A New Notion of Effective Resistance for Directed Graphs—Part I: Definition and Properties

George Forrest Young, *Member, IEEE*, Luca Scardovi, *Member, IEEE*, and Naomi Ehrich Leonard, *Fellow, IEEE*

***Abstract*—The graphical notion of effective resistance has found wide-ranging applications in many areas of pure mathematics, applied mathematics and control theory. By the nature of its construction, effective resistance can only be computed in undirected graphs and yet in several areas of its application, directed graphs arise as naturally (or more naturally) than undirected ones. In Part I of this work, we propose a generalization of effective resistance to directed graphs that preserves its control-theoretic properties in relation to consensus-type dynamics. We proceed to analyze the dependence of our algebraic definition on the structural properties of the graph and the relationship between our construction and a graphical distance. The results make possible the calculation of effective resistance between any two nodes in any directed graph and provide a solid foundation for the application of effective resistance to problems involving directed graphs.**

***Index Terms*—Directed graphs, effective resistance, graph theory, networked control systems, networks.**

I. Introduction

THE concept of *effective resistance* has been used in relation to graphs for some time [1]. This concept stems from considering a graph, consisting of a set of nodes connected by weighted edges, to represent a network of resistors (one resistor corresponding to each edge) with resistances equal to the inverse of the corresponding edge weights. Then, the effective resistance between nodes k and j, denoted $r_{j,k}$, can be found by the resistance offered by the network when a voltage source is connected between these two nodes. One of the useful properties of the effective resistance is that it defines a distance function on a graph that takes into account all paths between two nodes, not just the shortest path [1]. This allows the effective resistance to be used in place of the shortest-path distance to analyze problems involving random motion, percolation and flows over networks.

Effective resistance has proven to have a number of interpretations and applications over a wide variety of fields. One of the earliest interpretations was in the study of random walks and Markov chains on networks [2]–[5], where the effective resistance between a pair of nodes was related to expected commute, cover and hitting times and the probabilities of a random walk reaching a node or traversing an edge. More direct applications have arisen in the study of power dissipation and time delays in electrical networks [6]. In addition, the effective resistance has been shown to have combinatorial interpretations, relating to spanning trees and forests [7] as well as the number of nodes and edges in the graph [1]. Following the work of Klein and Randić [1], there has been a substantial literature investigating the use of effective resistance and the Kirchhoff index in the study of molecular graphs [8]–[12].

More recently, effective resistance has arisen in control theory, in the study of control, estimation and synchronization over networks. Barooah and Hespanha described in [13] how effective resistance can be used to measure the performance of collective formation control, rendezvous and estimation problems. They developed the theory of estimation from relative measurements further in [14], [15]. A number of authors have demonstrated the use of effective resistance in investigating the problem of selecting leaders to maximize agreement or coherence in a network [16]–[19]. Dörfler and Bullo used effective resistance in their study of synchronization of oscillators in power networks [20], and subsequently developed a theoretical analysis involving resistance for a graph reduction technique [21]. We have also used the concept of effective resistance to measure the robustness to noise of linear consensus over networks [22], [23] as well as the performance of nodes in networks of stochastic decision-makers [24], [25].

By the nature of its definition, effective resistance is restricted to undirected graphs, and in many applications, including the study of molecular graphs and electrical networks, it is natural to focus solely on undirected graphs. However, in many other applications, including random walks and networked control systems, directed graphs arise just as naturally as undirected ones (which can be thought of as special cases of directed graphs in which every edge also exists in the opposite direction and with equal weight). For example, if the nodes in a graph represent agents and the edges interactions, directed edges result from interactions in which one agent reacts to another but the second either has no reaction to the first or reacts with a different strength or rule. Only in highly constrained circumstances would such a network result in an undirected graph.

Accordingly, it would be particularly useful if the concept of effective resistance could be extended to apply to any directed or undirected graph, so that analysis that is currently

Manuscript received October 18, 2013; revised August 20, 2014, June 15, 2015, and August 18, 2015; accepted August 23, 2015. Date of publication September 25, 2015; date of current version June 24, 2016. This work was supported in part by AFOSR Grant FA9550-07-1-0-0528, ONR Grants N00014-09-1-1074, N00014-14-1-0635, ARO Grants W911NG-11-1-0385, W911NF-14-1-0431, and by the Natural Sciences and Engineering Research Council (NSERC) of Canada. Recommended by Associate Editor A. Nedich.

G. F. Young and N. E. Leonard are with the Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08544 USA (e-mail: gfyoung@princeton.edu; naomi@princeton.edu).

L. Scardovi is with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: scardovi@scg.utoronto.ca).

Digital Object Identifier 10.1109/TAC.2015.2481978

0018-9286 © 2015 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission.
See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.

only applicable to undirected graphs could be applied in the more general case. Indeed in [13]–[15], the authors investigated directed measurement graphs, but assumed undirected communication in order to analyze their systems using effective resistance. Similarly, in [22]–[24], we began our investigations using directed graphs and then specialized to undirected graphs when we used effective resistance.

In the present work we propose a generalized definition of effective resistance for any graph, constructed in such a way that it preserves the connection between effective resistance and networked control and decision systems [22]–[24]. This new definition produces a well-defined pairwise property of nodes that depends only on the connections between the nodes. Although it is no longer always a metric on the nodes of a graph, our notion of effective resistance does allow for the construction of a resistance-based metric for any graph. This is in contrast with a (perhaps) more intuitive generalization based on the use of pseudoinverses, which does not yield a resistance-based metric in the general case. Further, this suggests that our construction should prove to be useful for applications other than those we have presented here. In the companion paper we explore some of the implications of our new approach by computing effective resistances in several canonical directed graphs.

This paper is organized as follows. In Section II we provide an overview of the notation and definitions used throughout the paper. In Section III we define our extended notion of effective resistance. In Section IV we analyze some basic properties of our definition, including its well-definedness, its dependence on connections between nodes and its relationship to a metric. We conclude and offer some final thoughts in Section V.

II. Background and Notation

In this section we provide a summary of the notation used throughout this two-part paper and then explain some of the basic definitions and terminology of graph theory, as it applies to this work. There are competing definitions for many of the basic concepts of graph theory, mainly due to varying scope amongst authors. For our purposes, we restrict our attention to directed graphs as they may arise in control theory, and so our definitions mostly follow [26], with some taken from [27]. The notable exceptions are our definition of connectivity, which falls between the standard notions of weak and strong connectivity but is more applicable to control over graphs [22], [28], our definition of *connections* in directed graphs, which have interesting parallels to paths in undirected graphs, and our definitions of *reachable* and *connection* subgraphs.

Throughout this paper we will represent matrices using capital letters. A matrix M can also be written $[m_{i,j}]$, where $m_{i,j}$ denotes the scalar entry in the (i,j)th position. The identity matrix in $\mathbb{R}^{n\times n}$ will be denoted by I_n, and a zero matrix (with dimensions inferred from context) will be denoted by $\emptyset$. Entries of a matrix whose values are not relevant may be denoted by $*$. We will represent vectors using bold, lower case letters and their scalar entries with the same (non-bold) lower case letter with a single subscript. In particular, we use $\mathbf{e}_n^{(k)}$ to denote the kth standard basis vector of $\mathbb{R}^n$. That is, $\mathbf{e}_n^{(k)}$ contains a zero in every position except the kth position, which is a 1. In addition, we use $\mathbf{1}_n$ to denote the vector in $\mathbb{R}^n$ containing a 1 in every entry and $\mathbf{0}$ to denote the zero vector (with size inferred from context). A diagonal matrix (with the elements of a vector $\mathbf{v}$ along the main diagonal and zeros elsewhere) will be denoted by $\mathrm{diag}(\mathbf{v})$.

