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Robustness of Noisy Consensus Dynamics with Directed Communication

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***Abstract*— In this paper we study robustness of consensus in networks of coupled single integrators driven by white noise. Robustness is quantified as the H_2 norm of the closed-loop system. In particular we investigate how robustness depends on the properties of the underlying (directed) communication graph. To this end several classes of directed and undirected communication topologies are analyzed and compared. The trade-off between speed of convergence and robustness to noise is also investigated.**

I. INTRODUCTION

The study of consensus problems has gained much interest in recent years, in particular with the application to multi-agent autonomous systems [1]–[4]. In this setting, consensus refers to every agent reaching agreement about some common or shared quantity. Two major areas where consensus is required are collective decision-making (such as deciding a common direction of travel) and collective sensing (such as reaching agreement about a measured environmental parameter). Both of these frameworks apply to biological systems such as bird flocks [5] and fish schools [6], and a number of models have been proposed to explain the methods by which animal groups reach consensus [7]–[9]. In an engineering context, the same consensus problems must be solved by autonomous groups of aerial, ground or underwater vehicles [10].

For the purposes of achieving consensus, most of the important details of a multi-agent system are encoded by the communication graph of the system. In this way, the performance of the consensus protocol can be related to the properties of the communication graph. This provides a more general setting to investigate consensus and allows for the application of graph-theoretic notions and tools. In particular, it is well known that the properties of the Laplacian matrix of the graph are intimately related to the performance of the consensus protocol [1], [3], [11], [12].

Since autonomous systems must operate in uncertain environments without direct supervision, it is important that such systems be robust. Multi-agent systems should be robust with respect to several different parameters, including component or individual agent failure, environmental uncertainty and communication uncertainty. This means that there are a number of different ways in which the robustness of consensus can be measured. Robustness to failure can be measured by the node and edge connectivities of the communication graph, while robustness to uncertainty and noise can be related to the H_∞ norm, H_2 norm, or L_2 gain of the consensus system. H_∞ robustness has been investigated in [13]–[15] in relation to uncertainty in the communication graph and to non-ideal communication channels. L_2 robustness has been considered in [12], [16] in relation to bounded (in the L_2 sense) inputs or errors. A notion of robustness equivalent to the H_2 norm for discrete-time consensus on undirected graphs has been used in [17] and for a particular undirected graph in [18], however little work appears to have been done on H_2 robustness of consensus for directed graphs.

In this paper we study the robustness of consensus to communication noise for directed communication topologies, which we show is naturally characterized by the H_2 norm of the system. In this context, the H_2 norm measures the expected steady-state dispersion of the agents under unit-intensity white noise. Thus systems with lower H_2 norms will remain closer to consensus despite the presence of noise. It should be noted that we do not consider the "accuracy" of the final consensus value, merely how well it is maintained.

The paper is organized as follows. In Section II notations are summarized. In Section III the model is introduced. In Section IV we introduce the robustness measure and we provide a closed-form expression for a particular class of graphs. Finally, in Section V, several families of directed and undirected graphs are compared and the trade-off between speed of convergence and robustness is investigated.

II. PRELIMINARIES AND NOTATION

The state of the system is given by $x = [x_1, x_2, \dots, x_N] \in \mathbb{R}^N$, where x_i is the state of agent i. For each agent i we define the set of neighbors, $\mathcal{N}_i$, to be the set of agents which supply information to agent i.

We call the state of the system a *consensus* state when $x = \gamma 1_N$, where $1_N = [1, 1, \dots, 1]^T \in \mathbb{R}^N$ and $\gamma \in \mathbb{R}$. Let Π be the projection matrix onto the subspace of $\mathbb{R}^N$ orthogonal to 1_N. Thus $\Pi = I_N - \frac{1}{N} 1_N 1_N^T$, where I_N is the N-dimensional identity matrix. Then the system is in consensus if and only if $\Pi x = 0$.

