrightarrow \bar{r}_{jk} \in \bar{\Gamma}_{jk}.$$

Because sensing over a period of motion depends upon the relative phases of motion between neighbors, we must calculate an equivalent sensing region for each *pair* of agents k and j, $j \neq k$, which implies $N^2 - N$ sensing regions. However, when sensing is isotropic and hence undirected (for example in the case of a limited sensing range only), we need only compute $(N^2 - N)/2$ sensing regions.

Recall that $j \in \mathcal{N}_k(t)$ if and only if $r_j(t) \in \Gamma_k(t)$, which (by definition) is equivalent to

$$r_{jk}(t) \in \Gamma e^{i\theta_k(t)}. \tag{6}$$

Given that $r_{jk}(t) = \bar{r}_{jk}(t) + (r_{jk} - \bar{r}_{jk})$, we can rewrite (6) as

$$\bar{r}_{jk} \in \Gamma e^{i\theta_k(t)} - (r_{jk}(t) - \bar{r}_{jk}).$$

Because relative motions are periodic, the communication graph is also periodic. For uniform connectivity of this graph we consider the associated time-invariant graph $\bar{\mathcal{G}}_T = \{\mathcal{V}, \bar{\mathcal{E}}, \bar{A}\}$. Assuming that the relative trajectories are suitably smooth and that Γ is an open set, $(j,k) \in \bar{\mathcal{E}}$ if and only if

$$\bar{r}_{jk} \in \bigcup_{t\in[0,T)} \left(\Gamma e^{i\theta_k(t)} - (r_{jk}(t) - \bar{r}_{jk})\right) \triangleq \bar{\Gamma}_{jk} \tag{7}$$

where we define membership in the union of time-varying regions $R(t)$ over a time interval I as $x \in \bigcup_{t\in I} R(t) \;\Leftrightarrow\; x \in R(t)$ for some t in the interval I.

We call the time-invariant region $\bar{\Gamma}_{jk} \subset \mathbb{C}$ the *effective sensing region* for the pair (j,k). This region can be found by taking the union, over any period of motion T, of the template sensing region Γ aligned with the velocity of agent k and shifted (negatively) by $r_{jk}(t) - \bar{r}_{jk}$.

We have shown the following.

Theorem 1: Consider N agents undergoing periodic relative motion with period T. Let the sensing region template for each agent be given by Γ, which determines the corresponding periodically time-varying graph $\mathcal{G}(t)$. The associated time-invariant graph over any time interval $[t, t+T]$, denoted $\bar{\mathcal{G}}_T = \{\mathcal{V}, \bar{\mathcal{E}}, \bar{A}\}$, is determined by computing the $N^2 - N$ edges ($(N^2-N)/2$ if Γ defines isotropic sensing) as follows:

$$(j,k) \in \bar{\mathcal{E}} \Leftrightarrow \bar{r}_{jk} \in \bar{\Gamma}_{jk}.$$

Further, $\mathcal{G}(t)$ is uniformly connected if $\bar{\mathcal{G}}_T$ is strongly connected.

IV. STRAIGHT-LINE MOTIONS WITH SINUSOIDAL SPEED

Here we apply the above methods to the case of straight-line motions with sinusoidally oscillating speed: we use the model (1), (3) with $\dot{\theta}_k = 0$. We consider communication or sensing that is limited by a distance $\rho > 0$, i.e., $(j,k) \in \mathcal{E}(t)$ if and only if $|r_{jk}(t)| < \rho$. The group is assumed to be arranged initially with parallel headings. Without loss of generality, let $\theta_1(0) = \theta_2(0) = \ldots = \theta_N(0) = 0$. We apply the formula (7) to obtain effective sensing region $\bar{\Gamma}_{jk}$.