A *graph* (also *directed graph* or *digraph*) $\mathcal{G}$ consists of the triple $(\mathcal{V},\mathcal{E},A)$, where $\mathcal{V}=\{1,2,\ldots,N\}$ is the set of nodes, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of edges and $A\in\mathbb{R}^{N\times N}$ is a weighted adjacency matrix with non-negative entries $a_{i,j}$. Each $a_{i,j}$ will be positive if and only if $(i,j)\in\mathcal{E}$, otherwise $a_{i,j}=0$. Note that by viewing $\mathcal{E}$ as a subset of $\mathcal{V}\times\mathcal{V}$, $\mathcal{G}$ can contain at most one edge between any ordered pair of nodes. In addition, we restrict our attention to graphs which do not contain any self-cycles (edges connecting a node to itself).

Our definition of a graph corresponds to a "sensing" convention for edges—that is, edge (i,j) indicates that node i can "sense" or receive information from node j. An alternative, "information flow" convention exists, in which edge (i,j) indicates that information from node i can be transmitted to node j. In this case, edge (i,j) would correspond to entry $a_{j,i}$ in A. Everything presented in this paper can be reformulated for the "information flow" convention by transposing all matrices related to the graph.

The graph $\mathcal{G}$ is said to be *undirected* if $(i,j)\in\mathcal{E}$ implies $(j,i)\in\mathcal{E}$ and $a_{i,j}=a_{j,i}$. Thus, a graph will be undirected if and only if its adjacency matrix is symmetric. We use the term *undirected edge* to refer to a pair of edges between two nodes (one in each direction), with equal weights.

The *out-degree* (respectively *in-degree*) of node k is defined as $d_k^{\mathrm{out}}=\sum_{j=1}^{N}a_{k,j}$ (respectively $d_k^{\mathrm{in}}=\sum_{j=1}^{N}a_{j,k}$).

$\mathcal{G}$ has an associated *Laplacian* matrix L, defined by $L=D-A$, where D is the diagonal matrix of node out-degrees, that is $D=\mathrm{diag}(d_1^{\mathrm{out}},d_2^{\mathrm{out}},\ldots,d_N^{\mathrm{out}})$. The row sums of the Laplacian matrix are zero, that is $L\mathbf{1}_N=\mathbf{0}$. Thus 0 is always an eigenvalue of L with corresponding eigenvector $\mathbf{1}_N$. It can be shown that all the eigenvalues of L are either 0 or have positive real part [29]. A graph will be undirected if and only if its Laplacian matrix is symmetric, and then all the eigenvalues of L will be real and non-negative.

The *set of neighbors* of node k, denoted $\mathcal{N}_k$, is the set of nodes j for which the edge $(k,j)\in\mathcal{E}$.

A *path* in $\mathcal{G}$ is a (finite) sequence of nodes such that each node is a neighbor of the previous one. A path is called *simple* if no internal nodes (i.e., other than the initial or final nodes) are repeated. If a path exists with initial node k and final node j, node j is said to be *reachable* from node k. The *length* of a path is the number of edges traversed. Thus a single node is considered to be a path of length 0. A directed (respectively, undirected) *path graph* on N nodes is a graph containing exactly $N-1$ directed (respectively, undirected) edges and which admits a path of length $N-1$ containing every node in the graph.

A *cycle* in $\mathcal{G}$ is a non-trivial closed path. That is, a cycle is a path of length greater than zero in which the initial and final nodes are the same. A *simple cycle* is a non-trivial closed simple path. Since we are only considering graphs which do not contain self-cycles, the minimum length of a cycle is two.

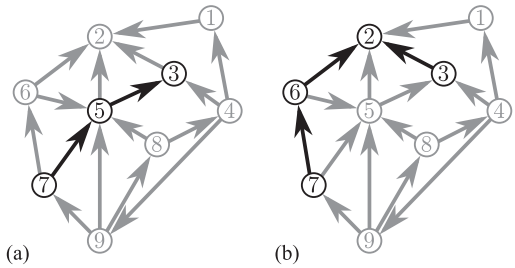

Fig. 1. A directed graph on 9 nodes with: (a) a direct connection (i.e., a path) between nodes 3 and 7 highlighted, and (b) an indirect connection between nodes 3 and 7 highlighted.

A directed (respectively, undirected) *cycle graph* on N nodes is a graph containing exactly N directed (respectively, undirected) edges and which admits a simple cycle of length N containing every node in the graph. Note however, that while a directed cycle graph on 2 nodes is possible, the minimum number of nodes in an undirected cycle graph is 3.

We define a *connection* in $\mathcal{G}$ between nodes k and j to consist of two paths, one starting at k and the other at j and which both terminate at the same node. A *direct connection* between nodes k and j is a connection in which one path is trivial (i.e., either only node k or only node j)—thus a direct connection is equivalent to a path. Conversely, an *indirect connection* is one in which the terminal node of the two paths is neither node k nor node j. Examples of direct and indirect connections are shown in Fig. 1. A *simple connection* is a connection that consists of two simple paths. Note that in both a connection and a simple connection, multiple nodes may be common between the two paths.

The graph $\mathcal{G}$ is *connected* if for each pair of nodes in the graph, there exists a connection between them. Equivalently, $\mathcal{G}$ is connected if and only if it contains a globally reachable node k; i.e., there exists a node k such that there is a path in $\mathcal{G}$ from i to k for every node i. This definition of connectivity lies in between the classical notions of strong connectivity and weak connectivity.

The eigenvalue 0 of the Laplacian matrix is simple if and only if $\mathcal{G}$ is connected [29]. An undirected graph is connected if and only if there is a path between any pair of nodes.

The *reachable subgraph*, denoted $\mathcal{R}_{\mathcal{G}}(k,j)$, of nodes k and j in the graph $\mathcal{G}$ is the graph formed by every node in $\mathcal{G}$ that is reachable from node k or node j and every edge in $\mathcal{G}$ between these nodes. As we demonstrate in the following lemma, if $\mathcal{G}$ is connected the reachable subgraph of nodes k and j is precisely the subgraph formed by every connection between them.

Lemma 1: If $\mathcal{G}$ is connected, then for any pair of nodes k and j,

1) $\mathcal{R}_{\mathcal{G}}(k,j)$ is connected,
2) Every node in $\mathcal{R}_{\mathcal{G}}(k,j)$ is part of a connection between nodes k and j,
3) Every edge in $\mathcal{R}_{\mathcal{G}}(k,j)$ is part of a connection between nodes k and j, and
4) Every connection between nodes k and j is contained in $\mathcal{R}_{\mathcal{G}}(k,j)$.

Proof:

1) Since $\mathcal{G}$ is connected, there is a node, ℓ, in $\mathcal{G}$ which is reachable from every other node. Since ℓ is reachable from nodes k and j, it is also in $\mathcal{R}_{\mathcal{G}}(k,j)$. Now, suppose that m is a node in $\mathcal{R}_{\mathcal{G}}(k,j)$. Then there is a path in $\mathcal{G}$ from m to ℓ. Since m is reachable from either k or j (or both), every node along this path is as well. Thus, this path is contained in $\mathcal{R}_{\mathcal{G}}(k,j)$ and so ℓ is reachable (in $\mathcal{R}_{\mathcal{G}}(k,j)$) from every node in $\mathcal{R}_{\mathcal{G}}(k,j)$.
2) Let m be a node in $\mathcal{R}_{\mathcal{G}}(k,j)$. Then m must be reachable from either k or j (or both). Without loss of generality, suppose that m is reachable from k. Then [as we saw in part (i)] there must be a path in $\mathcal{R}_{\mathcal{G}}(k,j)$ from m to the globally reachable node ℓ as well as a path from j to ℓ. Thus m is part of a connection between k and j.
3) Let (m,n) be an edge in $\mathcal{R}_{\mathcal{G}}(k,j)$. Without loss of generality, suppose that m is reachable from k. Then n is also reachable from k. Then [as we saw in part (i)] there must be a path in $\mathcal{R}_{\mathcal{G}}(k,j)$ from n to the globally reachable node ℓ as well as a path from j to ℓ. Thus (m,n) is part of a connection between k and j.
4) Every node along a path is reachable from the node where the path started. Thus, every node in a connection between k and j is reachable from either k or j and hence in $\mathcal{R}_{\mathcal{G}}(k,j)$. Since $\mathcal{R}_{\mathcal{G}}(k,j)$ contains every edge in $\mathcal{G}$ between its nodes, every edge in the connection must also be in $\mathcal{R}_{\mathcal{G}}(k,j)$.