We associate to the system a (directed) *communication graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and $A \in \mathbb{R}^{N \times N}$ is a weighted adjacency matrix with nonnegative entries $a_{i,j}$. A is defined such that $a_{i,j} > 0$ if and only if $(i, j) \in \mathcal{E}$. Every node in the graph corresponds to an agent in our system, while the graph contains edge (i, j) when $j \in \mathcal{N}_i$. That is, every directed edge in $\mathcal{G}$ points from an agent receiving

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information to the agent supplying the information. Then $a_{i,j}$ is the weight given by agent i to the information from agent j. Note that according to our definition, $\mathcal{G}$ will contain at most one edge between any pair of nodes and will not contain any self-cycles (edges connecting a node to itself).

The *out-degree* (respectively *in-degree*) of node k is defined as $d_k^{out} = \sum_{j=1}^{N} a_{k,j}$ (respectively $d_k^{in} = \sum_{j=1}^{N} a_{j,k}$). A graph is said to be *balanced* if for every node, the out-degree and in-degree are equal.

$\mathcal{G}$ has an associated *Laplacian* matrix L, defined by $L = D - A$, where $D = \operatorname{diag}\left(d_1^{out}, d_2^{out}, \ldots, d_N^{out}\right)$ is the diagonal matrix of node out-degrees. The row sums of the Laplacian matrix are zero, that is $L1_N = 0$. Thus 0 is always an eigenvalue of L with corresponding eigenvector 1_N. Balanced graphs have the particular properties that $1_N^T L = 0$ and, as the Laplacian is then a hyperdominant matrix with zero excess, $L + L^T \geq 0$ [19].

A path in $\mathcal{G}$ is a (finite) sequence of nodes such that each node is a neighbor of the previous one. The graph $\mathcal{G}$ is *connected* if it contains a globally reachable node k; i.e. there exists a node k such that there is a path in $\mathcal{G}$ from i to k for every node i. It can be shown that all the eigenvalues of L are either 0 or have positive real part, and 0 will be a simple eigenvalue of L if and only if $\mathcal{G}$ is connected [20].

To allow for meaningful comparisons between directed communication graphs, we can *normalize* a graph by scaling the edge weights so that each node has an out-degree of either 1 or 0. That is, for every non-zero $a_{i,j}$, we replace $a_{i,j}$ with $\frac{1}{d_i^{out}} a_{i,j}$. This notion is similar to the way graph Laplacians are defined in [13]. Physically, this corresponds to each agent in our system taking as an input a weighted average of the differences between its own variable and those of its neighbors. Throughout the rest of this paper, we will assume that every directed graph has been normalized.

The graph $\mathcal{G}$ is said to be *undirected* if $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$ and $a_{i,j} = a_{j,i}$. We can generate an equivalent undirected graph for any directed graph. We first replace every edge (i,j) with the two edges (i,j) and (j,i), each with weight $\frac{a_{i,j}}{2}$. Then, if there are multiple edges between two nodes, the edges are combined by summing their weights. Equivalently, the adjacency matrix of the undirected graph will be the symmetric part of the directed adjacency matrix. Note that a graph will be undirected if and only if its adjacency and Laplacian matrices are symmetric.

III. NOISY CONSENSUS DYNAMICS

In consensus dynamics, noise can be introduced to each agent through communication errors, spurious measurements and other means. For simplicity, we assume that every agent is independently affected by white noise of the same intensity. The resulting dynamics are

$$\dot{x}(t) = -Lx(t) + \xi(t) \tag{1}$$

with $x \in \mathbb{R}^N$ and where $\xi(t) \in \mathbb{R}^N$ is a random signal with $E[\xi(t)] = 0$, $E[\xi(t)\xi^T(\tau)] = \frac{\alpha}{2} I_N \delta(t-\tau)$ and $E[x(0)\xi^T(\tau)] = 0$. $\delta(t)$ is the Dirac delta function and $\alpha > 0$ is the intensity of the noise.

Since (1) is only marginally stable in the noise-free case, we only consider the dynamics on the subspace of $\mathbb{R}^N$ orthogonal to the subspace spanned by 1_N. We let $Q \in \mathbb{R}^{(N-1)\times N}$ be a matrix whose rows form an orthonormal basis of this subspace. This is equivalent to requiring that

$$\begin{aligned} &Q1_N = 0, \\ &QQ^T = I_{N-1} \text{ and } Q^TQ = I_N - \frac{1}{N}1_N1_N^T = \Pi. \end{aligned} \tag{2}$$

Note that $LQ^TQ = L\left(I_N - \frac{1}{N}1_N1_N^T\right) = L$ as $L1_N = 0$. Next, we define $y := Qx$. Then $y = 0$ if and only if $x = \gamma 1_N, \gamma \in \mathbb{R}$. A measure of the distance from consensus is the *dispersion* of the system $||y(t)|| = \left(y^T(t)y(t)\right)^{\frac{1}{2}}$. Note that the projection of (1) onto the subspace spanned by 1_N will give the dynamics of the mean of x. These dynamics remain marginally stable (in the noise-free case).