Simply integrating the dynamics (1) with speed (3) gives the position

$$r_k(t) = t + \bar{r}_k + \frac{\mu}{\Omega}\sin\phi_k(t) \tag{8}$$

where $\phi_k(t) = \Omega t + \phi_k(0)$ and $\bar{r}_k = r_k(0) - \frac{\mu}{\Omega}\sin\phi_k(0)$ is the average location of $r_k(t) - t$. The position of particle j relative to particle k is

$$r_{jk}(t) = r_j(t) - r_k(t) = \bar{r}_{jk} + \frac{\mu}{\Omega}(\sin\phi_j(t) - \sin\phi_k(t))$$

where $\bar{r}_{jk} = \bar{r}_j - \bar{r}_k$ is the average relative position as defined by (5) over the period $T = 2\pi/\Omega$. Let $\phi_{jk} = \phi_j(t) - \phi_k(t) = \phi_j(0) - \phi_k(0)$ and $\varphi_{jk}(t) = \phi_j(t) + \phi_k(t) = 2\Omega t + \phi_j(0) + \phi_k(0)$, then

$$r_{jk}(t) - \bar{r}_{jk} = 2\frac{\mu}{\Omega}\sin\frac{\phi_{jk}}{2}\cos\frac{\varphi_{jk}(t)}{2}. \tag{9}$$

Because the headings of all agents are equal and zero, we need only take the union in (7) over the relative translations. Thus, by (9), the center of Γ translates along the real axis with an oscillation of amplitude $2\frac{\mu}{\Omega}\sin\frac{\phi_{jk}}{2}$ centered about the origin. The union over these translations gives the region illustrated in Fig. 2.

The description of $\bar{\Gamma}_{jk}$ is in terms of relative speed phase ϕ_{jk}, speed oscillation frequency Ω and amplitude μ and radius ρ of sensing region template Γ. By Theorem 1,

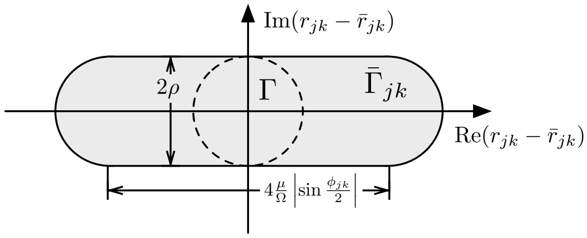


Fig. 2. Illustration of the effective sensing region $\bar{\Gamma}_{jk}$ for motion as described in Sec. IV as a function of relative speed phase ϕ_{jk}, speed oscillation frequency Ω and amplitude μ and radius ρ of sensing region template Γ (also shown).

$(j,k) \in \bar{\mathcal{E}}$ if and only if $\bar{r}_{jk} \in \bar{\Gamma}_{jk}$ and by isotropy of the sensing region, $(j,k) \in \bar{\mathcal{E}}$ if and only if $(k,j) \in \bar{\mathcal{E}}$. Thus, by checking these $(N^2 - N)/2$ conditions we can determine if $\mathcal{G}(t)$ is uniformly connected.

V. CONNECTIVITY AND CONSENSUS IN STRAIGHT-LINE MOTION WITH SINUSOIDAL SPEED

Here we apply the results of Section IV to two scenarios. In the first, motion takes place in a highly ordered configuration and we analyze consensus convergence rates in terms of motion parameters. In the second, agents are distributed randomly and we observe the effects of relative motion on connectivity.

A. Connectivity of Ordered Agents Oscillating in a Line

Consider N agents arranged along a straight line such that their average positions are evenly spaced. Their velocity is described by (1) with speed (3). The position of each agent is given by (8) with $\bar{r}_k = (k-1)a$ for $k = 1, \ldots, N$ and some constant $a > 0$ (see Fig. 3(a)). We assume that

$$2\frac{\mu}{\Omega} + \rho > a > \max\left\{\frac{\mu}{\Omega} + \frac{\rho}{2}, \rho\right\}, \tag{10}$$

so that 1) $a > \rho$ guarantees that constant connectivity of the graph is impossible, 2) $a > \frac{\mu}{\Omega} + \frac{\rho}{2}$ guarantees that each agent can become a neighbor of only the agents immediately to its left and right, and 3) $a < 2\frac{\mu}{\Omega} + \rho$ guarantees that connectivity is at least possible.