■

A *connection subgraph* between nodes k and j in the graph $\mathcal{G}$ is a maximal connected subgraph of $\mathcal{G}$ in which every node and edge form part of a connection between nodes k and j in $\mathcal{G}$. That is, a connection subgraph is formed from the union of connections between nodes k and j, and the addition of any other connections would make the subgraph disconnected. If only one connection subgraph exists in $\mathcal{G}$ between nodes k and j, it is referred to as *the* connection subgraph and is denoted by $\mathcal{C}_{\mathcal{G}}(k,j)$.

A directed (respectively, undirected) *tree* is a connected graph on N nodes that contains exactly $N-1$ directed (respectively, undirected) edges. A *leaf* in a directed tree is any node with zero in-degree, and a leaf in an undirected tree is any node with only one neighbor. The *root* of a directed tree is a node with zero out-degree; note that every directed tree will contain precisely one such node. A *branch* of a directed tree is a path from a leaf to the root.

Let $\Pi := I_N - (1/N)\mathbf{1}_N\mathbf{1}_N^T$ denote the orthogonal projection matrix onto the subspace of $\mathbb{R}^N$ perpendicular to $\mathbf{1}_N$. We will make a slight abuse of notation and use $\mathbf{1}_N^{\perp}$ to denote this subspace, instead of $\text{span}\{\mathbf{1}_N\}^{\perp}$. The matrix Π is symmetric and since $L\mathbf{1}_N = \mathbf{0}$, $L\Pi = L$, and $\Pi L^T = L^T$ for any graph.

Let $Q \in \mathbb{R}^{(N-1)\times N}$ be a matrix whose rows form an orthonormal basis for $\mathbf{1}_N^{\perp}$. This is equivalent to requiring that $[(1/\sqrt{N})\mathbf{1}_N \quad Q^T]$ is an orthogonal matrix, or more explicitly

$$Q\mathbf{1}_N = \mathbf{0},\ QQ^T = I_{N-1},\ \text{and}\ Q^TQ = \Pi. \tag{1}$$

Using these properties, it follows that $Q\Pi = Q$ and $\Pi Q^T = Q^T$.

III. An Extended Definition of Effective Resistance

We now proceed to examine the derivation of effective resistance for connected undirected graphs and compare the matrices involved to those that arise in control-theoretic applications. Using this comparison, we propose a generalization of effective resistance to connected directed graphs that preserves key control-theoretic properties related to consensus-type dynamics.

A complete derivation of the standard notion of effective resistance is given in [1], in which the effective resistance between two nodes in a connected undirected graph can be calculated by appropriately applying Kirchhoff's voltage and current laws. This calculation relies on what the authors call the "generalized inverse" of the Laplacian matrix, a matrix X which satisfies

$$XL = LX = \Pi \text{ and } X\Pi = \Pi X = X. \tag{2}$$

Then, if we let $X = [x_{i,j}]$, the effective resistance is given by

$$r_{k,j} = \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T X \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right) = x_{k,k} + x_{j,j} - 2x_{k,j}. \tag{3}$$

Although (2) are not the defining equations for the Moore-Penrose generalized inverse, it is easy to show that for a symmetric Laplacian matrix L, any solution to (2) will indeed be the Moore-Penrose generalized inverse of L (as well as the group inverse of L). In fact, it is standard practice to define the effective resistance in terms of the Moore-Penrose generalized inverse [30].

In [1], the authors describe X in the following way (with notation changed to match the present paper):

Definition 1 [Klein and Randić, [1]]: X is equal on $\mathbf{1}_N^\perp$ to the inverse of L and is otherwise $\emptyset$.

By this, Klein and Randić mean that $XL\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathbf{1}_N^\perp$ and that $X\mathbf{u} = \mathbf{0}$ for any $\mathbf{u} \in \text{span}\{\mathbf{1}_N\}$.

It is therefore instructive to characterize the action of L restricted to the subspace $\mathbf{1}_N^\perp$. Suppose we choose an orthonormal basis for $\mathbf{1}_N^\perp$ and let $Q \in \mathbb{R}^{(N-1)\times N}$ be the matrix formed with these basis vectors as rows. Then for any $\mathbf{v} \in \mathbb{R}^N$, $\overline{\mathbf{v}} := Q\mathbf{v}$ is a coordinate vector (with respect to the chosen basis) of the orthogonal projection of $\mathbf{v}$ onto $\mathbf{1}_N^\perp$ and for any $M \in \mathbb{R}^{N\times N}$, $\overline{M} := QMQ^T$ is the $(N-1)\times(N-1)$ matrix whose action on $\mathbf{1}_N^\perp$ is identical to M, in the sense that $\overline{M}\overline{\mathbf{v}} = \overline{M\mathbf{v}}$ for any $\mathbf{v} \in \mathbf{1}_N^\perp$.

Thus on $\mathbf{1}_N^\perp$, the Laplacian matrix is equivalent to

$$\overline{L} = QLQ^T, \tag{4}$$

which we refer to as the *reduced Laplacian*. We can see that $\overline{L}$ is a symmetric matrix if and only if the graph is undirected. The reduced Laplacian has the same eigenvalues as L except for a single 0 eigenvalue [22]. Hence, for a connected graph, $\overline{L}$ is invertible. For undirected graphs, this allows us to give an explicit construction for X as

$$X = Q^T\overline{L}^{-1}Q, \tag{5}$$

which satisfies Definition 1 since $X\mathbf{1}_N = \mathbf{0}$ and $\overline{X\mathbf{v}} = \overline{L}^{-1}\overline{\mathbf{v}}$ for any $\mathbf{v} \in \mathbb{R}^N$. Furthermore, we can use (1) and the fact that $L = L\Pi = \Pi L$ for undirected graphs to show that (5) satisfies (2) when the graph is undirected.

It should be noted that $\overline{L}$ is not unique, since it depends on the choice of Q. However, if Q and Q' both satisfy (1), we can define $P := Q'Q^T$. Then $Q' = PQ$ and P is orthogonal. Hence $X' := Q'^T(Q'LQ'^T)^{-1}Q' = Q^TP^T(PQLQ^TP^T)^{-1}PQ = X$ and thus the computation of X in (5) is independent of the choice of Q.

These multiple ways [the Moore-Penrose generalized inverse, Definition 1, (2) and (5)] to describe the matrix X no longer agree when the graph is directed. While (5) still satisfies Definition 1, it does not satisfy (2) (specifically, LX no longer equals Π). Furthermore, the Moore-Penrose generalized inverse satisfies neither (2) nor (5) for non-symmetric Laplacian matrices. Thus, instead of seeking to extend the notion of effective resistance to directed graphs using one of the above descriptions (which all arose through an analysis of electrical networks that were, by definition, undirected), we draw inspiration from a different context not as fundamentally tied to electrical networks.