Differentiating $y(t)$, we obtain

$$\dot{y}(t) = -\bar{L}y(t) + Q\xi(t) \tag{3}$$

where $\bar{L} = QLQ^T$ is the *reduced Laplacian* matrix.

Note that $\bar{L}$ is not unique, since we can compute it using any matrix Q that satisfies (2). However, if Q and Q' both satisfy (2), we can define $P := Q'Q^T$. Then $Q' = PQ$ and P is orthogonal. Therefore, if $y'(t) := Q'x(t) = Py(t)$, we have that $y'^T(t)y'(t) = y^T(t)P^TPy(t) = y^T(t)y(t)$ and thus the dispersion is invariant to the choice of Q.

Lemma 1: $\bar{L}$ has the same eigenvalues as L but the zero eigenvalue.

Proof: Define the matrix

$$V = \begin{bmatrix} Q \\ \frac{1}{\sqrt{N}}1_N^T \end{bmatrix}$$

The matrix V is orthogonal as

$$VV^T = \begin{bmatrix} Q \\ \frac{1}{\sqrt{N}}1_N^T \end{bmatrix} \begin{bmatrix} Q^T & \frac{1}{\sqrt{N}}1_N \end{bmatrix} = \begin{bmatrix} I_{N-1} & 0 \\ 0 & 1 \end{bmatrix} = I_N$$

and

$$\begin{aligned} V^TV &= \begin{bmatrix} Q^T & \frac{1}{\sqrt{N}}1_N \end{bmatrix} \begin{bmatrix} Q \\ \frac{1}{\sqrt{N}}1_N^T \end{bmatrix} \\ &= I_N - \frac{1}{N}1_N1_N^T + \frac{1}{N}1_N1_N^T = I_N. \end{aligned}$$

Consider the change of coordinates $v = Vx$, and the corresponding (noise-free) dynamics

$$\dot{v} = -VLV^Tv$$

where we used the fact that $V^{-1} = V^T$. Clearly the eigenvalues of L and VLV^T are the same. In particular we have that

$$VLV^T = \begin{bmatrix} QLQ^T & 0 \\ \frac{1}{\sqrt{N}}1_N^TLQ^T & 0 \end{bmatrix}, \tag{4}$$

where we used the fact that $L1_N = 0$. Since (4) is a block matrix, the eigenvalues of (4) are the solutions of

$$\lambda \det(\lambda I - QLQ^T) = 0.$$

We conclude that $\bar{L}$ has the same eigenvalues of L but the zero eigenvalue. ■

By Theorem 4 of [20], L has eigenvalue 0 with multiplicity 1 precisely when the communication graph is connected. Furthermore, Geršgorin's theorem guarantees that all other eigenvalues have real parts strictly positive. Thus for connected graphs, all eigenvalues of $\bar{L}$ have positive real part, and so $-\bar{L}$ is Hurwitz. In the rest of this paper we will assume that $\mathcal{G}$ is connected and thus (in the absence of noise), (3) converges exponentially to zero. The speed of convergence will be determined by the eigenvalue of $\bar{L}$ with smallest real part, or equivalently, by the eigenvalue of L with second-smallest real part. For undirected graphs this corresponds to the standard definition of the *algebraic connectivity* of the graph. It is worth noting that the definition of algebraic connectivity for directed graphs, as given in [21], does not correspond to the speed of convergence.

