Optimising Robustness of Consensus to Noise on Directed Networks

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mechanical and Aerospace Engineering
Adviser: Naomi Leonard

January 2014

Abstract

A major area of study in recent years has been the development of robotic groups that are capable of carrying out complicated and useful tasks and yet are comprised of relatively simple individuals following relatively simple rules. Despite the evidence from natural groups of animals, birds, fish and insects that such behaviour is possible, many challenges remain in the attempt to translate it into engineered systems. One important aspect of understanding and designing group behaviour is the analysis of the communication structure within a group and its effect on overall group performance.

In this dissertation, we focus on understanding the role played by a directed communication graph in the ability of a group to maintain consensus in noisy environments. To this end, we relate a $\mathcal{H}_2$ norm that can be computed from a directed graph to the robustness of the group to noise. Using this relationship, we are able to compute bounds on the group robustness and analyse the capabilities of several families of graphs.

The robustness of consensus to noise on undirected graphs is intimately related to the concept of effective resistance. We present a generalisation of this concept to directed networks and confirm that our new notion of effective resistance is a graphical property that depends on the connections between nodes in the graph. Furthermore, in certain circumstances effective resistance in directed graphs behaves in a similar fashion to effective resistance in undirected graphs, while in other situations it behaves in unexpected ways.

We use effective resistance as a tool to analyse tree graphs, and derive rules by which local changes can be made that will guarantee that the robustness of the entire system will improve. These rules lead to the possibility of decentralised algorithms that allow individuals interacting over a tree graph to rearrange their connections and improve robustness without requiring knowledge of the entire group.

Finally, we use our measure of robustness to analyse a family of interaction strategies within flocks of starlings. This analysis demonstrates that the observed interactions between the starlings optimise the tradeoff between robust performance of the group and individual sensing cost.

Acknowledgements

I owe a deep debt of gratitude to my adviser, Naomi Leonard, for being a mentor, guide, teacher and colleague throughout my time at Princeton. When I arrived here I knew that I wanted to use nature as a source of inspiration for control and robotic applications, but I was very short on the details. Naomi showed me how to channel a sense of wonder about the natural world into practical and tractable problems and then how to use those problems to gain insight into the real world. None of the work presented here would have occurred without her guidance. Moreover, she gave me the freedom to pursue my own ideas while knowing, much better than I did, where I was going and how my work connected to the broader field. Many of my most enlightening and productive hours at Princeton were spent in meetings with Naomi and I hope that when I leave, I can carry with me some of the passion and dedication I have learned from her.

All of the work in this dissertation was carried out, to varying degrees, in collaboration with others. In particular, Luca Scardovi was instrumental in getting me started in my investigations and introducing me to effective resistance. Additionally, Andrea Cavagna and Irene Giardina generously shared their starling data with me and offered enormous insight into the analysis of physical systems.

My committee members, Phil Holmes and Simon Levin, have supported me in many different ways over the years. Their classes gave me a much deeper appreciation of the power and complexities of dynamical systems, and their assistance and advice has helped me to broaden the scope of my work. Finally, I greatly appreciate their time and care in reading my dissertation and providing feedback to improve it.

All of the graduate students and postdocs in the Leonard lab have made innumerable contributions to both my work and my life. Ben, Dan, Andy, Kendra, Darren, Stephanie, Tian, Paul, Katie, Will, Brendan, Luca, Carlos, Ioannis, Alex and Vaibhav have all provided me with advice, feedback, camaraderie and friendship. It has

also been a pleasure for me to work with several undergraduate or newly graduated students. In helping Kelsey, Alex, Aman and Doris with their projects, I have been able to widen my outlook and learn from their ideas and perspectives.

One of the many wonderful aspects of my experience at Princeton has been the opportunities for cross-disciplinary communication and collaboration. In particular, I would like to thank the members of the Couzin lab, the Levin lab and the Project 6 team for sharing their fascinating work.

The Mechanical and Aerospace Engineering department has been a second family to me and has greatly enriched my life both academically and non-academically. No acknowledgement would be complete with thanking Jill Ray and Jess O'Leary for their tireless work in all aspects of keeping this family together. The other PhD students who started in MAE with me - Abdel, Anand, Brandt, Dan, Elizabeth, Fujia, Johnny, Jonathan, Justin, Owen, Parthav, Peter, Thanh, Tian and Yu-Cheng - formed the most incredible group of people I could imagine to share this ride. There are too many other wonderful people in the department to name them all, but I want to give special thanks to the MAE softball and soccer teams for all they added to my experiences here.

Outside of my work, I have had the privilege of meeting many incredible people during my time here, particularly at the Graduate College and at Princeton United Methodist Church. Their friendship will be deeply missed when I leave.

My family has been a constant source of inspiration and support for me through the many good and few hard times. My parents, Forrest and Sandra, allowed me to dream of being whatever I wanted to be, and then provided me with everything I needed to turn that dream into a reality. My siblings, Elizabeth, John and Rosemary have shared so much of my life with me, even as we have all followed our separate paths. In addition, my new family, Richard, D'Ann and Jo have welcomed me as one of their own.

Finally, I would like to express my deepest gratitude and steadfast love to my wonderful wife Elizabeth. She has given me so much more than I ever expected to find when I made the journey over to the USA. Every facet of my work and life is enriched by her love, her understanding and her support.

This dissertation carries the number T-3274 in the records of the Department of Mechanical and Aerospace Engineering.

For my parents and my wife.

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Chapter 1

Introduction

Over the course of the 20$^{\text{th}}$ century, robotics and automation became a critical part of modern society. Today, robots are in widespread use in the manufacturing, logistics, aerospace, medical and military industries. Yet despite their ubiquity in certain areas, most modern robots require highly structured environments and predictable conditions in which to operate. This severely limits the range of tasks that a robot can be used for, and it has long been a goal of researchers to develop robots capable of operating in more natural, varied and unpredictable environments. One approach to this problem is to build more complicated and sophisticated robots, carrying more sensors and having greater computing power. An alternative approach is to build small and cheap robots that, through communication and collaboration, are capable of performing sophisticated tasks *as a group*. The use of robotic groups as an alternative to individual robots potentially offers the benefit of reduced costs, but more significantly it should provide the ability to create far more *robust* and *adaptable* systems [15]. Motivation for this second option can also be found by observing the natural world, where herds of animals, flocks of birds, schools of fish and swarms of insects are *collectively* capable of behaviours beyond the abilities of any individual within the group [23, 95].

Despite its elegance as seen in the natural world, collective behaviour is difficult to engineer. As well as the hardware and cost constraints imposed by the need to build many robots, individual robots within a group must have the ability to *sense and communicate with* one another. Rather than have a supervisor instructing each individual, a group should be able to operate autonomously (or at least semi-autonomously). Although one member of the group could act as a leader and make decisions, such a scenario would negate most of the benefits (i.e., adaptability and robustness) envisioned for collective robotics [42]. Instead, a group of robots should implement a *decentralised* form of control, in which each individual makes their own decisions based on a (potentially) limited knowledge of the rest of the group [2, 58].

Due to the decentralised nature of collective control problems, one key ability for any group to have is the capacity to reach *consensus*, that is, a state in which every individual is in agreement with every other. At its most basic level, consensus is required to make decisions as a group, such as which task to perform next or in which direction to travel. More generally, a group must remain cohesive to enable any sort of collective behaviour, and this, too, requires consensus [85]. Additionally, many more complicated group behaviours, such as formation control [89], task sharing [67] and sensor fusion [61, 76], can be reformulated as consensus problems. Because of this, consensus is widely seen to be one of the most prototypical forms of collective behaviour [77].

For almost any collective system, the *communication structure* - that is, who is communicating with whom - plays a determining role in the behaviour of the group [62, 72]. In fact, under a set of reasonable simplifying assumptions, the outcome of a consensus procedure is *entirely* governed by the communication structure. Therefore, in order to build a group of robots that can collectively perform a task, it is not enough to make them simply capable of communication. The way that they communicate, including whom they choose to communicate with and what they do with the

information they receive, is also a critical design feature. In keeping with our goal of decentralised control, this means that each individual should have a strategy for finding, keeping and (potentially) changing neighbours in a way that will result in the desired outcome for the entire group.