In previous work on distributed consensus-formation and evidence-accumulation for decision-making ([22]–[25]), effective resistances arose due to a correspondence between covariance matrices and the matrix X as described above (in the case of undirected graphs). These applications involved stochastic systems evolving on graphs with dynamics driven by the Laplacian matrix, and covariance matrices were sought to describe the distribution of node values. In fact, the dynamics in these applications represented the simplest possible stable and marginally stable systems driven by a graph Laplacian matrix, with the addition of Wiener processes. Therefore, the covariance matrices that arose reflected fundamental properties of the graphs, rather than being tied to the particular applications. For general (i.e., directed or undirected) graphs, these covariance matrices were computed using integrals of the form

$$\Sigma_1 = \int_0^\infty Q^T e^{-\overline{L}t} e^{-\overline{L}^T t} Q \, dt, \tag{6}$$

$$\Sigma = \int_0^\infty e^{-\overline{L}t} e^{-\overline{L}^T t} \, dt. \tag{7}$$

Now, we can observe that $\Sigma_1 = Q^T\Sigma Q$, and that Σ can also be expressed as the solution to the Lyapunov equation [31]

$$\overline{L}\Sigma + \Sigma\overline{L}^T = I_{N-1}. \tag{8}$$

It should be noted that (8) has a unique positive definite solution when all the eigenvalues of $\overline{L}$ have positive real part, (i.e., when the graph is connected) [31]. It is then clear that for undirected graphs (where $\overline{L}$ is symmetric)

$$\Sigma = \frac{1}{2}\overline{L}^{-1} \tag{9}$$

and so [using (5)]

$$\Sigma_1 = \frac{1}{2}X.$$

It is this relationship that links these covariance matrices to the generalized inverse X, and hence to effective resistances. Since these covariance matrices arise naturally from directed graphs as well as undirected graphs, we use their solutions to define effective resistances on directed graphs. Thus, for any connected directed graph, we let Σ be the unique solution to the Lyapunov equation (8). Then, we let

$$X := 2Q^T \Sigma Q \tag{10}$$

and notice that X will be symmetric for any graph because Σ is always symmetric. Finally we can use (3) to define the effective resistance between any two nodes in the graph.

Definition 2: Let $\mathcal{G}$ be a connected directed graph with N nodes and Laplacian matrix L. Then the *effective resistance between nodes k and j in $\mathcal{G}$* is defined as

$$\begin{aligned} r_{k,j} &= \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T X \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right) \\ &= x_{k,k} + x_{j,j} - 2x_{k,j}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} X &= 2Q^T \Sigma Q, \\ \overline{L}\Sigma + \Sigma \overline{L}^T &= I_{N-1}, \\ \overline{L} &= QLQ^T, \end{aligned} \tag{12}$$

and Q is a matrix satisfying (1).

By summing all distinct effective resistances in a directed graph, we define the *Kirchhoff index*, K_f, of the graph, that is

$$K_f := \sum_{k<j} r_{k,j}. \tag{13}$$

Our definition of K_f generalizes to directed graphs the Kirchhoff index defined for undirected graphs [30].

Remark 1: This generalized notion of Kirchhoff index has been used in [22], [23] to characterize the robustness of consensus dynamics with respect to white noise disturbances. In addition, our definition of effective resistance has been used to compute a performance measure for stochastic decision-makers that are connected via a balanced network[1] [24].

We conclude the section by deriving an equivalent expression for the effective resistance. Using the expression for $\overline{L}$ given in (4), the integral form for Σ given in (7), and the properties of Q from (1), we observe that the matrix X can be expressed as

$$X = 2\int_0^\infty \Pi e^{-Lt} e^{-L^T t} \Pi \, dt. \tag{14}$$

Since $(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)})^T \Pi = (\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)})^T$, we can express the effective resistance as

$$r_{k,j} = 2\int_0^\infty \left\| \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T e^{-Lt} \right\|_2^2 dt. \tag{15}$$

[1]A graph is said to be *balanced* if for every node, the out-degree and in-degree are equal.

IV. Basic Properties of our Definition

Although Definition 2 ensures that effective resistance maintains our desired relationship with some control-theoretic properties, such as those discussed above, by itself it remains an algebraic construction that yields little insight into the ways in which effective resistance depends on the graph structure. We now proceed to analyze our definition to understand some of its fundamental properties. In Section IV-A we verify that Definition 2 results in a well-defined property of pairs of nodes in a connected directed graph. In Section IV-B we investigate how effective resistances depend on connections in the graph and extend Definition 2 further to apply to disconnected graphs. Finally in Section IV-C we determine that effective resistance is a distance-like function and explore the limitations of the triangle inequality for effective resistances in directed graphs.

A. Effective Resistance is Well-Defined

By construction, (11) will yield the standard effective resistance for any undirected graph. However, we must confirm that our concept of effective resistance for directed graphs is well-defined. This is achieved by the following two lemmas.

Lemma 2: The value of the effective resistance between two nodes in a connected directed graph is independent of the choice of Q.

Proof: Since neither (14) nor (15) involves Q, the choice of Q must not affect the value of the effective resistance. ■

From Lemma 2 we see that no matter how it is computed, the effective resistance between two nodes will be the same unique number. Next, we will show in Lemma 3 that the effective resistance is a property of a pair of nodes, irrespective of the way in which they are labelled.

Lemma 3: The value of the effective resistance between two nodes in a connected directed graph is independent of the labeling of the nodes.

Proof: Any two labelings of the nodes in a graph can be related via a permutation. Suppose L and L' are two Laplacian matrices associated with the same graph, but with different labelings of the nodes. Then L' can be found from L by permuting its rows and columns. That is, there exists an $N \times N$ permutation matrix P such that $L' = PLP^T$. Note that as a permutation matrix, there is exactly one 1 in every row and column of P with every other entry equal to 0. Furthermore, $P^{-1} = P^T$, $P\mathbf{1}_N = \mathbf{1}_N$, and $\mathbf{1}_N^T P = \mathbf{1}_N^T$. Thus we can observe that $QP = \overline{P}Q$, $PQ^T = Q^T\overline{P}$ and $\overline{P}^{-1} = \overline{P}^T$, where, as usual, $\overline{P} = QPQ^T$.

Now, we can use (4) and the properties of P to write $\overline{L}' = \overline{P}\,\overline{L}\,\overline{P}^T$. Then the solution to the Lyapunov equation associated with $\overline{L}'$ becomes $\Sigma' = \overline{P}\Sigma\overline{P}^T$. Hence, we observe that $X' = PXP^T$.

Thus if P permutes node k to node m and node j to node ℓ, we find that $r'_{m,\ell} = r_{k,j}$. ■

B. Effective Resistance Depends on Connections Between Nodes

Next we consider which features of a directed graph will affect the effective resistance between a given pair of nodes. For

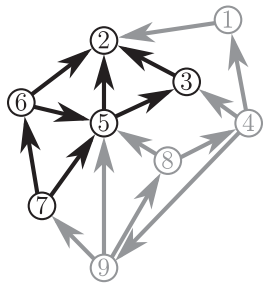

Fig. 2. Directed graph on 9 nodes with the reachable subgraph of nodes 3 and 7 highlighted.

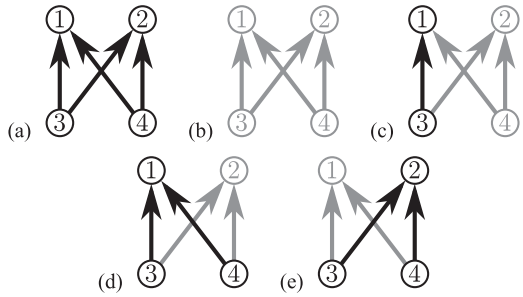

Fig. 3. (a) A disconnected directed graph on 4 nodes. (b) There are no connection subgraphs between nodes 1 and 2. (c) The connection subgraph between nodes 1 and 3 is highlighted. (d) One connection subgraph between nodes 3 and 4 is highlighted. (e) A second connection subgraph between nodes 3 and 4 is highlighted. In this example, the effective resistance between nodes 3 and 4 is undefined.

undirected graphs, we know that effective resistances depend on every path between a pair of nodes [1]. The situation becomes more complicated with directed graphs since there can exist pairs of nodes in a connected directed graph with no path between them. Instead of looking at paths between nodes, we therefore have to consider connections. To incorporate all of the connections between two nodes, we examine the reachable subgraph, an example of which is shown in Fig. 2.

We proceed to show in Theorem 1 that the effective resistance between two nodes in a connected directed graph can only depend on the connections between them.

Theorem 1: The effective resistance between nodes k and j in a connected directed graph $\mathcal{G}$ is equal to the effective resistance between nodes k and j in $\mathcal{R}_{\mathcal{G}}(k,j)$.