IV. ROBUSTNESS AND THE H_2 NORM

A. The H_2 norm as a measure of robustness

We now seek to describe the robustness of the consensus dynamics to white noise inputs. Since the dispersion of the system $||y(t)||$ defines the distance to consensus, we measure the robustness of the system by $H := \lim_{t\to\infty} E\left[||y(t)||\right]$. Since $y^T(t)y(t)$ is a scalar quantity, $y^T(t)y(t) = \mathrm{tr}\left(y^T(t)y(t)\right) = \mathrm{tr}\left(y(t)y^T(t)\right)$. Therefore, if we let $\Sigma(t) := E[y(t)y^T(t)]$, $H = \lim_{t\to\infty} \left[\mathrm{tr}\left(\Sigma(t)\right)\right]^{\frac{1}{2}} = \left[\mathrm{tr}\left(\lim_{t\to\infty}\Sigma(t)\right)\right]^{\frac{1}{2}} =: \left[\mathrm{tr}\left(\Sigma_{ss}\right)\right]^{\frac{1}{2}}$. Note that this definition corresponds to the steady-state mean-square deviation used in [17].

For the state-space system $\dot{y} = Ay + Bu, z = Cy$ with A Hurwitz, the H_2 norm is $\left[\mathrm{tr}(CXC^T)\right]^{\frac{1}{2}}$, where X is the solution of the Lyapunov equation $AX + XA^T + BB^T = 0$. It is well known that our definition of H is equal to the H_2 norm of system (3) with output equation $z(t) = Iy(t)$. However, we include a proof of this for completeness.

Lemma 2: For unit-intensity white noise, Σ_{ss} is the solution to the Lyapunov equation $\bar{L}\Sigma_{ss} + \Sigma_{ss}\bar{L}^T = I$. That is, H is equal to the H_2 norm of system (3) with output equation $z(t) = Iy(t)$.

Proof: Differentiating $\Sigma(t)$ with respect to time, we obtain

$$\begin{aligned}\dot{\Sigma}(t) &= E\left[\dot{y}(t)y^T(t) + y(t)\dot{y}^T(t)\right]\\ &= E\left[-\bar{L}y(t)y^T(t) + Q\xi(t)y^T(t) - y(t)y^T(t)\bar{L}^T\right.\\ &\qquad \left.+y(t)\xi^T(t)Q^T\right]\\ &= -\bar{L}\Sigma(t) - \Sigma(t)\bar{L}^T + QE\left[\xi(t)y^T(t)\right]\\ &\qquad + E\left[y(t)\xi^T(t)\right]Q^T.\end{aligned}$$

Notice that the solution of (3) is $y(t) = \mathrm{e}^{-\bar{L}t}y(0) + \int_0^t \mathrm{e}^{-\bar{L}(t-\tau)}Q\xi(\tau)\,\mathrm{d}\tau$, which leads to

$$\begin{aligned}E\left[y(t)\xi^T(t)\right] &= \mathrm{e}^{-\bar{L}t}QE\left[x(0)\xi^T(t)\right]\\ &\qquad + \int_0^t \mathrm{e}^{-\bar{L}(t-\tau)}QE\left[\xi(\tau)\xi^T(t)\right]\,\mathrm{d}\tau\\ &= \int_0^t \frac{1}{2}\mathrm{e}^{-\bar{L}(t-\tau)}Q\delta(t-\tau)\,\mathrm{d}\tau\\ &= \frac{1}{2}Q.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\dot{\Sigma}(t) &= -\bar{L}\Sigma(t) - \Sigma(t)\bar{L}^T + \frac{1}{2}QQ^T + \frac{1}{2}QQ^T\\ &= -\bar{L}\Sigma(t) - \Sigma(t)\bar{L}^T + I.\end{aligned}$$

And in steady state (i.e. as $t\to\infty$) $\dot{\Sigma}(t) = 0$, so we get

$$\bar{L}\Sigma_{ss} + \Sigma_{ss}\bar{L}^T = I \tag{5}$$

Thus, $\left[\mathrm{tr}\left(\Sigma_{ss}\right)\right]^{\frac{1}{2}}$ is the H_2 norm of system (3). ■

B. H_2 norm for graphs with normal reduced Laplacian matrices

We can use Lemma 2 to derive a relationship between the eigenvalues of L and the H_2 norm of system (3) when the reduced Laplacian matrix $\bar{L}$ is *normal*, that is when $\bar{L}\bar{L}^T = \bar{L}^T\bar{L}$. Normal matrices are particularly easy to work with since we know by the Spectral Theorem that they can be diagonalized using unitary matrices. We can use this fact to solve the Lyapunov equation (5) and obtain an expression for the H_2 norm of system (3).