Since one of the goals of collective robotics is to make each individual as simple as possible, it is important to understand how to achieve desired collective behaviour with as little sensing and communication as is practical. This implies that we are not ultimately concerned with *absolute* performance, but rather with *efficient* performance. In particular, for groups of robots or animals, the capability to detect and follow many neighbours comes at a cost - either directly, through the need for such communication capacity, or indirectly, through the loss of time or energy to perform other tasks. Therefore, every individual within a group should be able to balance the performance gain from more interactions with the cost of additional communication.

An important distinction exists between *directed* and *undirected* communication. In undirected communication between two individuals, *both* individuals receive information from the other, and the information transmitted is accorded the *same importance* by each individual. In contrast, directed communication either involves information flowing in only *one direction*, or information flowing in both directions that is given *different importance* by each individual. Thus undirected communication requires not only the ability for both individuals to transmit and receive information but also some prior or active agreement about how to use that information. Directed communication, on the other hand, covers a much wider range of situations [2, 16, 70].

Despite the greater generality of studying directed communication, undirected communication is much easier to study and analyse. In part this is due to the symmetry required for undirected communication, but it is also due to the fact that many of the mathematical approaches and tools used to study communication networks have arisen from the study of other networks that are fundamentally undirected. Thus

additional tools are required in order to study more general directed communication networks. Tools that relate specific aspects of network structure to group-level properties are particularly useful since they can be used to guide decentralised control strategies.

This dissertation makes contributions to both the *analysis* and *synthesis* of directed communication networks for the purpose of consensus in uncertain environments. To do so, we first concentrate on relating the robustness of a consensus system to its communication network and then examine in more detail how to relate structural features of the network to overall performance. With this understanding, we are able to address the question of how an individual should choose their neighbours to efficiently achieve robust performance in certain engineered and biological settings.

1.1 Robust Consensus

The study of consensus has been a central part of the investigation of multi-agent autonomous systems [12, 17, 70, 75, 85, 92]. Two major areas where consensus is required are collective decision-making (such as deciding a common direction of travel) and collective sensing (such as reaching agreement about a measured environmental parameter). Both of these frameworks apply to biological systems such as bird flocks [4] and fish schools [95], and a number of models have been proposed to explain how animal groups reach consensus [54, 71, 99]. In an engineering context, the same consensus problems must be solved by autonomous groups of aerial, ground or underwater vehicles [89]. In addition to multi-agent systems, consensus problems arise in networks of coupled oscillators [63, 94] and in distributed computation [67].

For the purposes of achieving consensus, most of the important details of a multi-agent system are encoded by the communication graph of the system. In this way,

the performance of the consensus protocol can be related to the properties of the communication graph. This provides a more general setting to investigate consensus and allows for the application of graph-theoretic notions and tools. In particular, it is well known that the properties of the Laplacian matrix of the graph are intimately related to the performance of the consensus protocol [75, 85, 88, 109].

Since autonomous systems must operate in uncertain environments without direct supervision, it is important that such systems be robust. Multi-agent systems should be robust with respect to several different parameters, including component or individual agent failure, environmental uncertainty and communication uncertainty. This means that there are a number of different ways in which the robustness of consensus can be measured. Robustness to failure can be measured by the node and edge connectivities of the communication graph, while robustness to uncertainty and noise can be related to the $\mathcal{H}_\infty$ norm, $\mathcal{H}_2$ norm, or $\mathcal{L}_2$ gain of the consensus system. $\mathcal{H}_\infty$ robustness has been investigated in [60, 64, 100] in relation to uncertainty in the communication graph and to non-ideal communication channels. $\mathcal{L}_2$ robustness has been considered in [88, 101] in relation to bounded (in the $\mathcal{L}_2$ sense) inputs or errors. A slightly different notion of robustness to bounded inputs has been investigated in [65, 102]. In addition, robustness in relation to model uncertainty has been studied in [53] and in relation to input time delays in [98]. $\mathcal{H}_2$ robustness to noisy inputs has been studied on undirected graphs in [5, 119, 120], and a similar notion for discrete-time consensus in undirected graphs has been used in [111]. However, little work appears to have been done on $\mathcal{H}_2$ robustness of consensus for directed graphs.

1.2 Effective Resistance

The concept of *effective resistance* has been used in relation to graphs for some time [56]. This concept stems from considering a graph, consisting of a set of nodes

connected by weighted edges, to represent a network of resistors (one resistor corresponding to each edge) with resistances equal to the inverse of the corresponding edge weights. Then, the effective resistance between two nodes can be found by the resistance offered by the network when a voltage source is connected between this pair of nodes. One of the useful properties of effective resistance is that it defines a distance function on a graph that takes into account all paths between two nodes, not just the shortest path [56]. This allows effective resistance to be used in place of the shortest-path distance to analyse problems involving random motion, percolation and flows over networks.

Effective resistance has proven to have a number of interpretations and applications over a wide variety of fields. One of the earliest interpretations was in the study of random walks and Markov chains on networks [19, 36, 78, 97], where the effective resistance between a pair of nodes was related to expected commute, cover and hitting times and the probabilities of a random walk reaching a node or traversing an edge. More direct applications have arisen in the study of power dissipation and time delays in electrical networks [45]. In addition, effective resistance has been shown to have combinatorial interpretations, relating to spanning trees and forests [90] as well as the number of nodes and edges in the graph [56]. Following the work of Klein and Randić [56], there has been a substantial literature investigating the use of effective resistance and the Kirchhoff index in the study of molecular graphs [14, 48, 55, 66, 121].

More recently, effective resistance has arisen in control theory, in the study of control, estimation and synchronisation over networks. Barooah and Hespanha described in [7] how effective resistance can be used to measure the performance of collective formation control, rendezvous and estimation problems. They developed the theory of estimation from relative measurements further in [8, 9]. A number of authors have demonstrated the use of effective resistance in investigating the problem of selecting leaders to maximise agreement or coherence in a network [22, 41, 43, 82]. Dörfler

and Bullo used effective resistance in their study of synchronisation of oscillators in power networks [34], and subsequently developed a theoretical analysis involving effective resistance for a graph reduction technique [35]. We have also connected the concept of effective resistance to the robustness to noise of linear consensus over networks [114, 115] as well as the performance of nodes in networks of stochastic decision-makers [84].

By the nature of its definition, effective resistance is restricted to undirected graphs, and in many applications, including the study of molecular graphs and electrical networks, it is natural to focus solely on undirected graphs. However, in many other applications, including random walks and networked control systems, directed graphs arise just as naturally as undirected ones.

Accordingly, it would be particularly useful if the concept of effective resistance could be extended to apply to any directed or undirected graph, so that analyses that are currently only applicable to undirected graphs could be applied in the more general case. Indeed in [7, 8, 9], the authors investigated directed measurement graphs, but assumed undirected communication in order to analyse their systems using effective resistance. Similarly, in [84, 114, 115], we began our investigations using directed graphs and then specialised to undirected graphs when we used effective resistance.

1.3 Bird Flocking

Flocks of birds and schools of fish exhibit striking and robust collective behaviours despite the challenging environments in which they live [4, 11, 24, 26, 27, 51, 80, 81, 95]. These collective behaviours are believed to emerge from simple, local interactions among the individuals [25, 29, 47, 73, 79]. Significantly, such groups are able to maintain cohesion and coherence even when every individual is subject to uncertain information about the behaviour of its neighbours (those in the group that it can

sense) as well as disturbances from the environment. However, it is not well understood if and how this robustness to uncertainty depends on the structure of the interaction network of individuals, that is, on who is sensing and responding to whom.