Proof: Let $\mathcal{G}_1 = \mathcal{R}_{\mathcal{G}}(k,j)$. Let N_1, A_1, D_1, and L_1 be the number of nodes, the adjacency matrix, the matrix of node out-degrees and the Laplacian matrix of $\mathcal{G}_1$, respectively. Let $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ satisfy (1). Since $\mathcal{G}_1$ is connected by Lemma 1, we can find matrices $\overline{L}_1$, Σ_1, and X_1 from (12) for $\mathcal{G}_1$.

Let $\mathcal{G}_2$ be the subgraph of $\mathcal{G}$ formed by every node in $\mathcal{G}$ which is not in $\mathcal{G}_1$ and every edge in $\mathcal{G}$ between these nodes. Then $\mathcal{G}_2$ will contain N_2 nodes and have associated matrices A_2, D_2, L_2, Q_2, and $\overline{L}_2$.

Now, if there was an edge (m,n) in $\mathcal{G}$ from a node m in $\mathcal{G}_1$ to a node n in $\mathcal{G}_2$, then n would be reachable from either k or j (as m is reachable from one of these nodes). Thus there are no edges in $\mathcal{G}$ leading from nodes in $\mathcal{G}_1$ to nodes in $\mathcal{G}_2$. Therefore, if we order the nodes in $\mathcal{G}$ with the nodes in $\mathcal{G}_1$ listed first, followed by the nodes in $\mathcal{G}_2$, then the adjacency matrix of $\mathcal{G}$ can be written as

$$A = \begin{bmatrix} A_1 & \emptyset \\ A_{2,1} & A_2 \end{bmatrix},$$

where $A_{2,1} \in \mathbb{R}^{N_2 \times N_1}$ contains the edge weights for all edges in $\mathcal{G}$ leading from nodes in $\mathcal{G}_2$ to nodes in $\mathcal{G}_1$. Similarly, we can write the matrix of node out-degrees as

$$D = \begin{bmatrix} D_1 & \emptyset \\ \emptyset & D_2 + D_{2,1} \end{bmatrix},$$

where $D_{2,1}$ is the diagonal matrix containing the row sums of $A_{2,1}$ along its diagonal. Utilizing these two expressions, we find that the Laplacian matrix of $\mathcal{G}$ can be written as

$$L = \begin{bmatrix} L_1 & \emptyset \\ -A_{2,1} & L_2 + D_{2,1} \end{bmatrix}. \tag{16}$$

Since (16) is lower-block triangular, and $k, j \leq N_1$

$$\left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T L^n = \left(\mathbf{e}_{N_1}^{(k)} - \mathbf{e}_{N_1}^{(j)}\right)^T L_1^n$$

for any $n \geq 0$. By using (15), and the definition of the matrix exponential, we conclude that

$$\begin{aligned} r_{k,j} &= 2\int_0^\infty \left\| \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T e^{-Lt} \right\|_2^2 dt \\ &= 2\int_0^\infty \left\| \left(\mathbf{e}_{N_1}^{(k)} - \mathbf{e}_{N_1}^{(j)}\right)^T e^{-L_1 t} \right\|_2^2 dt \end{aligned}$$

thus proving the claim. ■

Remark 2: We can use Theorem 1 to partially extend the definition of effective resistance to disconnected directed graphs. To do this, we must examine connection subgraphs. From Lemma 1 we know that $\mathcal{C}_{\mathcal{G}}(k,j) = \mathcal{R}_{\mathcal{G}}(k,j)$ if $\mathcal{G}$ is connected. However, a disconnected directed graph may contain 0, 1 or more connection subgraphs between a pair of nodes. There will be no connection subgraphs precisely when there are no connections between nodes k and j in $\mathcal{G}$. However, there may also be multiple connections between a pair of nodes that lead to multiple connection subgraphs. A simple example of connection subgraphs in a disconnected directed graph is shown in Fig. 3.

By definition, whenever it exists, $\mathcal{C}_{\mathcal{G}}(k,j)$ will be connected, and we can thus compute effective resistances in it. We can now define effective resistances between some node pairs in any digraph, whether or not it is connected, as follows

Definition 3: The effective resistance between nodes k and j in a directed graph $\mathcal{G}$ is

$$r_{k,j} = \begin{cases} \infty & \text{if there are no connections} \\ & \text{between nodes } k \text{ and } j \\ r_{k,j} \text{ in } \mathcal{C}_{\mathcal{G}}(k,j) & \text{if } \mathcal{C}_{\mathcal{G}}(k,j) \text{ exists} \\ & (\text{computed using } (11)) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

By Theorem 1, this new definition specializes to our original definition of effective resistance for connected directed graphs.

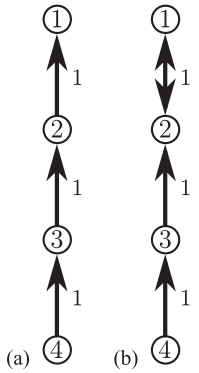

Fig. 4. Two simple 4-node graphs: (a) $\mathcal{G}_4^{\text{path}}$, a 4-node directed path graph with unit edge weights, and (b) $\mathcal{G}^{\text{line}}$, similar to $\mathcal{G}_4^{\text{path}}$ but with the directed edge (2, 1) replaced by an undirected edge.

For certain applications, there may be an appropriate way to handle pairs of nodes with multiple connection subgraphs, but that falls outside the scope of the present work.

In undirected graphs, we know that the effective resistance between two nodes does not depend on edges that do not lie on any simple path between the nodes [1]. Unfortunately, the situation is not as straightforward for directed graphs. Consider the 4-node graphs shown in Fig. 4. In $\mathcal{G}_4^{\text{path}}$, we can compute that $r_{3,4} = 2$. However in $\mathcal{G}^{\text{line}}$, even though no new simple connections were introduced between nodes 3 and 4, the effective resistance is now $r_{3,4} = 16/9$.

Despite this (perhaps) unexpected behavior, we are able to show that certain parts of the connection subgraph do not affect the effective resistance between two nodes.

Theorem 2: Suppose $\mathcal{G}_1$ is a connected directed graph containing only one globally reachable node, and let $\mathcal{G}$ be the graph formed by connecting the globally reachable node in $\mathcal{G}_1$ to an additional node via a directed edge of arbitrary weight. Then the effective resistance between any two nodes in $\mathcal{G}_1$ is equal to the effective resistance between them in $\mathcal{G}$.

The proof of this theorem relies on Lemma 4, which can be found in Appendix A.

Proof of Theorem 2: Let N_1, A_1, D_1, and L_1 be the number of nodes, the adjacency matrix, the matrix of node out-degrees and the Laplacian matrix of $\mathcal{G}_1$, respectively. Let $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ satisfy (1). Using Q_1, we can compute $\overline{L}_1$ from (4) and since $\mathcal{G}_1$ is connected, we can find matrices Σ_1 and X_1 from (12). Without loss of generality, suppose that the globally reachable node in $\mathcal{G}_1$ is node 1. Since this is the only globally reachable node in $\mathcal{G}_1$, no other nodes can be reached from node 1 and hence $d_1^{\text{out}} = 0$. Thus

$$L_1^T \mathbf{e}_{N_1}^{(1)} = \mathbf{0}. \tag{17}$$

Let the additional node in $\mathcal{G}$ be node $N = N_1 + 1$. We can see that since node 1 is globally reachable in $\mathcal{G}_1$ and node N is reachable from node 1, node N is globally reachable in $\mathcal{G}$. Thus $\mathcal{G}$ is connected.