First, we want to characterize those graphs with normal reduced Laplacians.

Lemma 3: $\bar{L}$ is normal if and only if ΠL is normal, that is, if and only if $\Pi LL^T\Pi = L^T\Pi L$.

Proof: By definition, ΠL is normal if $\left(\Pi L\right)\left(\Pi L\right)^T = \left(\Pi L\right)^T\left(\Pi L\right)$, that is, if $\Pi LL^T\Pi = L^T\Pi^2 L$ (as Π is symmetric). However, Π is a projection matrix, so $\Pi^2 = \Pi$. Thus ΠL being normal is equivalent to

$$\Pi LL^T\Pi = L^T\Pi L. \tag{6}$$

Now, since $\bar{L} = QLQ^T$, $\bar{L}$ being normal is equivalent to $QLQ^TQL^TQ^T = QL^TQ^TQLQ^T$, which reduces to

$$QLL^TQ^T = QL^T\Pi LQ^T \tag{7}$$

since $Q^TQ = \Pi$ and $L\Pi = L$.

Suppose $\bar{L}$ is normal. Then, pre-multiplying (7) by Q^T and post-multiplying by Q gives us $\Pi LL^T\Pi = \Pi L^T\Pi L\Pi$. However, $L\Pi = L$ and thus $\Pi L^T = L^T$ also. Therefore, $\Pi LL^T\Pi = L^T\Pi L$, and so ΠL is normal.

Suppose ΠL is normal. Then, pre-multiplying (6) by Q and post-multiplying by Q^T gives us $QQ^TQLL^TQ^TQQ^T = QL^T\Pi LQ^T$. But $QQ^T = I_{N-1}$, and so $QLL^TQ^T = QL^T\Pi LQ^T$. Thus, $\bar{L}$ is normal. ■

Although Lemma 3 gives us the most general condition on L for $\bar{L}$ to be normal, it is instructive to consider some special graphs that have normal reduced Laplacians. Incidentally, Lemma 3 also shows that the normality of $\bar{L}$ does not depend on a particular choice of matrix Q.

Lemma 4: Let $\mathcal{G}$ be a connected graph with normal Laplacian matrix. Then $\mathcal{G}$ is balanced.

Proof: Suppose L is the Laplacian matrix of a connected graph $\mathcal{G}$ and that $LL^T = L^TL$. Since $\mathcal{G}$ is connected, the 0 eigenvalue of L has multiplicity 1. Hence the only non-zero vectors v for which $Lv = 0$ are $v = \beta 1_N, \beta\in\mathbb{R}$. Since $L^TL1_N = 0$ we have that $LL^T1_N = 0$ and we conclude that

$$L^T1_N = \beta 1_N, \quad \beta\in\mathbb{R}. \tag{8}$$

Premultiplying both sides of (8) by 1_N^T we obtain

$$\beta 1_N^T 1_N = 0$$

Thus $1_N^T L = 0$, and so the graph must be balanced. ■

Lemma 5: If L is a normal Laplacian matrix of a connected graph then $\bar{L}$ is also normal (as a complex matrix).

Proof: Suppose L is the Laplacian matrix of a connected graph and that $LL^T = L^T L$. By Lemma 4, we know that $L^T 1_N = 0$, or $1_N^T L = 0$. Then $\Pi L = \left(I - \frac{1}{N} 1_N 1_N^T\right) L = L$. Thus ΠL is normal and so, by Lemma 3, $\bar{L}$ is normal as well. ■

Any undirected graph will have a symmetric Laplacian matrix, which is trivially normal. Thus we know that $\bar{L}$ will be normal for all undirected graphs. In addition to this, there exist directed graphs with normal Laplacians, such as all circulant graphs (for example, see Section V-B), which by Lemma 5 have normal reduced Laplacians. Furthermore, some directed graphs do not have normal Laplacians but still satisfy the condition of Lemma 3 (see Section V-D).

Now we are able to derive a formula for the H_2 norm of a system with normal $\bar{L}$ in terms of the non-zero eigenvalues of the Laplacian matrix.