Recent analysis of position [4] and velocity [11] correlations in empirical data collected for large flocks of starlings (*Sturnus vulgaris*) has shown that each bird responds to a fixed number, approximately seven, of its nearest neighbours. This work suggests that following a topological interaction rule (i.e. interacting with a fixed number of neighbours) provides important robustness benefits for group cohesion compared to a metric rule (i.e. interacting with neighbours within a fixed distance) [4]. In addition, work is underway on techniques that can reveal in greater detail the precise nature of the inter-individual interactions [13, 40]. However, these analyses do not yield an explanation for why the starlings interact with seven neighbours, rather than some other number.

1.4 Outline of Dissertation

Motivated by the desire to design robotic groups capable of robust collective behaviour, in this dissertation we address some of the open questions discussed in §1.1, §1.2 and §1.3. In Chapter 2 we provide some of the necessary mathematical background and survey some known results. We then proceed in Chapter 3 to characterise the robustness of consensus to noisy inputs on directed networks. This discussion begins with a rigorous description of how to measure robustness and how this measure can be formulated as a $\mathcal{H}_2$ norm related to the communication graph. Next, we are able to relate this $\mathcal{H}_2$ norm to the eigenstructure of the Laplacian matrix of the graph. The chapter concludes with an investigation of several simple families of graphs.

Although the eigenvalue-based approach developed in Chapter 3 can measure the performance of a graph, it provides little insight into how the graph structure affects

the group performance. However, this desired structural insight can be gained for undirected graphs through the notion of effective resistance, which is intimately related to the $\mathcal{H}_2$ norm. In Chapter 4, we propose a new notion of effective resistance that applies to directed graphs as well as undirected ones, and bears the same relationship to the $\mathcal{H}_2$ norm. We follow our new definition with an investigation of its basic properties, including its well-definedness, its dependence on connections in the graph and its relationship to a metric. In Chapter 5 we proceed further and calculate the effective resistance of some of the fundamental connections that arise in directed graphs. This analysis reveals situations where our new definition of effective resistance behaves in much the same way as effective resistance in undirected graphs, as well as situations where it behaves in unexpected and unintuitive ways.

With the understanding developed in Chapters 4 and 5, we investigate in Chapter 6 the ways in which a certain family of graphs, namely trees, can be rearranged in order to improve their robustness. Our investigation begins with undirected trees, and using the traditional notion of effective resistance we are able to develop a partial ordering by robustness. We then use the insight gained to propose a decentralised algorithm that can be guaranteed to improve the robustness of an undirected tree on each application. Following this, we use our new notion of effective resistance to derive a similar partial ordering for directed trees, and discuss a decentralised algorithm that can monotonically improve robustness in this case as well.

The applicability of the theory developed in Chapter 6 rests on an assumption made much earlier - that is, that our measure of robustness is an appropriate and relevant measure of performance for real-world groups. In Chapter 7, we turn our attention to one such group - flocks of starlings. We proceed to compute the robustness of both actual (to the best of our knowledge) and hypothetical interaction networks based on measurements of the positions of each bird within a flock. These computations reveal that not only do the observed interactions within starling flocks perform

well according to our measure, but they actually optimise the tradeoff between maximising robustness to noise and minimising communication costs. This result offers both a validation of our measure of group performance as well as a potential explanation for the observed behaviour of the birds.

The organisation of the body of this dissertation is summarised in Figure 1.1. The main definitions and results of each chapter are stated, and the dependencies between chapters are indicated by arrows. Chapter 2 provides background material for all the later chapters. Chapters 3 to 6 form a linear sequence, with each chapter building on the material previously presented. In contrast, Chapter 7 can be read following Chapter 3.

We complete this dissertation in Chapter 8 by summing up our results and conclusions, as well as looking forward to more questions that remain to be answered.

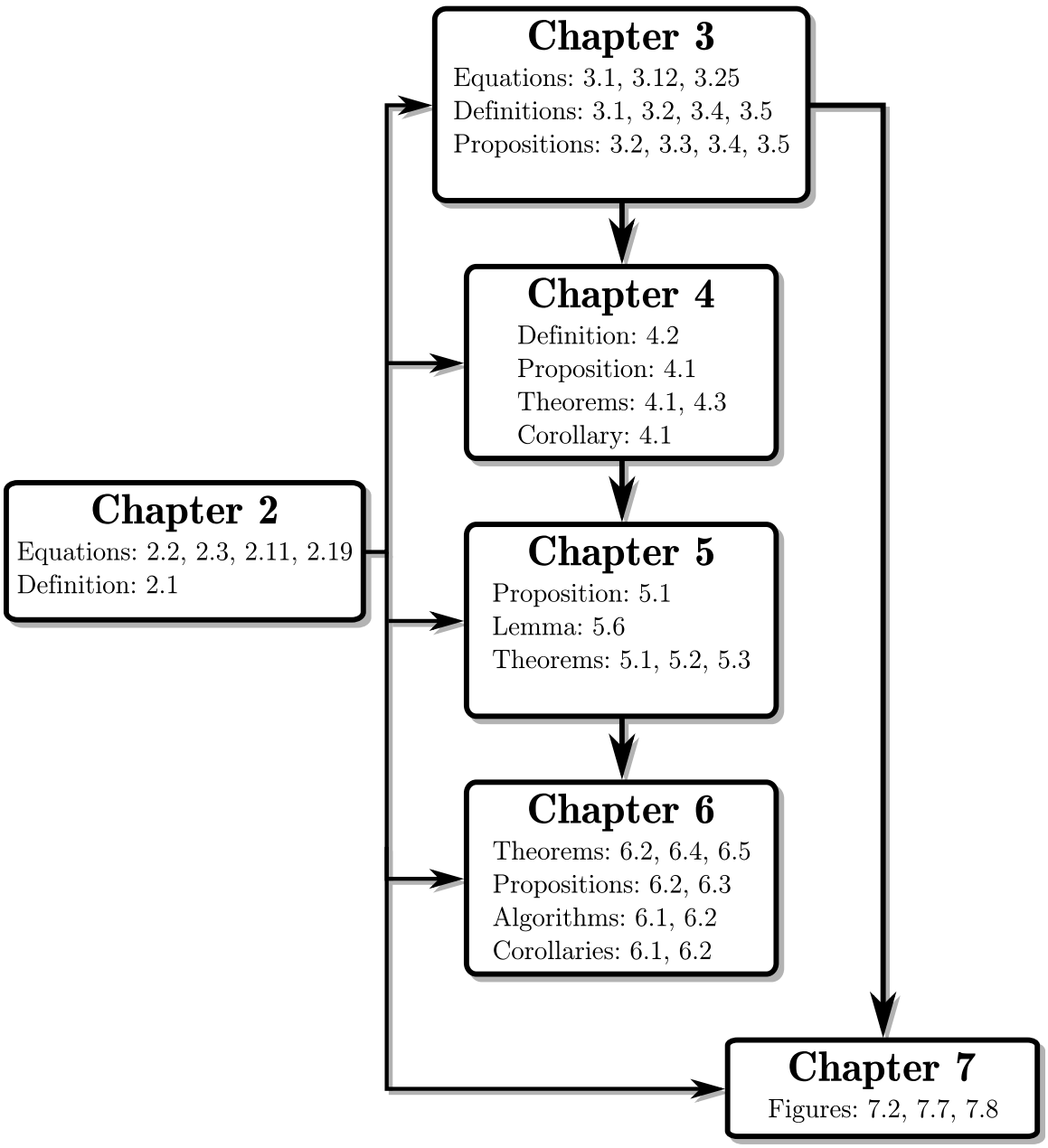

Figure 1.1: Organisation of the body of this dissertation. Significant definitions and results are listed for each chapter. An arrow between chapters indicates that the material in the earlier chapter is required for the later chapter.

Chapter 2

Background and Notation

In this chapter we outline some of the mathematical notation and definitions that will be used throughout this dissertation. In addition, we attempt to summarise some of the relevant details from the fields of work on which our contribution is based. Each of these fields is an important area of study in its own right, with many more applications than those we discuss. We make no attempt to be comprehensive in our treatment of them. Instead, we hope to introduce the tools that will be used throughout this dissertation and provide a brief glimpse of the extensive work that has preceded us.