Due to the ordered nature of the problem, it is possible to generalize a case of small N. Accordingly, we specialize to the case of $N = 4$ agents. Additionally we assume that the first and third agents' speeds are synchronized and the second and fourth agents' speeds are synchronized. Without loss of generality, we let $\phi_1(0) = \phi_3(0) = 0$ and $\phi_2(0) = \phi_4(0) = \phi$.

Let $x_k(t) \in \mathbb{R}$ be the value at time t from agent k of a scalar quantity of interest, and consider the linear consensus dynamics (4), where $L(t)$ is the Laplacian matrix generated by the graph $\mathcal{G}(t)$ corresponding to communication limited by the sensing radius ρ. We show here that the convergence rate of (4) is maximized if nearest neighbors are exactly out of phase ($\phi = \pi$) and that there is a nontrivial frequency Ω, for $\phi = \pi$, that maximizes convergence rate. All of

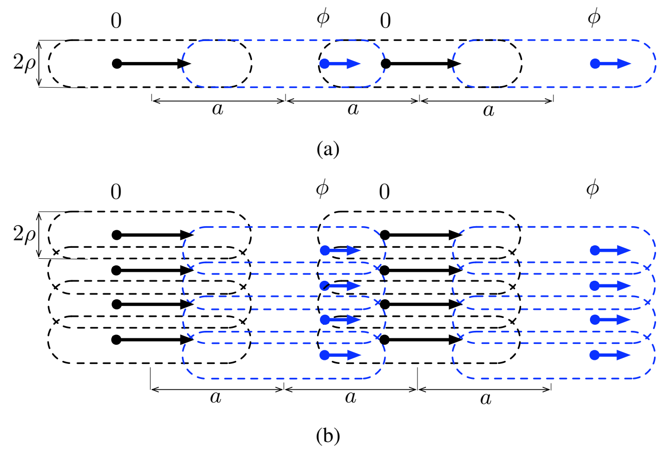


Fig. 3. (a) Illustration of the ordered in-line setup of Section V-A. Agents with sensing radius ρ are evenly distributed with spacing a along a line, every other agent is phase locked, and the phase difference between the first and second agents is ϕ. (b) Modification of the setup (a) in which a two-dimensional lattice is generated by vertical offsets. Note that in these figures the regions enclosed by dashed lines represent the area covered by a single agent as it traverses its path and *not* the effective sensing region (although they share a common shape).

these results are with respect to the following definition of convergence rate:

Definition 1: Let $L(t)$ be the Laplacian matrix of an undirected graph that is uniformly connected so that the consensus dynamics (4) converge to a vector $x_f = \mathbf{1}\bar{x}$ where $\mathbf{1}$ is the vector of ones and $\bar{x} \in \mathbb{R}$ [5]. The *convergence rate* σ is the supremum over all $\bar{\sigma} > 0$ for which there exists a $\beta > 0$ such that

$$\|x(t) - x_f\| \leq \beta e^{-\bar{\sigma} t} \tag{11}$$

for all initial conditions $x(0) \in \mathbb{R}^N$.

We next review stochastic matrices. A square matrix M is *stochastic* if it has nonnegative elements and $M\mathbf{1} = \mathbf{1}$. The class of $N \times N$ stochastic matrices is closed under multiplication [17]. Additionally, for a finite sequence of graphs L_i, $i = 1, \ldots, n$, with $\cup_i L_i$ strongly connected, the matrix product $e^{-L_1 a_1} \cdots e^{-L_n a_n}$ for scalars $a_i \geq 0$, $i = 1, \ldots, n$, is a stochastic matrix with a single largest eigenvalue equal to 1 and the remaining eigenvalues in the interior of the unit circle in the complex plane [18].

We can show the following.