Proposition 1: Suppose L is a Laplacian matrix of a connected graph with eigenvalues $\lambda_1 = 0 < \mathrm{Re}\{\lambda_2\} \leq \mathrm{Re}\{\lambda_3\} \leq \ldots \leq \mathrm{Re}\{\lambda_N\}$, and that ΠL is normal. Then the H_2 norm of system (3) is

$$H = \left(\sum_{i=2}^{N} \frac{1}{2\mathrm{Re}\{\lambda_i\}} \right)^{\frac{1}{2}} \tag{9}$$

Proof: By Lemma 3, we know that $\bar{L}$ is normal. Furthermore, by Lemma 1 the eigenvalues of $\bar{L}$ are $\lambda_2, \lambda_3, \ldots, \lambda_N$. Therefore, by the Spectral Theorem, we can find a unitary matrix V (i.e. $VV^* = V^*V = I$, where * is the Hermitian operator) such that $\bar{L} = V\Lambda V^*$, where $\Lambda = \mathrm{diag}\{\lambda_2, \ldots, \lambda_N\}$. Note that as $\bar{L}$ is real, $\bar{L}^T = \bar{L}^*$. Therefore, to find the H_2 norm, we must solve the Lyapunov equation

$$V\Lambda V^* \Sigma + \Sigma V \Lambda^* V^* = I.$$

Rearranging, we get

$$\Lambda V^* \Sigma V + V^* \Sigma V \Lambda^* = I.$$

Now, let $\Gamma = V^* \Sigma V$, and note that $\mathrm{tr}\,(\Sigma) = \mathrm{tr}\,(V\Gamma V^*) = \mathrm{tr}\,(V^* V \Gamma) = \mathrm{tr}\,(\Gamma)$. Thus, $H = [\mathrm{tr}(\Gamma)]^{\frac{1}{2}}$, where

$$\Lambda\Gamma + \Gamma\Lambda^* = I.$$

Since Λ and I are both diagonal matrices, Γ must be a diagonal matrix as well. Thus Γ and Λ commute and we can write

$$\Gamma\Lambda + \Gamma\Lambda^* = \Gamma\left(2\mathrm{Re}\{\Lambda\}\right) = I,$$

which implies

$$\Gamma = \left(2\mathrm{Re}\{\Lambda\}\right)^{-1}.$$

Thus Γ is the diagonal matrix with entries $\frac{1}{2\mathrm{Re}\{\lambda_2\}}, \ldots, \frac{1}{2\mathrm{Re}\{\lambda_N\}}$ and we conclude that

$$H = \left(\sum_{i=2}^{N} \frac{1}{2\mathrm{Re}\{\lambda_i\}} \right)^{\frac{1}{2}}.$$

■

Remark 1: Equation (9) does not hold for every graph (see Section V-C). However, based on numerical results, we conjecture that (3) is a lower bound for the H_2 norm associated to any graph.

Remark 2: One well-known property of undirected graphs is the *effective resistance* K_f, or *Kirchhoff index* [22]. The effective resistance of a graph can be related to the power dissipated by the graph when it is considered as a resistor network, and also to the expected commute time between any two nodes for a random walk with transition probabilities governed by the edge weights [23]. This is related to the eigenvalues of the graph Laplacian by the formula $K_f = N \sum_{j=2}^{N} \frac{1}{\lambda_j}$, leading to the relationship

$$H = \left(\frac{K_f}{2N} \right)^{\frac{1}{2}}.$$

V. PROPERTIES OF FAMILIES OF GRAPHS

A graph with good convergence speed will have a large value for the real part of the second-smallest eigenvalue, while a graph with good robustness will have a small value of the H_2 norm. In this section we compare the properties of families of directed and undirected graphs to investigate the trade-off between speed of convergence and robustness to noise.

A. Complete graphs

The complete graph on N nodes contains an edge connecting every pair of nodes. In its usual form, every edge has unit weight and so L would equal $NI - 1_N 1_N^T = N\Pi$. In its normalized form, every edge has a weight of $\frac{1}{N-1}$ and thus the complete graph has Laplacian matrix $L = \frac{N}{N-1}\Pi$. The complete graph is undirected so L is symmetric, and hence normal, with eigenvalues $0, \frac{N}{N-1}, \frac{N}{N-1}, \ldots, \frac{N}{N-1}$. Thus the complete graph has convergence speed $\frac{N}{N-1}$ and H_2 norm $\frac{N-1}{\sqrt{2N}}$.