2.1 Basic Notation and Definitions

In this dissertation, we will generally represent scalar quantities using lower case Roman or Greek letters. The only exceptions will be the capital letter N, which we use to denote the number of agents in a multi-agent system or the number of nodes in a graph, the capital letter W, which we use to denote a Wiener process, and (with a subscript f) the capital letter K_f, which we use to denote the Kirchhoff index of a graph. The set of real numbers is denoted by $\mathbb{R}$, the set of complex numbers by

$\mathbb{C}$, the set of integers by $\mathbb{Z}$ and the set of natural numbers[‡] by $\mathbb{N}$. The imaginary unit (i.e. the square root of -1) will be denoted by a non-italic i, the real part of a complex number $z \in \mathbb{C}$ will be denoted by $\mathrm{Re}\{z\}$ and the imaginary part of z by $\mathrm{Im}\{z\}$.

We will represent vectors (elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ for some $n \in \mathbb{N}$) using bold Roman or Greek letters. These will typically be lower case, with the exception of vector-valued Wiener processes that will be denoted by $\mathbf{W}$. Their scalar entries will be denoted with the same, non-bold, letter with a single subscript denoting the position of the entry in the vector. When a vector carries a subscript or superscript, its scalar entries will carry these additional sub- and superscripts as well. Thus the i^{th} entry of the vector $\mathbf{x}$ is x_i, while the i^{th} entry of the vector $\mathbf{x}_k$ is $x_{k\,i}$. We will use a fixed notation for several vectors that commonly occur in this dissertation. In particular, we use $\mathbf{e}_n^{(k)}$ to denote the k^{th} standard basis vector of $\mathbb{R}^n$. That is, $\mathbf{e}_n^{(k)}$ contains a zero in every position except the k^{th} position, which is a 1. In addition, we use $\mathbf{1}_n$ to denote the vector in $\mathbb{R}^n$ containing a 1 in every entry and $\mathbf{0}$ to denote the zero vector (with size inferred from context). For the purposes of calculation, vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ will be treated as matrices with n rows and one column. Thus a vector can be transposed to form a matrix with a single row.

Matrices will be represented using capital Roman or Greek letters[◇]. A matrix M can also be written $[m_{i,j}]$, where $m_{i,j}$ denotes the scalar entry in the i^{th} row and j^{th} column of M. A capital letter with an additional subscript or superscript, such as M_p, will still refer to a matrix, and the scalar entries of this matrix will carry the additional sub- and/or superscript(s) - i.e., $M_p = [m_{p\,i,j}]$. Entries of a matrix whose values are not relevant may be denoted by the symbol $\star$. The identity matrix in $\mathbb{R}^{n\times n}$ will be denoted by I_n, and a zero matrix (with dimensions inferred from context) will be denoted by $\emptyset$. We will use $\mathrm{diag}^{(k)}(\mathbf{v})$ to denote a k-diagonal matrix, with the

[‡]We will take 0 to be a natural number, and use $\mathbb{N}^+$ to denote the positive integers.

[◇]With the exceptions of N, W and K_f, which as described above have particular uses.

entries of $\mathbf{v}$ along the k^{th} diagonal[‡] and zeros elsewhere (and the dimensions inferred from the length of $\mathbf{v}$ and k). A diagonal matrix can be denoted by $\text{diag}(\mathbf{v})$.

Other mathematical objects (such as graphs or sets) will generally be denoted using calligraphic capital letters.

We use $\lambda_i(A)$ to refer to the i^{th} eigenvalue of the (square) matrix A, when arranged in order of ascending real part (and counting algebraic multiplicity). Similarly, $\sigma_i(B)$ refers to the i^{th} singular value of the matrix B when arranged in ascending order.

The transpose of a matrix will be denoted by the exponent T, while the Hermitian transpose will be denoted by the exponent $*$. The *trace* of the (square) matrix A is the sum of its diagonal elements and is denoted by $\text{tr}\,(A)$. One property of the trace function is that for an $n \times m$ matrix B and an $m \times n$ matrix C, $\text{tr}\,(BC) = \text{tr}\,(CB)$ [52, §1.2, Problem 2].

A *unitary matrix* is a square matrix $U \in \mathbb{C}^{n\times n}$ with an inverse equal to its Hermitian transpose, that is $U^*U = UU^* = I_n$. A real unitary matrix is called an *orthogonal matrix*. A square matrix $A \in \mathbb{R}^{n\times n}$ is called *normal* if it commutes with its transpose, that is $AA^T = A^TA$. Alternatively, a matrix is normal if and only if it has an orthonormal set of n eigenvectors in $\mathbb{C}^n$ (and hence can be diagonalised by a unitary matrix). This characterisation of normal matrices is known as the *Spectral Theorem* [52, Theorem 2.5.4].

An important identity involving matrix inverses can be derived by examining the inverse of a 2×2 block matrix. This identity is often referred to as the *Matrix Inversion Lemma*, and states that for matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$, for which A^{-1}, D^{-1} and $(A + BD^{-1}C)^{-1}$ all exist, [107]

$$\left(A + BD^{-1}C\right)^{-1} = A^{-1} - A^{-1}B\left(D + CA^{-1}B\right)^{-1}CA^{-1}. \tag{2.1}$$

[‡]The 0^{th} diagonal of a matrix is the main diagonal, from the top-left to bottom-right corners. Diagonals are numbered in increasing order above the main diagonal and in decreasing order below.

The subspace of $\mathbb{R}^n$ spanned by the vector $\mathbf{1}_n$ is referred to as the *consensus subspace*, since vectors in this subspace represent consensus states (see Definition 2.1 in §2.2). The orthogonal complement of this subspace, known as the *disagreement subspace*, is of particular interest in this dissertation. In the following chapters, we will often make use of particular matrices related to the disagreement subspace. We will make a slight abuse of notation and use $\mathbf{1}_n^\perp$ to denote the disagreement subspace, instead of $\text{span}\{\mathbf{1}_n\}^\perp$. Let

$$\Pi := I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T \tag{2.2}$$

denote the orthogonal projection matrix onto $\mathbf{1}_n^\perp$. Since Π is an orthogonal projection matrix (or by examining (2.2)), we can observe that it is symmetric.

It will often be instructive to characterise the action of a matrix restricted to the subspace $\mathbf{1}_n^\perp$. Suppose we choose an orthonormal basis for $\mathbf{1}_n^\perp$ and let $Q \in \mathbb{R}^{(n-1)\times n}$ be the matrix formed with these basis vectors as rows. This is equivalent to requiring that $\begin{bmatrix} \frac{1}{\sqrt{n}}\mathbf{1}_n & Q^T \end{bmatrix}$ is an orthogonal matrix, or more explicitly,

$$Q\mathbf{1}_n = \mathbf{0},\ QQ^T = I_{n-1} \text{ and } Q^TQ = \Pi. \tag{2.3}$$

Using these properties, it follows that $Q\Pi = Q$ and $\Pi Q^T = Q^T$. Then for any vector $\mathbf{v} \in \mathbb{R}^n$,

$$\overline{\mathbf{v}} := Q\mathbf{v} \tag{2.4}$$

is a coordinate vector (with respect to the chosen basis) of the orthogonal projection of $\mathbf{v}$ onto $\mathbf{1}_N^\perp$. Similarly, for any matrix $M \in \mathbb{R}^{n\times n}$,

$$\overline{M} := QMQ^T \tag{2.5}$$

is the $(n-1)\times(n-1)$ matrix whose action on $\mathbf{1}_n^\perp$ is identical to M, in the sense that $\overline{M}\overline{\mathbf{v}} = \overline{M\mathbf{v}}$ for any $\mathbf{v} \in \mathbf{1}_n^\perp$.