Theorem 2: Let $\mathcal{G}(t)$ be a graph that is piecewise constant and periodic with period $T = \frac{2\pi}{\Omega}$. Define the finite sequence of Laplacians $L_1, \ldots, L_n$ and time intervals $\Delta t_1, \ldots, \Delta t_n$ with $\sum_{i=1}^{n} \Delta t_i = T$ such that for some $t_i \in [0, T]$, we have $L\left(\bar{\mathcal{G}}_{[t_i+\ell T, t_{i+1}+\ell T)}\right) = L_i$ where $\ell = 0, 1, \ldots,$ $t_i = t_{i-1} + \Delta t_i$, $i = 0, 1, \ldots, n-1$, and $\mathcal{G}(t)$ is constant over $[t_i + \ell T, t_{i+1} + \ell T)$. Assume that $\cup_i L_i$ is strongly connected. The convergence rate of (4) calculated over $\mathcal{G}(t)$ is given by

$$\sigma = -\frac{\Omega}{2\pi} \log m_2 \tag{12}$$

where m_2 is the second largest eigenvalue of the matrix $M = e^{-L_n \Delta t_n} \cdots e^{-L_1 \Delta t_1}$.

Proof: The solution to (4) for $qT < t < (q+1)T$ is given by $x(t) = M_2 M^q M_1 x(0)$, where $M = e^{-L_n \Delta t_n} \dots e^{-L_1 \Delta t_1}$ and M_1 and M_2 account for the fractions of a period before and after the q complete cycles over the n graphs. For large q, the term M^q dominates the convergence rate [3]. Since $\cup_i L_i$ is strongly connected, $\lim_{q\to\infty} M^q x(0) = \mathbf{1}\bar{x}$ for some scalar $\bar{x}$. Let $L = -\frac{\Omega}{2\pi}\log M$, then $\lim_{q\to\infty} e^{-LqT}x(0) = \mathbf{1}\bar{x}$ and hence the convergence rate in time is determined by the second smallest eigenvalue λ_2 of L, where $\lambda_2 = -\frac{\Omega}{2\pi}\log m_2$. ■

Depending upon the relative phasing ϕ, there are either 1 or 3 piecewise constant graphs over a single period of the above-described motion. Referring to Fig. 2, we see that there is no connectivity when $\left|\sin\frac{\phi}{2}\right| \le \frac{\Omega(a-\rho)}{2\mu}$, in which case the communication graph is constant and null. For $\left|\sin\frac{\phi}{2}\right| > \frac{\Omega(a-\rho)}{2\mu}$, the communication graph cycles between the null graph $L_1 = 0$, the graph L_2 with undirected edges $(1,2)$ and $(3,4)$, and the graph L_3 with the single undirected edge $(2,3)$. The case of zero connectivity is uninteresting, so we consider only the latter two cases.

To apply Theorem 2, we need also to compute the times Δt_2 and Δt_3 corresponding to the time-durations of L_2 and L_3. To do so we use the effective sensing region calculation method. From (9) the relative positions r_{21} and r_{43} oscillate with amplitude $2\frac{\mu}{\Omega}\sin\frac{\phi}{2}$ with frequency Ω about the mean distance a. Since the phase difference between agents 2 and 3 is $2\pi-\phi$ and $\sin\frac{2\pi-\phi}{2} = \sin\frac{\phi}{2}$, r_{32} undergoes an identical oscillation with a different phase. Therefore $\Delta t_2 = \Delta t_3$. The corresponding phase length $\Delta\phi = \Omega\Delta t_2 = \Omega\Delta t_3$ can be determined by finding where $r_{21} = \rho$. By our choice of range for a, $0 < \Delta\phi < \pi$ with the range being evenly distributed about the minimum distance. Therefore, we have

$$\Delta t_2 = \Delta t_3 = \frac{\Delta\phi}{\Omega} = \frac{2}{\Omega}\cos^{-1}\left(\frac{\Omega(a-\rho)}{2\mu\sin\frac{\phi}{2}}\right), \qquad (13)$$

where we take only the solution of $\cos^{-1}$ in the range $(0,\pi)$. Note that (13) has a solution for all of the cases of interest (i.e. where there is connectivity).

We can now state the following:

Corollary 1: Consider $N = 4$ agents with sinusoidal motion defined by (8) with $\phi_1(0) = \phi_3(0) = 0$ and $\phi_2(0) = \phi_4(0) = \phi$. Each agent senses in a circle of radius ρ where ρ satisfies (10). The rate of convergence of consensus dynamics (4) is maximized with respect to the phase offset ϕ when $\phi = \pi$, i.e., when every agent's oscillations are exactly out of phase with its nearest neighbors.