Now, we can write the adjacency matrix of $\mathcal{G}$ as

$$A = \begin{bmatrix} A_1 & a\mathbf{e}_{N_1}^{(1)} \\ \mathbf{0}^T & 0 \end{bmatrix},$$

where $a > 0$ is the weight on edge $(1, N)$ in $\mathcal{G}$. Similarly, we can write the matrix of node out-degrees as

$$D = \begin{bmatrix} D_1 + a\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}.$$

Utilizing these two expressions, we find that the Laplacian matrix of $\mathcal{G}$ can be written as

$$L = \begin{bmatrix} L_1 + a\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T} & -a\mathbf{e}_{N_1}^{(1)} \\ \mathbf{0}^T & 0 \end{bmatrix}.$$

Now, let

$$Q = \begin{bmatrix} Q_1 & \mathbf{0} \\ \alpha \mathbf{1}_{N_1}^T & -\beta \end{bmatrix}$$

where $\alpha = (1/\sqrt{N_1(N_1+1)})$ and $\beta = (N_1/\sqrt{N_1(N_1+1)})$. Then Q satisfies (1). Substituting this matrix into (4) gives us

$$\overline{L} = \begin{bmatrix} \overline{L}_1 + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T & a(\alpha+\beta)Q_1\mathbf{e}_{N_1}^{(1)} \\ \alpha\mathbf{1}_{N_1}^T L_1 Q_1^T + a\alpha\mathbf{e}_{N_1}^{(1)T}Q_1^T & \frac{a}{N_1} \end{bmatrix}.$$

In order to compute effective resistances in $\mathcal{G}$, we must find the matrix Σ which solves (8). Since we have partitioned $\overline{L}$ into a 2×2 block matrix, we will do the same for Σ. Let

$$\Sigma = \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix},$$

where $S \in \mathbb{R}^{(N_1-1)\times(N_1-1)}$ is a symmetric matrix, $\mathbf{t} \in \mathbb{R}^{N_1-1}$ and $u \in \mathbb{R}$. Then Lemma 4 gives a solution to (8) using $\overline{L}$ and our desired form of Σ. However, since $\mathcal{G}$ is connected, we know that there must be a unique solution. Thus (22) from Lemma 4 is the unique solution to the Lyapunov equation, and in particular

$$S = \Sigma_1.$$

Now, recall that effective resistances in $\mathcal{G}_1$ can be found from $X_1 = 2Q_1^T\Sigma_1 Q_1$ as

$$r_{1\ k,j} = x_{1\ k,k} + x_{1\ j,j} - 2x_{1\ k,j}.$$

Next, we must examine $X = 2Q^T\Sigma Q$ in order to compute effective resistances in $\mathcal{G}$. Using the matrices given above, and letting $\mathbf{p} := 2\alpha Q_1^T\mathbf{t} = [p_i]$, we obtain

$$X = \begin{bmatrix} X_1 + \mathbf{1}_{N_1}\mathbf{p}^T + \mathbf{p}\mathbf{1}_{N_1}^T + 2\alpha^2 u \mathbf{1}_{N_1}\mathbf{1}_{N_1}^T & * \\ * & * \end{bmatrix}.$$

Finally, since nodes k and j are both in $\mathcal{G}_1$, we observe that

$$\begin{aligned} r_{k,j} &= x_{1\ k,k} + 2p_k + 2\alpha^2 u + x_{1\ j,j} + 2p_j + 2\alpha^2 u - 2x_{1\ k,j} \\ &\quad - 2(p_k + p_j) - 4\alpha^2 u \\ &= r_{1\ k,j}. \end{aligned}$$

Thus, the effective resistance between two nodes in $\mathcal{G}_1$ is equal to the effective resistance between the same two nodes in $\mathcal{G}$. ■

Corollary 1: Suppose $\mathcal{C}_{\mathcal{G}}(k,j)$ consists of a subgraph $\mathcal{C}'_{\mathcal{G}}(k,j)$ that is connected via a single edge of arbitrary weight to the leaf node of a directed path. Then the effective resistance between nodes k and j in the directed graph $\mathcal{G}$ is equal to the effective resistance between nodes k and j in $\mathcal{C}'_{\mathcal{G}}(k,j)$.

Proof: This follows by simply applying Theorem 2 repeatedly to "prune" away the nodes in the directed path. ■

We can see from the graphs shown in Fig. 4 that this "pruning" operation can only be applied to directed edges in general. There may, however, be other graphical structures that also do not affect the effective resistance between two nodes.

C. Effective Resistance is a Distance-Like Function and its Square Root is a Metric

One useful property of effective resistance in undirected graphs is that it is a metric on the nodes of the graph [1]. This allows effective resistance to substitute for the shortest-path distance in various graphical indices and analyses, as well as offering an alternative interpretation of effective resistance that does not rely on an electrical analogy. Although many of the requirements of a metric follow from the algebraic construction of the effective resistance, the triangle inequality depends on Kirchhoff's laws [1]. Consequently, we shall see that the effective resistance does not satisfy the triangle inequality on general directed graphs. Importantly, however, the square root of the effective resistance is a metric. Therefore, if a true metric on directed graphs is sought which incorporates information about all connections between two nodes, the square root of the effective resistance is a valid option. In contrast, if the effective resistance had been generalized using the Moore-Penrose generalized inverse instead of our definition, then neither would it be a metric nor would its square root be a metric.

We note that the only difference in the conditions for a metric between a function and its square root lies in the triangle inequality. Furthermore, if a function $d(\cdot,\cdot)$ is a metric, then $\sqrt{d(\cdot,\cdot)}$ is by necessity a metric too.

Theorem 3: The square root of the effective resistance is a metric on the nodes of any connected directed graph. That is

$$r_{k,j} \geq 0 \quad \forall \text{ nodes } k \text{ and } j \tag{18}$$

$$r_{k,j} = 0 \;\Leftrightarrow\; k = j \tag{19}$$

$$r_{k,j} = r_{j,k} \tag{20}$$

$$\sqrt{r_{k,\ell}} + \sqrt{r_{\ell,j}} \geq \sqrt{r_{k,j}} \quad \forall \text{ nodes } k,\ j,\ \text{and } \ell. \tag{21}$$

Furthermore, the effective resistance itself is not a metric since it fails to satisfy the triangle inequality.

Proof: From (14) we know that the square root of the effective resistance is

$$\sqrt{r_{k,j}} = \sqrt{2\int_0^\infty \left\| \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T e^{-Lt} \right\|_2^2 dt}.$$

Notice that, modulo a multiplicative factor, this is the L_2 distance between the functions $\mathbf{e}_N^{(k)^T} e^{-Lt}$ and $\mathbf{e}_N^{(j)^T} e^{-Lt}$. The properties (18), (20), and (21) are therefore trivially satisfied. The second property (19), follows by the full-rank property of e^{-Lt}.

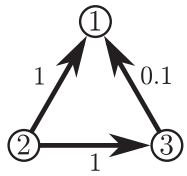

Fig. 5. Simple 3-node directed graph, $\mathcal{G}^{\text{triangle}}$,where the triangle inequality fails.

Finally, to show that $r_{k,j}$ is not a metric, we consider the 3-node graph shown in Fig. 5. In this case, we find that $r_{1,3} = 20$, $r_{1,2} = (131/21) \approx 6.24$, and $r_{2,3} = (37/7) \approx 5.29$. Thus $r_{1,3} > r_{1,2} + r_{2,3}$ and the triangle inequality does not hold. ■

Remark 3: The counterexample in the proof of Theorem 3 also demonstrates why the triangle inequality could be expected to fail for any effective resistance definition that respects edge direction. We can observe that there should be a "low" effective resistance between nodes 1 and 2 due to the connecting edge with unit weight. Likewise, nodes 2 and 3 should have a "low" effective resistance between them for the same reason. But node 2 does not belong to $\mathcal{C}_{\mathcal{G}^{\text{triangle}}}(1,3)$ and so there should be a "high" effective resistance between nodes 1 and 3 due to their only connection being an edge with low weight. Thus, the sum of the effective resistances between nodes 1 and 2 and between nodes 2 and 3 should be lower than the effective resistance between nodes 1 and 3. For example, if effective resistance was defined by setting X equal to the Moore-Penrose generalized inverse of L, then the triangle inequality for $r_{k,j}$ would fail in the graph shown in Fig. 5. However, in this case the triangle inequality would also fail for $\sqrt{r_{k,j}}$, while it does not for our definition.

V. Conclusion

We have generalized the concept of effective resistance to directed graphs in a way that maintains the connection between effective resistances and control-theoretic properties relating to consensus-type dynamics. Despite the algebraic nature of our generalization, we have shown that effective resistances in directed graphs bear a fundamental relationship to the structure of the connections between nodes. Moreover, the square root of effective resistance provides a well-defined metric on connected directed graphs, allowing for a notion of distance between nodes, even in cases where neither node is reachable from the other.