In Chapters 5 and 6, we make use of *binomial coefficients*, defined as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\; n, k \in \mathbb{N},\; 0 \leq k \leq n. \tag{2.6}$$

Note that by this definition, we can say

$$\binom{n}{0} = 1,\; \binom{n}{1} = n, \text{ and } \binom{n}{2} = \frac{n(n-1)}{2}.$$

2.2 Linear Consensus Dynamics

The fundamental dynamical system studied in this dissertation describes multiple individuals reaching consensus. To pose this formally, we consider a set of N agents in which each individual agent is a control system, with the goal of consensus being to drive every system to the same state. The most common approach in the analysis of consensus is to assume that each agent can be described by a scalar state, x_i for the i^{th} agent, and consider single-integrator control systems of the form[‡]

$$\dot{x}_i = u_i, \tag{2.7}$$

where u_i is the control signal for agent i [12, 70, 75, 85], although more complicated systems are sometimes analysed too [5, 42, 88, 98]. There are a number of reasons to focus on such a simple description. Firstly, any analysis of (2.7) can be easily extended to a system consisting of a k-dimensional state with a corresponding k-dimensional control signal. Next, this approach allows us to focus on the effects of communication without conflating the effects of individual system dynamics. In fact, more complicated control systems can often depend on network properties in a very similar way to this simple system [42]. Furthermore, if the state of the system is a

[‡]Or the equivalent discrete-time form of $x_i[k+1] = x_i[k] + \Delta t\, u_i[k]$.

"virtual" quantity, such as an estimate of an environmental parameter, the location of a target or some other decision variable, then there is no reason to impose any more complicated dynamics on each individual. Finally, even if the agents are trying to reach consensus on some physical state, such as a direction of travel, (2.7) can be used if we assume that each agent can directly (or at least quickly) control the derivative of this physical state. For example, if the state in question is a direction of travel, we assume that each agent can control their own angular velocity. This could be achieved if each agent runs a control loop on their angular velocity that operates at a faster time scale than the time scale on which the desired angular velocity changes. For systems such as these, the term consensus is formalised by the definition below. Note that according to our convention given in §2.1, we will use N to denote the number of agents in a system.

Definition 2.1. *A system containing N agents, each represented by a scalar state $x_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, N\}$, is* in consensus *if*

$$x_i = x_j \; \forall \; i, j \in \{1, 2, \ldots, N\} \, . \tag{2.8}$$

Equivalently, if the system is represented by the state vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T$, then the system is in consensus if

$$\mathbf{x} = \gamma \mathbf{1}_N, \tag{2.9}$$

for some $\gamma \in \mathbb{R}$.

The second key assumption in the study of consensus is that each agent is able to access the *relative* state values of some set of neighbours. That is, if agent i can access the state of agent j (through sensing, broadcast or other means), then agent i can only measure $x_j - x_i$, and not the absolute value of x_j. The distinction here may seem

semantic since if agent i knows x_i as well (which is always the case if x_i is a virtual quantity and presumably possible in almost any other case) then it can compute x_j. The distinction, however, lies in the fact that a reference point is needed to measure absolute values, while it is not needed for relative measurements. Thus agent i could compute state values for all of its neighbours, but these values will not necessarily agree with the neighbours' values of their own states. For example, if x_i represented a direction, agent i might measure directions relative to magnetic north, while agent j might have a biased compass and measure directions relative to some other heading. Nevertheless, consensus is still possible because the relative measurement of the two agents' directions is the same regardless of their respective reference points.

In order to reach consensus, each agent must choose a control law, u_i, that is a function of the relative measurements, $x_j - x_i$, to which it has access. The goal of the control law is to drive all individual states to the same value, although for *pure consensus* no conditions are placed on what this value should be. There are variations of the consensus problem, including *average-consensus* (convergence to the average of the agents' initial states), *min-consensus* (convergence to the minimum of the agents' initial states) and *max-consensus* (convergence to the maximum of the agents' initial states), in which the goal is to drive all individual states to some function of the initial states of each agent [75]. There are a number of *consensus protocols* that have been proposed, including nonlinear ones [103]. However, the most common is the *linear consensus protocol* [75, 85, 88, 101], which is defined as

$$u_i = \sum_{j \in \mathcal{N}_i} a_{i,j} \left(x_j - x_i \right), \tag{2.10}$$

where $\mathcal{N}_i$ is the set of neighbours of agent i and $a_{i,j}$ is a positive weight that agent i assigns to the information from agent j.

By combining the states of all agents into a single vector, $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^T$,

we can express (2.7) with control law given by (2.10) in matrix form as

$$\dot{\mathbf{x}} = -L\mathbf{x}, \tag{2.11}$$

where L is known as the Laplacian matrix of the communication graph (see §2.4). If the neighbours of any agent change over time, or if the weights chosen by any agent change, then the matrix L in (2.11) will be a function of time. Conversely, if all sets of neighbours and weights are fixed, L will be a constant matrix.

The linear consensus protocol is popular to study due to its simplicity and the fact that it allows any analysis to focus on the effects of the communication structure. Additionally, when all communication is undirected[‡], the linear consensus protocol can be found as the negative gradient of the associated positive-definite *Laplacian potential* [74],

$$\Psi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T L\mathbf{x} = \frac{1}{2}\sum_{i,j=1}^{N} a_{i,j}\left(x_j - x_i\right)^2,$$

where $a_{i,j} = 0$ if i and j are not neighbours. Thus, in this case, (2.11) is a gradient dynamical system. The Laplacian potential can be thought of as measuring the (weighted) disagreement between all pairs of neighbouring agents. This same potential function can be used to analyse a range of nonlinear consensus protocols [74], implying that any results from the linear protocol could be locally applied to these nonlinear cases as well.

Much of the early work in the study of consensus focussed on understanding the conditions under which (2.11) converges to consensus. Since (2.10) produces no control signals when the system is in consensus, it is clear that L (whether static or time-varying) will always have an eigenvalue of 0 associated with the eigenvector $\mathbf{1}_N$. When L is constant, (2.11) will then converge to a consensus state if every other eigenvalue of L has a positive real part [85]. This is precisely the case when the

[‡]That is, when i is a neighbour of j if and only if j is a neighbour of i and $a_{i,j} = a_{j,i}$.

corresponding (static) communication graph is connected (see §2.4). When L is a function of time, the system will have a corresponding time-varying communication graph (i.e. a communication graph associated with the system at each point in time) and the analysis becomes much more complicated. Various assumptions can be made about the time-varying nature of L - for example, that it is piecewise constant or periodic [70, 75, 85]. In general, however, to ensure consensus there must be infinitely many consecutive bounded time intervals such that the graph formed by the union of the communication graphs across any such interval is connected [85]. Since this dissertation focuses on static communication graphs, we will not present any more details of the time-varying case.

2.3 Stochastic Differential Equations

The concept of *noise* is ubiquitous in science and engineering, where many physical processes are affected by fundamentally random events (usually on a very small scale) or unknown/unknowable events (on almost any scale). In control systems, noise usually arises as a result of imperfect measurements, physical disturbances or unmodelled dynamics. One common notion in the modelling of noise is the *(Gaussian) white noise process* [3, 44]. This is understood to be a scalar, stationary, Gaussian process, $\xi(t)$, with zero mean and constant spectral density over the whole real axis[‡]. Using this idea, noise can be introduced to a control system as an additional input signal. For example, a state space system may be described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B_1\mathbf{u}(t) + B_2\boldsymbol{\xi}(t), \tag{2.12}$$

where $\boldsymbol{\xi}(t)$ is a vector-valued white noise process, with each element being an independent scalar white noise process $\xi_i(t)$. Unfortunately, equations of the form of (2.12)

[‡]That is, a white noise process contains equal power at every frequency.

present mathematical difficulties, which can sometimes be ignored without harm but actually demand a more careful approach.