Proof: We have shown that, when connectivity is established, the system cycles periodically through three graphs $L_1 = 0$, L_2 and L_3 where L_2 has only the edge $(2,3)$ and L_3 has edge set $(1,2),\ (3,4)$. The second and third graphs have time durations $\Delta t_2 = \Delta t_3$ given by (13). Because L_1 is the zero matrix, $M = e^{-L_2\Delta t_2}e^{-L_3\Delta t_3}$. Since $M^T = e^{-L_3\Delta t_3}e^{-L_2\Delta t_2}$, the eigenvalues are independent of the order of graphs L_2 and L_3. We compute

$$m_2 = e^{-2\Delta t_2}\left(\cosh^2\Delta t_2 + \sqrt{\cosh^4\Delta t_2 - 1}\right). \qquad (14)$$

By Theorem 2, the convergence rate is given by $\sigma = -\frac{\Omega}{2\pi}\log m_2$, which by (14) is monotonically increasing in Δt_2. For a fixed Ω, Δt_2, and likewise the convergence rate, is maximized when $\phi = \pi$. ■

From (12), (14), and (13), we see that the convergence rate and the frequency of motion Ω are closely related. In fact, we may state the following.

Corollary 2: Given the same hypotheses as Corollary 1, the rate of convergence to consensus is maximized with respect to the frequency of motion Ω for some $\frac{a-\rho}{2\mu} \le \Omega < 2\frac{\mu\sin\frac{\phi}{2}}{a-\rho}$.

Proof: As $\Omega \to 0$, (12) shows that the convergence rate decays to zero. For $\Omega = 2\frac{\mu\sin\frac{\phi}{2}}{a-\rho}$, there is no solution to (13), that is, connectivity is lost and hence the convergence rate is zero. For all intermediate frequencies, the convergence rate is positive. Therefore, at least one local maximum must exist between $\Omega = 0$ and $\Omega = 2\frac{\mu\sin\frac{\phi}{2}}{a-\rho}$. The lower bound $\frac{a-\rho}{2\mu}$ prevents failure of the assumption that communication is only with immediately adjacent neighbors. ■

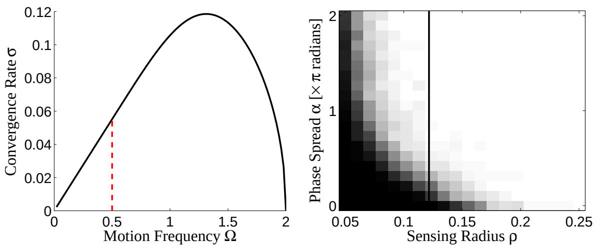
Fig. 4(a) shows an example convergence rate σ as a function of Ω for $a = 1$, $\mu = 0.5$, $\rho = 0.5$, and $\phi = \pi$. The lower bound $\frac{a-\rho}{2\mu}$ is indicated by the dashed vertical line. In this case the optimal value of Ω is above the lower bound. In case Ω at the peak of the curve is less than the lower bound, the lower bound is the optimal Ω. We note that the presence of an optimal frequency is a consequence of the tradeoff between duration and frequency of communication.

Remark. The results from this section can be generalized to $N > 4$ and also to ordered distributions of agents in the plane as illustrated in Figure 3(b). In particular, one may use this as a building block to describe the spatial flow of information along such lattices.

B. Connectivity of Randomly Distributed Agents with Randomly Distributed Speed Phase

In this section, we study numerically the connectivity of randomly distributed agents with periodic relative motion. We utilize effective sensing regions to translate between uniform connectivity of the periodically time-varying graph and strong connectivity of a static graph. The resulting framework is similar to the application of percolation theory [8] to the study of *phase transition* in connectivity of random geometric graphs [9], [8]. Through this study we present a measure of the improvement in connectivity offered by relative motion.