B. Cycle graphs

The (directed) cycle graph on N nodes consists of a closed "chain" of nodes with each node connected to the next node in the chain. When every edge has unit weight, the cycle graph will have Laplacian matrix

$$L = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Now, L is a circulant matrix, and all circulant matrices are normal. Furthermore, the characteristic equation of L is $(\lambda-1)^N - (-1)^N = 0$ and so L has eigenvalues $1 + \mathrm{e}^{i\pi\left(1-\frac{2k}{N}\right)},\ k = 0, 1, \ldots, N-1$. Thus the convergence speed is $\mathrm{Re}\left\{1 + \mathrm{e}^{i\pi\left(1-\frac{2}{N}\right)}\right\} = 1 + \cos\left(\pi - \frac{2\pi}{N}\right) = 2\sin^2\left(\frac{\pi}{N}\right)$.

Since L is a normal matrix, we can apply Lemma 5 and Proposition 1 to obtain

$$
\begin{aligned}
H &= \left[\sum_{k=1}^{N-1} \frac{1}{2\mathrm{Re}\left\{1 + \mathrm{e}^{i\pi\left(1-\frac{2k}{N}\right)}\right\}}\right]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left[\sum_{k=1}^{N-1} \frac{1}{1 + \cos\left(\pi\left(1-\frac{2k}{N}\right)\right)}\right]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left[\sum_{k=1}^{N-1} \frac{1}{2\sin^2\left(\frac{k\pi}{N}\right)}\right]^{\frac{1}{2}}
\end{aligned}
$$

Since

$$
\sum_{k=1}^{N-1} \csc^2\left(\frac{k\pi}{N}\right) = \frac{N^2-1}{3}
$$

we conclude that

$$
H = \sqrt{\frac{N^2-1}{12}}.
$$

We can also consider the undirected form of the cycle graph. As outlined in Section II, the adjacency matrix of the undirected cycle is the symmetric part of the adjacency matrix of the directed cycle. Then, the undirected cycle will have Laplacian matrix

$$
L = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.
$$

Now L is symmetric, so Proposition 1 applies. Furthermore, the eigenvalues of L are the real parts of the eigenvalues of the directed cycle. Thus the undirected cycle has convergence speed $2\sin^2\left(\frac{\pi}{N}\right)$ and H_2 norm $\sqrt{\frac{N^2-1}{12}}$.

C. Path graphs

The path graph on N nodes consists of an open "chain" of nodes with each node connected to the next. It is equivalent to the cycle graph with one edge removed. When every edge has unit weight, the path graph will have Laplacian matrix

$$
L = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.
$$

In this case it is clear that L is not balanced, and so by Lemma 4, L is not normal.

Since L is upper triangular, its eigenvalues are just its diagonal entries, namely $0, 1, 1, \ldots, 1$. Therefore, the convergence speed of the path graph is 1, and the formula from Proposition 1 produces the following result: $\left(\sum_{i=2}^{N} \frac{1}{2\mathrm{Re}\{\lambda_i\}}\right)^{\frac{1}{2}} = \sqrt{\frac{N-1}{2}}$. However, numerical calculations show that (at least for $N \leq 50$), the H_2 norm of the path graph is $\sqrt{\frac{N^2-1}{6}}$.

We can consider the undirected path graph by finding the symmetric part of the adjacency matrix of the directed path. This produces the following Laplacian matrix

$$
L = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.
$$

This matrix has eigenvalues $2\sin^2\frac{k\pi}{2N},\ k = 0, 1, \ldots, N-1$ and since it is symmetric we can apply Proposition 1. Then, using the fact that $\sum_{k=1}^{N-1}\csc^2\left(\frac{k\pi}{2N}\right) = \frac{2}{3}\left(N^2-1\right)$, we have that the undirected path has convergence speed $2\sin^2\frac{\pi}{2N}$ and H_2 norm $\sqrt{\frac{N^2-1}{6}}$.