To demonstrate some of the issues with the idealisation of white noise, we will first introduce a concept intimately related to the notion of white noise - the Wiener process[‡] W. The Wiener process is a mathematical concept that was created to describe Brownian motion in the absence of friction [3, §3.1]. Precise definitions of W may be found in [3, §2.3] and [44, §3.8.1], but intuitively, a Wiener process is a continuous Markov process[◇] with a Gaussian probability density function that has zero mean and variance growing linearly with time. Thus the value of a Wiener process at time t follows the probability density function

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}}.$$

Importantly, the Markov nature of the Wiener process means that increments of W are independent - that is, for times $t_1 < t_2 < t_3$, $W(t_1)$, $W(t_2) - W(t_1)$ and $W(t_3) - W(t_2)$ are independent random variables [44, §3.8.1]. Furthermore, the increments of W are stationary. That is, the distribution of $W(t_2) - W(t_1)$ depends only on $t_2 - t_1$ (it is, in fact, Gaussian with zero mean and variance of $t_2 - t_1$), and not on the particular values of t_1 or t_2. In fact, it is possible to define the Wiener process as a process that is almost certainly continuous, with independent and stationary increments that are Gaussian with zero mean and variance equal to the time interval of the increment [3, §3.1].

An analysis of the white noise process defined earlier shows that [3, Equation 3.2.3b]

$$\int_0^t \xi(\tau)\, d\tau = W(t), \tag{2.13}$$

[‡]Since the Wiener process is a function of time, it can also be denoted by $W(t)$ or W_t.

[◇]A process is *Markov* if, roughly speaking, knowing the entire past history of the process provides no more information about its future values than knowing just its present value.

which suggests that the white noise process should be the derivative of the Wiener process. Unfortunately, the Wiener process as described above has sample functions that are nowhere differentiable [44, §3.8.1]. Thus, white noise does not exist in the way that we usually understand functions or random processes to exist. Although it is possible to extend our notions of functions and random processes to include white noise, the larger issue is that any function $\mathbf{x}(t)$ which satisfies the integral form of (2.12) becomes nowhere differentiable, and so (2.12) is not a valid differential equation. However, the integral form remains valid - that is, we can find a continuous random process $\mathbf{x}(t)$ that satisfies

$$\mathbf{x}(t) - \mathbf{x}(0) = A \int_0^t \mathbf{x}(\tau)\, d\tau + B_1 \int_0^t \mathbf{u}(\tau)\, d\tau + B_2 \int_0^t \boldsymbol{\xi}(\tau)\, d\tau, \tag{2.14}$$

since by (2.13), the final integral can be interpreted as a vector with each entry being an independent Wiener process.

The desire to integrate white noise processes, as seen in (2.13) and (2.14), led to the development of the Ito calculus. First, we make the formal substitution

$$dW(t) := \xi(t)\, dt,$$

after which we can write a general integral of a white noise process as

$$\int_0^t b(\tau)\xi(\tau)\, d\tau = \int_0^t b(\tau)\, dW(\tau) = \int_0^t b(\tau)\, dW, \tag{2.15}$$

where the τ has been dropped from the final expression for convenience. Then, the Ito stochastic integral is defined as

$$\int_{t_1}^{t_2} b(\tau)\, dW := \underset{n\to\infty}{\text{ms-lim}} \left\{ \sum_{i=1}^{n} b(\tau_{i-1}) \left[W(\tau_i) - W(\tau_{i-1}) \right] \right\}, \tag{2.16}$$

where $\tau_0 = t_1 < \tau_1 < \tau_2 < \ldots < \tau_n = t_2$ and ms-lim stands for the mean-square limit [44, §4.2.2]. In an obvious way, these definitions can be easily extended to k-dimensional Wiener processes $\mathbf{W}$ in which each element is an independent scalar Wiener process.

The use of the Ito calculus allows equations like (2.14) to be written as

$$\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t A\mathbf{x}(\tau) + B_1\mathbf{u}(\tau)\, d\tau + \int_0^t B_2\, d\mathbf{W},$$

or equivalently as the *stochastic differential equation*

$$d\mathbf{x} = (A\mathbf{x} + B_1\mathbf{u})\, dt \; + B_2\, d\mathbf{W}\,.$$

More generally, a stochastic differential equation has the form

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\, dt \; + G(\mathbf{x}, t)\, d\mathbf{W}, \tag{2.17}$$

with $\mathbf{f}$ a vector-valued function of $\mathbf{x}$ and t, and G a matrix-valued function of $\mathbf{x}$ and t. Then (2.17) describes the evolution of a random process $\mathbf{x}(t)$ which satisfies

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{f}(\mathbf{x}, \tau)\, d\tau + \int_{t_0}^t G(\mathbf{x}, \tau)\, d\mathbf{W}. \tag{2.18}$$

Intuitively, (2.17) describes the increment in $\mathbf{x}$ for an infinitesimal increment in time and corresponding increment in a Wiener process.

Since stochastic differential equations describe random variables, any analysis of solutions to (2.17) should focus on statistical properties of the solutions, rather than sample solutions. The most complete way to do this is to study the probability density function of $\mathbf{x}(t)$, $p(\mathbf{x}, t)$, which satisfies the *Fokker-Planck Equation* [44, §4.3.5]. For

the scalar version of (2.17), the Fokker-Planck equation is

$$\frac{\partial}{\partial t}\, p(x,t) = -\frac{\partial}{\partial x}\left[f(x,t)\, p(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g(x,t)^2\, p(x,t)\right],$$

which has the form of an advection-diffusion equation. However, the Fokker-Planck equation becomes increasingly more complicated in higher dimensions and its solution becomes harder to analyse. An alternative approach is to study only some of the moments of $p(\mathbf{x}, t)$, such as the mean and covariance. This approach is particularly attractive when it is known that $\mathbf{x}(t)$ is a Gaussian process, and is thus completely characterised at any point in time by its mean and covariance.

One set of sufficient conditions for (2.17) to describe a Gaussian process is for $\mathbf{f}(\mathbf{x}, t)$ to be linear in $\mathbf{x}$, $G(\mathbf{x}, t)$ to be independent of $\mathbf{x}$ and $\mathbf{x}(t_0)$ to be normally distributed or constant [3, Theorem 8.2.10]. In this case, (2.17) can be written as

$$d\mathbf{x} = \left[A(t)\mathbf{x} + \mathbf{a}(t)\right] dt \; + B(t)\, d\mathbf{W}\,. \tag{2.19}$$

Then, if we let $\boldsymbol{\mu}_{\mathbf{x}}(t) := E\left[\mathbf{x}(t)\right]$ (where $E\left[\cdot\right]$ denotes the *expected value* operator) be the mean of $\mathbf{x}$ and $\Sigma_{\mathbf{x}}(t) := E\left[\left(\mathbf{x}(t) - \boldsymbol{\mu}_{\mathbf{x}}(t)\right)\left(\mathbf{x}(t) - \boldsymbol{\mu}_{\mathbf{x}}(t)\right)^T\right]$ be its covariance, we can say by [3, Theorem 8.2.6] that $\boldsymbol{\mu}_{\mathbf{x}}$ is a deterministic function of time that satisfies the differential equation

$$\dot{\boldsymbol{\mu}}_{\mathbf{x}}(t) = A(t)\boldsymbol{\mu}_{\mathbf{x}}(t) + \mathbf{a}(t), \tag{2.20}$$

and $\Sigma_{\mathbf{x}}(t)$ is also a deterministic function of time that satisfies

$$\dot{\Sigma}_{\mathbf{x}}(t) = A(t)\Sigma_{\mathbf{x}}(t) + \Sigma_{\mathbf{x}}(t)A(t)^T + B(t)B(t)^T. \tag{2.21}$$

Thus for stochastic differential equations of the form of (2.19) (and with deterministic or Gaussian initial conditions) we can completely characterise their behaviour by studying (2.20) and (2.21).

2.4 Directed Graph Theory

When studying the effects of communication structure on collective behaviour, it is natural to represent the communication network as a graph. Individual agents can be represented by nodes and communication links by edges. In this way, a graph can provide a complete description of the essential elements of the communication between agents and allow for a formal analysis. Thus, the study of the effects of communication structure can be reduced to understanding properties of graphs, for which a rich literature already exists. In this section we define the terminology and concepts of graph theory that will be useful in our study of robust consensus.