Although it may have been tempting to use the Moore-Penrose generalized inverse of a directed graph's Laplacian matrix to define effective resistance, we have shown that not only would this approach ignore the complexity of the derivation of effective resistance for undirected graphs, but it would also fail to lead to a distance function for directed graphs. Instead, our generalization derives from an analysis of applications of effective resistance in which directed graphs arise naturally. We believe that this approach will allow for the application of this directed version of effective resistance in other situations than those we presented above.

In the companion paper [32], we demonstrate how to compute effective resistances in certain prototypical classes of directed graphs and we find cases where effective resistances in directed graphs behave analogously to effective resistances in undirected graphs as well as cases where they behave in unexpected ways.

APPENDIX A
THE SOLUTION TO A PARTICULAR LYAPUNOV EQUATION

Lemma 4: Suppose that $L_1 \in \mathbb{R}^{N_1 \times N_1}$ is the Laplacian matrix of a connected graph satisfying $L_1^T \mathbf{e}_{N_1}^{(1)} = \mathbf{0}$ and $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ satisfies (1). Let $\alpha = (1/\sqrt{N_1(N_1+1)})$, $\beta = (N_1/\sqrt{N_1(N_1+1)})$, and $a > 0$. If $\overline{L}_1 = Q_1 L_1 Q_1^T$, then a solution to the Lyapunov equation

$$\begin{aligned}
&\begin{bmatrix} \overline{L}_1 + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T & a(\alpha+\beta)Q_1\mathbf{e}_{N_1}^{(1)} \\ \alpha \mathbf{1}_{N_1}^T L_1 Q_1^T + a\alpha \mathbf{e}_{N_1}^{(1)T} Q_1^T & \frac{a}{N_1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix} + \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix} \\
&\quad \times \begin{bmatrix} \overline{L}_1^T + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T & \alpha Q_1 L_1^T \mathbf{1}_{N_1} + a\alpha Q_1 \mathbf{e}_{N_1}^{(1)} \\ a(\alpha+\beta)\mathbf{e}_{N_1}^{(1)T}Q_1^T & \frac{a}{N_1} \end{bmatrix} \\
&= \begin{bmatrix} I_{N_1-1} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}
\end{aligned}$$

is

$$\begin{aligned}
S &= \Sigma_1 \\
\mathbf{t} &= -N_1 \alpha \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} \\
u &= \frac{N_1}{2a} + \frac{N_1^2\alpha^2}{a}\left(\mathbf{1}_{N_1}^T L_1 + a\mathbf{e}_{N_1}^{(1)T}\right) Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} \qquad (22)
\end{aligned}$$

where Σ_1 is the solution to (8) for $\overline{L}_1$.

Proof: First we note that Σ_1 exists since L_1 is the Laplacian of a connected graph. Next, equating blocks of the given matrix equation gives us

$$\begin{aligned}
&\overline{L}_1 S + S\overline{L}_1^T + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T S + aSQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T \\
&\qquad + a(\alpha+\beta)\left(Q_1\mathbf{e}_{N_1}^{(1)}\mathbf{t}^T + \mathbf{t}\mathbf{e}_{N_1}^{(1)T}Q_1^T\right) = I_{N_1-1}, \qquad (23) \\
&\alpha S Q_1\left(L_1^T\mathbf{1}_{N_1} + a\mathbf{e}_{N_1}^{(1)}\right) + au(\alpha+\beta)Q_1\mathbf{e}_{N_1}^{(1)} \\
&\qquad + \left(\frac{a}{N_1}I_{N_1-1} + \overline{L}_1 + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T\right)\mathbf{t} = \mathbf{0}, \qquad (24) \\
&\frac{2au}{N_1} + 2\alpha\left(\mathbf{1}_{N_1}^T L_1 + a\mathbf{e}_{N_1}^{(1)T}\right)Q_1^T\mathbf{t} = 1. \qquad (25)
\end{aligned}$$

By directly substituting (22) into (23) and (25) (and noting that $N_1\alpha(\alpha+\beta) = 1$), we observe that (22) satisfies (23) and (25). Hence, we now focus our attention on (24). Substituting (22) into the left-hand side of (24) gives us

$$\begin{aligned}
\text{LHS} = \alpha\Sigma_1 Q_1 L_1^T\mathbf{1}_{N_1} - N_1\alpha\overline{L}_1\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)} + \frac{1}{2\alpha}Q_1\mathbf{e}_{N_1}^{(1)} \\
+ N_1\alpha Q_1\mathbf{e}_{N_1}^{(1)}\mathbf{1}_{N_1}^T L_1 Q_1^T\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)}. \qquad (26)
\end{aligned}$$

By (8), we know that $\overline{L}_1\Sigma_1 = I_{N_1-1} - \Sigma_1\overline{L}_1^T$. Therefore, using (1) and our assumption that $L_1^T\mathbf{e}_{N_1}^{(1)} = \mathbf{0}$, we have

$$\overline{L}_1\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)} = Q_1\mathbf{e}_{N_1}^{(1)} + \frac{1}{N_1}\Sigma_1 Q_1 L_1^T\mathbf{1}_{N_1}.$$

Thus $N_1\alpha\overline{L}_1\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)} = N_1\alpha Q_1\mathbf{e}_{N_1}^{(1)} + \alpha\Sigma_1 Q_1 L_1^T\mathbf{1}_{N_1}$ and (26) becomes

$$\text{LHS} = \left(\frac{1}{2\alpha} - N_1\alpha + N_1\alpha\mathbf{1}_{N_1}^T L_1 Q_1^T\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)}\right)Q_1\mathbf{e}_{N_1}^{(1)}. \qquad (27)$$

Next, if we define V to be the matrix $V := L_1 Q_1^T\Sigma_1 Q_1 + Q_1^T\Sigma_1 Q_1 L_1^T$, we have that $V = V^T$ and [using (1)]

$$\mathbf{1}_{N_1}^T V\mathbf{e}_{N_1}^{(1)} = \mathbf{1}_{N_1}^T L_1 Q_1^T\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)}.$$

But pre- and post-multiplying V by $\Pi = Q_1^T Q_1$ and using (8) gives us $\Pi V\Pi = \Pi$, and then by pre- and post-multiplying by $\mathbf{e}_{N_1}^{(1)T}$ and $\mathbf{e}_{N_1}^{(1)}$, we find

$$\begin{aligned}
&\mathbf{e}_{N_1}^{(1)T}V\mathbf{e}_{N_1}^{(1)} - \frac{2}{N_1}\mathbf{1}_{N_1}^T V\mathbf{e}_{N_1}^{(1)} + \frac{1}{N_1^2}\mathbf{1}_{N_1}^T V\mathbf{1}_{N_1} \\
&\qquad = \frac{N_1 - 1}{N_1} \text{ (since } V \text{ is symmetric).}
\end{aligned}$$

But since $L_1^T\mathbf{e}_{N_1}^{(1)} = \mathbf{0}$ and $Q_1\mathbf{1}_{N_1} = \mathbf{0}$, we know that both $\mathbf{e}_{N_1}^{(1)T}V\mathbf{e}_{N_1}^{(1)} = 0$ and $\mathbf{1}_{N_1}^T V\mathbf{1}_{N_1} = 0$. Thus

$$\mathbf{1}_{N_1}^T V\mathbf{e}_{N_1}^{(1)} = \frac{1-N_1}{2} \Rightarrow \mathbf{1}_{N_1}^T L_1 Q_1^T\Sigma_1 Q_1\mathbf{e}_{N_1}^{(1)} = \frac{1-N_1}{2}$$

and so (27) becomes LHS $= \mathbf{0}$. Thus (22) also satisfies (24) and is therefore a solution to the given matrix equation. ■

REFERENCES

[1] D. Klein and M. Randić, "Resistance distance," *J. Math. Chem.*, vol. 12, pp. 81–95, 1993.