D. Star graphs

The (imploding) star graph on N nodes consists of a "central" node with every other node connected to this central one. When every edge has unit weight, the star graph will have Laplacian matrix

$$
L = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.
$$

Now, we can write $L = I_N - 1_N e_N^T$, where $e_N = (0, \ldots, 0, 1)^T \in \mathbb{R}^N$. Then $\Pi L = \Pi\left(I_N - 1_N e_N^T\right) = \Pi$, since $\Pi 1_N = 0$. As Π is normal, Proposition 1 can be applied.

Since L is upper triangular, its eigenvalues are its diagonal entries, $0, 1, 1, \ldots, 1$. Therefore, the convergence speed of the star graph is 1 and by equation (9), the H_2 norm of the star graph is $\sqrt{\frac{N-1}{2}}$.

Once again we can consider the undirected form of this graph by finding the symmetric part of the adjacency matrix and then forming a new Laplacian. For the undirected star graph, we find that the Laplacian matrix is

$$
L = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & 0 & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & \frac{N-1}{2} \end{bmatrix}.
$$

Then L has eigenvalues $0, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{N}{2}$, and we can apply Proposition 1 to find the H_2 norm. Thus the undirected star has convergence speed $\frac{1}{2}$ and H_2 norm $\frac{N-1}{\sqrt{N}}$.

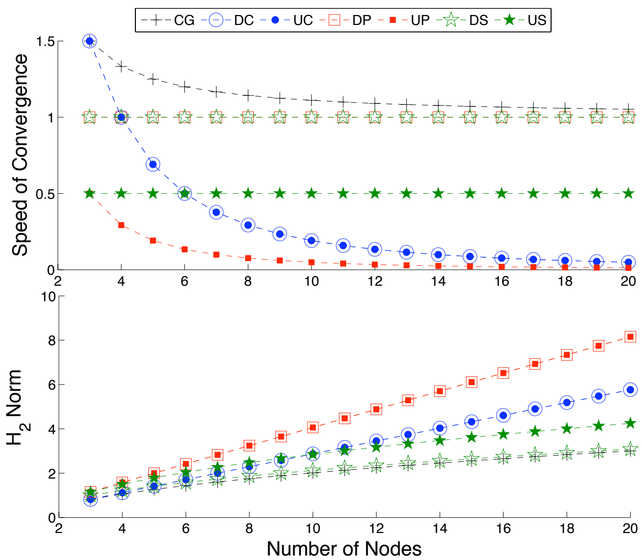

Fig. 1. Convergence speed and H_2 norm for certain directed and undirected graphs. The graphs shown are: CG - complete graph, DC - directed cycle, UC - undirected cycle, DP - directed path, UP - undirected path, DS - directed star and US - undirected star.

E. Comparison of complete, cycle, path and star graphs

Figure 1 shows the convergence speed and H_2 norm of the complete, cycle, path and star graphs. The complete graph has the best performance in both categories among these families. However, as the number of nodes increases, the directed path and directed star approach the complete graph in speed, and the directed star approaches it in robustness. This shows that the performance of the complete graph (which requires $(N-1)^2$ directed edges for N nodes) can be almost matched by certain graphs with many fewer edges (e.g. the star graph with $N-1$ edges).

These results also provide additional motivation for considering directed graphs for consensus problems. As well as the practical problems with maintaining undirected communication links in physical networks, undirected graphs can sometimes be out-performed in both speed and robustness by their directed counterparts. The directed and undirected path graphs have the same H_2 robustness, but the directed graph produces much higher convergence speeds. In addition, the directed star graph out-performs the undirected star in terms of both speed and robustness.

VI. CONCLUSIONS AND FUTURE WORK

We studied robustness of consensus in networks of coupled single integrators with directed communication topologies, perturbed by white noise. The robustness is measured by the H_2 norm of the system and is related to the underlying (directed) communication graph. A central result is a closed-form expression for the H_2 norm of a large class of graphs in terms of the eigenvalues of the graph Laplacian. A comparison of a number of simple families of graphs has shown that directed graphs can sometimes out-perform undirected graphs in both speed and robustness. In addition we have shown that for graphs with many nodes, performance similar to that of the complete graph can be achieved by graphs with many fewer edges. Application of this work to the study of animal aggregations is a subject of ongoing work. A fundamental challenge is the extension of the proposed technique to analyze networks with time-varying and periodic communication topologies.

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