There are competing definitions for many of the basic concepts of graph theory, mainly due to varying scope amongst authors. For our purposes, we restrict our attention to directed graphs as they may arise in control theory, and so our definitions mostly follow [75], with some taken from [6]. Two notable exceptions are our definition of connectivity, which falls between the standard notions of weak and strong connectivity but is more applicable to control over graphs [88, 114], and our definition of *connections* in directed graphs, which have interesting parallels to paths in undirected graphs.

A *graph* (also *directed graph* or *digraph*) $\mathcal{G}$ consists of the triple $(\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and $A \in \mathbb{R}^{N \times N}$ is a weighted adjacency matrix with non-negative entries $a_{i,j}$. Each $a_{i,j}$ will be positive if and only if $(i, j) \in \mathcal{E}$, otherwise $a_{i,j} = 0$. The edge (i, j) is referred to as an edge *from* node i *to* node j. Note that by viewing $\mathcal{E}$ as a subset of $\mathcal{V} \times \mathcal{V}$, $\mathcal{G}$ can contain at most one edge between any ordered pair of nodes. In addition, we restrict our attention to graphs which do not contain any self-cycles (edges connecting a node to itself).

Two graphs, $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, A_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, A_2)$, are *isomorphic* if there is a bijection $f : \mathcal{V}_1 \to \mathcal{V}_2$ such that $(i, j) \in \mathcal{E}_1$ with weight $a_{i,j}$ if and only if $(f(i), f(j)) \in \mathcal{E}_2$ with weight $a_{f(i),f(j)} = a_{i,j}$. That is, two graphs are isomorphic if

one can be constructed from the other by relabelling the nodes. In most applications of graph theory (including this dissertation), the labelling of the nodes is not significant and so we consider any isomorphic graphs to be the same graph.

The graph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s, A_s)$ is said to be a *subgraph* of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ if there is an injection $f : \mathcal{V}_s \to \mathcal{V}$ such that if $(i, j) \in \mathcal{E}_s$ with weight $a_{i,j}$ then $(f(i), f(j)) \in \mathcal{E}$ with weight $a_{f(i),f(j)} = a_{i,j}$. Thus a subgraph of $\mathcal{G}$ contains some of the nodes of $\mathcal{G}$ and some (or all) of the edges in $\mathcal{G}$ between these nodes, with edge weights equal to those in $\mathcal{G}$.

The graph $\mathcal{G}$ is said to be *undirected* if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ and $a_{i,j} = a_{j,i}$. Thus, a graph will be undirected if and only if its adjacency matrix is symmetric. We use the term *undirected edge* to refer to a pair of edges between two nodes (one in each direction), with equal weights.

Every directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ has an *underlying undirected graph* $\mathcal{G}_u = (\mathcal{V}_u, \mathcal{E}_u, A_u)$ (also called the *mirror graph* [75, 83]), with $\mathcal{V}_u = \mathcal{V}$, $\mathcal{E}_u$ such that $(i, j) \in \mathcal{E} \Rightarrow (i, j)$ and $(j, i) \in \mathcal{E}_u$ and $A_u = \frac{1}{2}\left(A + A^T\right)$. Conceptually, the underlying undirected graph can be found by first replacing every edge (i, j) with the two edges (i, j) and (j, i), each with weight $\frac{a_{i,j}}{2}$. Then, if this produces multiple edges between any pair of nodes, the edges are combined by summing their weights.

A graph can be drawn by representing each node with a distinct, non-overlapping circle and representing each edge (i, j) by a line joining the circles for nodes i and j. The direction of the edge is indicated by adding an arrow to the line pointing to the circle for node j. The edge weight can be written adjacent to the line representing an edge. If no weight is written next to an edge, it is assumed that the edge weight is 1. An undirected edge (corresponding to $(i, j) \in \mathcal{E}$, $(j, i) \in \mathcal{E}$ and $a_{i,j} = a_{j,i}$) can be represented by a single line either without any arrows or with arrows in both directions, and with the single edge weight written next to the line. A drawing of a directed graph is shown in Figure 2.1.

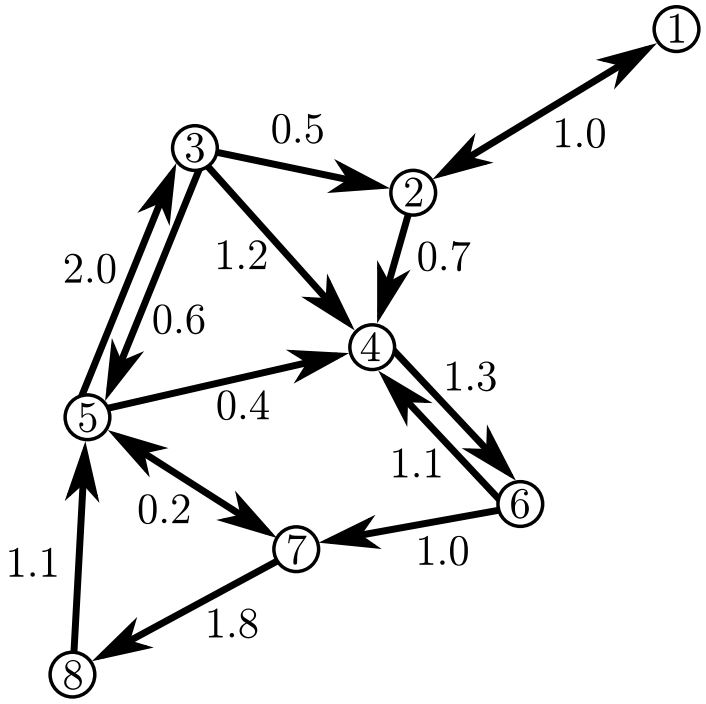

Figure 2.1: An example of a directed graph with edge weights indicated. This graph contains pairs of nodes with no edges between them (such as $\{1, 8\}$), pairs of nodes with a single directed edge between them (such as $\{2, 3\}$), pairs of nodes with two directed edges between them (such as $\{4, 6\}$) and pairs of nodes with an undirected edge between them (such as $\{5, 7\}$).

The *out-degree* (respectively *in-degree*) of node k is defined as

$$d_k^{out} = \sum_{j=1}^{N} a_{k,j}$$

(respectively $d_k^{in} = \sum_{j=1}^{N} a_{j,k}$). There are two competing definitions of out-degree or in-degree in the graph theory literature, with some authors using the terms to mean the *number* of edges leaving or incident to a node [6, §1.2], and other authors using the definition we gave above [75, 108]. These two definitions coincide for unweighted graphs (i.e. graphs with all edge weights equal to 1) but will not, in general, be equal for graphs with edge weights other than 1. For the purposes of this dissertation, the sum of the edge weights is a more useful concept of degree. A graph is said to be *balanced* if for every node, the out-degree and in-degree are equal. For balanced

graphs (including all undirected graphs), the term *degree* is used to refer to both the out-degree and in-degree.

$\mathcal{G}$ has an associated *Laplacian* matrix L, defined by $L = D - A$, where D is the diagonal matrix of node out-degrees, that is $D = \operatorname{diag}\left(\begin{bmatrix} d_1^{out} & d_2^{out} & \cdots & d_N^{out} \end{bmatrix}^T\right) = \operatorname{diag}(A\mathbf{1}_N)$. By construction, the row sums of the Laplacian matrix are always zero, that is $L\mathbf{1}_N = \mathbf{0}$. Thus 0 is always an eigenvalue of L with corresponding eigenvector $\mathbf{1}_N$. It can be shown that all the eigenvalues of L are either 0 or have positive real part [1]. A graph will be undirected if and only if its Laplacian matrix is symmetric, and then all the eigenvalues of L will be real and non-negative. Balanced graphs have the particular properties that $1_N^T L = 0$ and, as the Laplacian is then a hyperdominant matrix with zero excess, $L + L^T \geq 0$ [106, Theorem 3.7].