[2] P. Doyle and J. Snell, *Random Walks and Electric Networks*. Washington DC: The Mathematical Association of America, 1984.

[3] P. Tetali, "Random walks and the effective resistance of networks," *J. Theor. Prob.*, vol. 4, no. 1, pp. 101–109, 1991.

[4] A. Chandra, P. Raghavan, W. Ruzzo, R. Smolensky, and P. Tiwari, "The electrical resistance of a graph captures its commute and cover times," *Comput. Complex.*, vol. 6, no. 4, pp. 312–340, 1996.

[5] J. Palacios, "Resistance distance in graphs and random walks," *Int. J. Quantum Chem.*, vol. 81, no. 1, pp. 29–33, 2001.

[6] A. Ghosh, S. Boyd, and A. Saberi, "Minimizing effective resistance of a graph," *SIAM Rev.*, vol. 50, no. 1, pp. 37–66, 2008.

[7] L. Shapiro, "An electrical lemma," *Math. Mag.*, vol. 60, no. 1, pp. 36–38, 1987.

[8] D. Bonchev, A. Balaban, X. Liu, and D. Klein, "Molecular cyclicity and centricity of polycyclic graphs. I. Cyclicity based on resistance distances or reciprocal distances," *Int. J. Quantum Chem.*, vol. 50, no. 1, pp. 1–20, 1994.

[9] I. Gutman and B. Mohar, "The quasi-Wiener and the Kirchhoff indices coincide," *J. Chem. Inf. Comput. Sci.*, vol. 36, no. 5, pp. 982–985, 1996.

[10] H.-Y. Zhu, D. Klein, and I. Lukovits, "Extensions of the Wiener number," *J. Chem. Inf. Comput. Sci.*, vol. 36, no. 3, pp. 420–428, 1996.

[11] I. Lukovits, S. Nikolić, and N. Trinajstić, "Resistance distance in regular graphs," *Int. J. Quantum Chem.*, vol. 71, no. 3, pp. 217–225, 1999.

[12] D. Klein and O. Ivanciuc, "Graph cyclicity, excess conductance, resistance deficit," *J. Math. Chem.*, vol. 30, no. 3, pp. 271–287, 2001.

[13] P. Barooah and J. Hespanha, "Graph effective resistance and distributed control: Spectral properties and applications," in *Proc. CDC*, San Diego, CA, 2006, pp. 3479–3485.
[14] P. Barooah and J. Hespanha, "Estimation on graphs from relative measurements," *IEEE Contr. Syst. Mag.*, vol. 27, no. 4, pp. 57–74, 2007.
[15] P. Barooah and J. Hespanha, "Estimation from relative measurements: Electrical analogy and large graphs," *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 2181–2193, 2008.
[16] S. Patterson and B. Bamieh, "Leader selection for optimal network coherence," in *Proc. CDC*, Atlanta, GA, 2010, pp. 2692–2697.
[17] A. Clark and R. Poovendran, "A submodular optimization framework for leader selection in linear multi-agent systems," in *Proc. CDC-ECC*, Orlando, FL, 2011, pp. 3614–3621.
[18] M. Fardad, F. Lin, and M. Jovanović, "Algorithms for leader selection in large dynamical networks: Noise-free leaders," in *Proc. CDC-ECC*, Orlando, FL, 2011, pp. 7188–7193.
[19] K. Fitch and N. E. Leonard, "Information centrality and optimal leader selection in noisy networks," in *Proc. CDC*, Florence, Italy, 2013, pp. 7510–7515.
[20] F. Dörfler and F. Bullo, "Synchronization of power networks: Network reduction and effective resistance," in *IFAC Workshop Distrib. Est. Control Net. Syst.*, Annency, France, 2010, pp. 197–202.
[21] F. Dörfler and F. Bullo, "Kron reduction of graphs with applications to electrical networks," *IEEE Trans. Circuits Syst. I*, vol. 60, no. 1, pp. 150–163, 2013.
[22] G. Young, L. Scardovi, and N. Leonard, "Robustness of noisy consensus dynamics with directed communication," in *Proc. ACC*, Baltimore, MD, 2010, pp. 6312–6317.
[23] G. Young, L. Scardovi, and N. Leonard, "Rearranging trees for robust consensus," in *Proc. CDC-ECC*, Orlando, FL, 2011, pp. 1000–1005.
[24] I. Poulakakis, G. F. Young, L. Scardovi, and N. E. Leonard, "Information centrality and ordering of nodes for accuracy in noisy decision-making dynamics," *IEEE Trans. Autom. Control*, vol. 61, no. 4, pp. 1040–1045, 2016.
[25] V. Srivastava and N. Leonard, "Collective decision-making in ideal networks: The speed-accuracy tradeoff," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 1, pp. 121–130, 2014.
[26] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.
[27] J. Bang-Jensen and G. Gutin, *Digraphs Theory, Algorithms and Applications*. London, U.K.: Springer-Verlag, 2007.
[28] L. Scardovi and N. Leonard, "Robustness of aggregation in networked dynamical systems," in *Proc. ROBOCOMM*, Odense, Denmark, 2009, pp. 1–6.
[29] R. Agaev and P. Chebotarev, "On the spectra of nonsymmetric Laplacian matrices," *Lin. Alg. Appl.*, vol. 399, pp. 157–168, 2005.
[30] W. Xiao and I. Gutman, "Resistance distance and Laplacian spectrum," *Theor. Chem. Acc.*, vol. 110, no. 4, pp. 284–289, 2003.
[31] G. Dullerud and F. Paganini, *A Course in Robust Control Theory*. New York: Springer, 2000.
[32] G. Young, L. Scardovi, and N. E. Leonard, "A new notion of effective resistance for directed graphs—Part II: Computing resistances," *IEEE Trans. Autom. Control*, vol. 61, 2016.

George Forrest Young (M'12) received the B.E. degree in mechatronic engineering and the B.Math.Sci. degree in pure mathematics from the University of Adelaide, Adelaide, Australia, in 2007 and 2008, respectively, and the M.A. and Ph.D. degrees in mechanical and aerospace engineering from Princeton University, Princeton, NJ, in 2010 and 2014, respectively.

In 2014, he worked as a Systems Engineer at AgSmarts Inc., and is currently the Director of Engineering at AgSmarts Inc., Memphis, TN. In 2015 he was an Adjunct Professor at Rhodes College, Memphis, TN. His research and professional interests focus on robust control in uncertain environments with a particular regard to networked systems.

Luca Scardovi (M'06) received the Laurea degree and Ph.D. degree in electronic and computer engineering from the University of Genoa, Italy, in 2001 and 2005 respectively.

In 2005 he was an Adjunct Professor at the University of Salento, Lecce, Italy. He held research associate positions at the Department of Electrical Engineering and Computer Science, University of Liège, Belgium (2005–2007) and at the Department of Mechanical and Aerospace Engineering, Princeton University (2007–2009). From 2009 to 2011 he was an Assistant Professor at the Technische Universität München (TUM), Munich, Germany. He is currently an Assistant Professor in the Edward S. Rogers Sr. Department of Electrical and Computer Engineering at the University of Toronto. He is an Associate Editor for *Systems and Control Letters*. His research interests focus on dynamical systems with special emphasis in the analysis and control of emergent dynamics in networked dynamical systems.

Naomi Ehrich Leonard (F'07) received the B.S.E. degree in mechanical engineering from Princeton University, Princeton, NJ, in 1985 and the M.S. and Ph.D. degrees in electrical engineering from the University of Maryland, College Park, in 1991 and 1994, respectively.

From 1985 to 1989, she was an Engineer in the electric power industry. She is the Edwin S. Wilsey Professor of Mechanical and Aerospace Engineering and Director of the Council on Science and Technology, Princeton University. She is also an Associated Faculty Member of Princeton University's Program in Applied and Computational Mathematics. Her research and teaching are in control and dynamical systems with current interests in coordinated control for multi-agent systems, mobile sensor networks, collective animal behavior, and human decision-making dynamics.