Consider a random geometric graph with parameters N and ρ defined by placing N nodes independently, randomly, and uniformly on a unit square in $2D$ and adding edges between any pair of nodes with relative distance less than ρ [19]. Then for a fixed large number N, when one slowly increases ρ from zero, there exists a critical sensing radius ρ_c



(a) (b)

Fig. 4. (a) Consensus convergence rate λ_2 versus motion frequency Ω for $a = 1$, $\mu = 0.5$, $\rho = 0.5$, and $\phi = \pi$. The dashed vertical line corresponds to the value of Ω for which the assumption of nearest-neighbor-only connectivity fails. (b) Probability that a graph is connected, ranging from zero (black) to one (white). Probability is estimated for $N = 100$ agents, over 100 simulations at each of 400 grid points - 20 each for phase spread α and sensing radius ρ. The vertical line represents the value ρ_c predicted for connectivity of the static geometric graph with high probability.

for which the sudden dramatic change in connectivity happens, i.e. when $\rho > \rho_c$, a connected component containing a large number[1] of nodes appears with a positive probability. In this paper, we follow Balister et al. [10], who combined theoretical analysis with Monte Carlo simulation method and obtained the estimated bounds $\sqrt{\frac{1.43}{N}} < \rho_c < \sqrt{\frac{1.48}{N}}$ with probability 0.9999.

Consider N agents with sinusoidal straight-line motion as described by (8). We consider the average positions in the moving frame to be uniformly and independently randomly distributed over the unit square. That is, we draw $\mathrm{Re}(\bar{r}_{jk})$ and $\mathrm{Im}(\bar{r}_{jk})$ from identical and independent uniform distributions over the range $[0, 1]$. Similarly, we draw the initial speed phases $\phi_k(0)$ from a uniform distribution over the range $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$ with $\alpha \in [0, 2\pi]$ a parameter. We also consider as a parameter the sensing radius ρ. We consider a range of ρ about the critical value $\rho_c = \sqrt{\frac{1.48}{N}}$ suggested by [10].

For a given set of initial conditions and parameter values, we estimate probability of uniform connectivity of the graph $\mathcal{G}(t)$ by exploiting the equivalence of strong connectivity of the static graph $\bar{G}_T$ determined by effective sensing regions. We determine the Laplacian matrix $L(\bar{\mathcal{G}}_T)$ for $N = 100$ agents and compute the second smallest eigenvalue λ_2. This is carried out 100 times for each value of ρ and α. Probability of connectivity is estimated as the percentage of experiments where $\lambda_2 > 10^{-4}$. Fig. 4(b) shows the results for a grid of 20 values each for α and ρ, with the solid vertical line indicating the value of ρ_c.

The horizontal line in Fig. 4(b) corresponding to $\alpha = 0$ can be taken to represent the geometric graph case, as synchronized speed implies zero relative motion. Correspondingly, phase transition occurs near ρ_c for $\alpha = 0$. For subsequent horizontal lines, corresponding to larger α, phase transition occurs for smaller ρ. That is, the relative motion can be seen to either improve connectivity for a fixed sensing radius or to decrease the sensing radius necessary to maintain connectivity. The size of $\bar{\Gamma}_{jk}$ is maximized when $\phi_{jk} = \pi$, hence little additional improvement in connectivity is realized for $\alpha > \pi$.

[1]To be more precise, the number of nodes in this connected component is $\Theta(N)$ where $\Theta(\cdot)$ is the notation in computational complexity theory and $f(N) \in \Theta(g(N))$ means that f is bounded tightly by g asymptotically.

VI. Final Remarks

In this paper we present effective sensing regions as a tool to aid in the study of systems of agents with periodic relative motion. We demonstrate their utility by deriving results for two scenarios of straight-line motion with sinusoidally varying speed. In both cases, analysis is made tractable by reducing a time-varying graph determined by sensing regions to an equivalent static graph determined by effective sensing regions. In ongoing work, we are using the effective sensing region method to further explore the coupling between spatial dynamics and information passing of mobile networks and in particular the role of relative dynamics among agents.

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