To allow for meaningful comparisons between directed communication graphs, we can *normalise* a graph by scaling the edge weights so that each node has an out-degree of either 1 or 0. That is, for every non-zero $a_{i,j}$, we replace $a_{i,j}$ with $\frac{1}{d_i^{out}} a_{i,j}$. This notion is similar to the way graph Laplacians are defined in [42, 60] and edge weights chosen in [65]. Physically, this corresponds to each agent in our system taking as an input a weighted average of the differences between its own variable and those of its neighbours. Applying this procedure to an undirected graph may result in a directed graph since $a_{i,j}$ and $a_{j,i}$ could be scaled by different amounts. Conversely, the underlying undirected graph of a normalised directed graph will not necessarily be normalised. However, it is straightforward to see that the trace of the Laplacian of a directed graph will be equal to the trace of the Laplacian of its underlying undirected graph, and so the two graphs will have eigenvalues limited to the same range. Thus normalised directed graphs may be meaningfully compared to the underlying undirected graphs of normalised digraphs.

Since every Laplacian matrix shares the common zero eigenvector $\mathbf{1}_N$, $L\Pi = L$ and $\Pi L^T = L^T$ for any graph (where Π is the matrix defined in (2.2)). Furthermore,

$\Pi L = L$ for any balanced graph (including every undirected graph).

The *set of neighbours* of node k, denoted $\mathcal{N}_k$, is the set of nodes j for which the edge $(k, j) \in \mathcal{E}$. Since the set of neighbours depends on the direction of edges, it is possible for node i to be a neighbour of node j and j to not be a neighbour of node i.

A *path* in $\mathcal{G}$ is a (finite) sequence of nodes such that each node is a neighbour of the previous one. The edges in $\mathcal{G}$ between each consecutive pair of nodes in the path are considered to be the edges traversed by the path. A path is called *simple* if no internal nodes (i.e. other than the initial or final nodes) are repeated. The *length* of a path is the number of edges traversed. Thus a single node is considered to be a path of length 0. If a path exists in the graph $\mathcal{G}$ from node i to node j, node j is said to be *reachable* from node i. A *cycle* in $\mathcal{G}$ is a non-trivial closed path. That is, a cycle is a path of length greater than zero in which the initial and final nodes are the same. A *simple cycle* is a non-trivial closed simple path. Since we are only considering graphs which do not contain self-cycles, the minimum length of a cycle is two.

An *undirected path* in $\mathcal{G}$ is a path such that there is an undirected edge in $\mathcal{G}$ between each consecutive pair of nodes in the path. As above, an undirected path is simple if no internal nodes are repeated. Similarly, an *undirected cycle* is a non-trivial closed undirected path. A *simple undirected cycle* is a non-trivial closed simple undirected path that does not traverse any undirected edge more than once. Thus the minimum length of a simple undirected cycle is three.

We define a *connection* in $\mathcal{G}$ between nodes k and j to consist of two paths, one starting at k and the other at j and which both terminate at the same node. A *direct connection* between nodes k and j is a connection in which one path is trivial (i.e. either only node k or only node j) - thus a direct connection is equivalent to a path. Conversely, an *indirect connection* is one in which the terminal node of the two paths is neither node k nor node j. Examples of direct and indirect connections are shown in Figure 2.2. A *simple connection* is a connection that consists of two simple

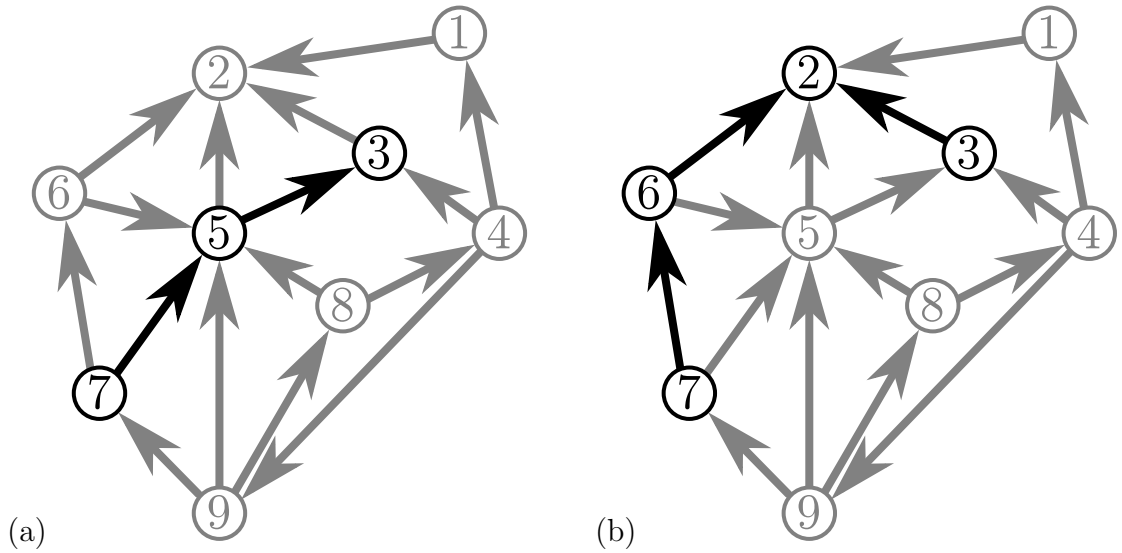

Figure 2.2: A directed graph on 9 nodes with: (a) a direct connection (i.e. a path) from node 7 to node 3 highlighted, and (b) an indirect connection between nodes 3 and 7 (i.e. a path from node 7 to node 2 and a path from node 3 to node 2) highlighted.

paths. Note that in both a connection and a simple connection, multiple nodes may be common between the two paths.

The graph $\mathcal{G}$ is *connected* if it contains a globally reachable node k; i.e. there exists a node k such that there is a path in $\mathcal{G}$ from i to k for every node i. Equivalently, $\mathcal{G}$ is connected if and only if every pair of nodes has a connection between them. 0 will be a simple eigenvalue of L if and only if $\mathcal{G}$ is connected [1]. An undirected graph is connected if and only if every pair of nodes has an undirected path between them.

Other notions of connectivity also exist for directed graphs. A graph is *strongly connected* if every ordered pair of nodes has a path between them. Every node in a strongly connected graph is globally reachable, and thus a strongly connected graph is also connected. A graph is *weakly connected* if its underlying undirected graph is connected. In the underlying undirected graph of a connected graph there will be a path from any node to the globally reachable node and a path from the globally reachable node to any other node. Thus a connected graph is also weakly connected.

A *complete graph* on N nodes is a graph containing every possible edge.

A directed (respectively, undirected) *path graph* on N nodes is a graph containing exactly $N-1$ directed (respectively, undirected) edges that admits a simple (respectively, simple undirected) path of length $N-1$ containing every node in the graph. The underlying undirected graph of a directed path graph is an undirected path graph.

A directed (respectively, undirected) *cycle graph* on N nodes is a graph containing exactly N directed (respectively, undirected) edges that admits a simple (respectively, simple undirected) cycle of length N containing every node in the graph. The underlying undirected graph of a directed cycle graph is an undirected cycle graph[‡].

A directed (respectively, undirected) *tree* is a connected graph on N nodes that contains exactly $N-1$ directed (respectively, undirected) edges. The underlying undirected graph of a directed tree is an undirected tree. A *leaf* in a directed tree is any node with zero in-degree, and a leaf in an undirected tree is any node with only one neighbour. The *root* of a directed tree is a node with zero out-degree; note that every directed tree will contain precisely one such node. A *branch* of a directed tree is a path from a leaf to the root. The *diameter* of an undirected tree is the length of the longest simple path in the tree. The *depth* of a directed tree is the length of the longest branch.

In a directed tree, every node other than the root has one outgoing edge, and thus one neighbour. The neighbour of node i is called its *parent*, and any nodes which have node i as their parent are the *children* of node i. Two nodes that share the same parent are said to be *siblings*. The *depth of a node* in a directed tree is the length of the path between that node and the root.

A directed *star graph* (also called an *imploding star* graph) is a directed tree with depth equal to one. An undirected star graph is an undirected tree with diame-

[‡]Except for a directed cycle on 2 nodes, which has an undirected path as its underlying undirected graph

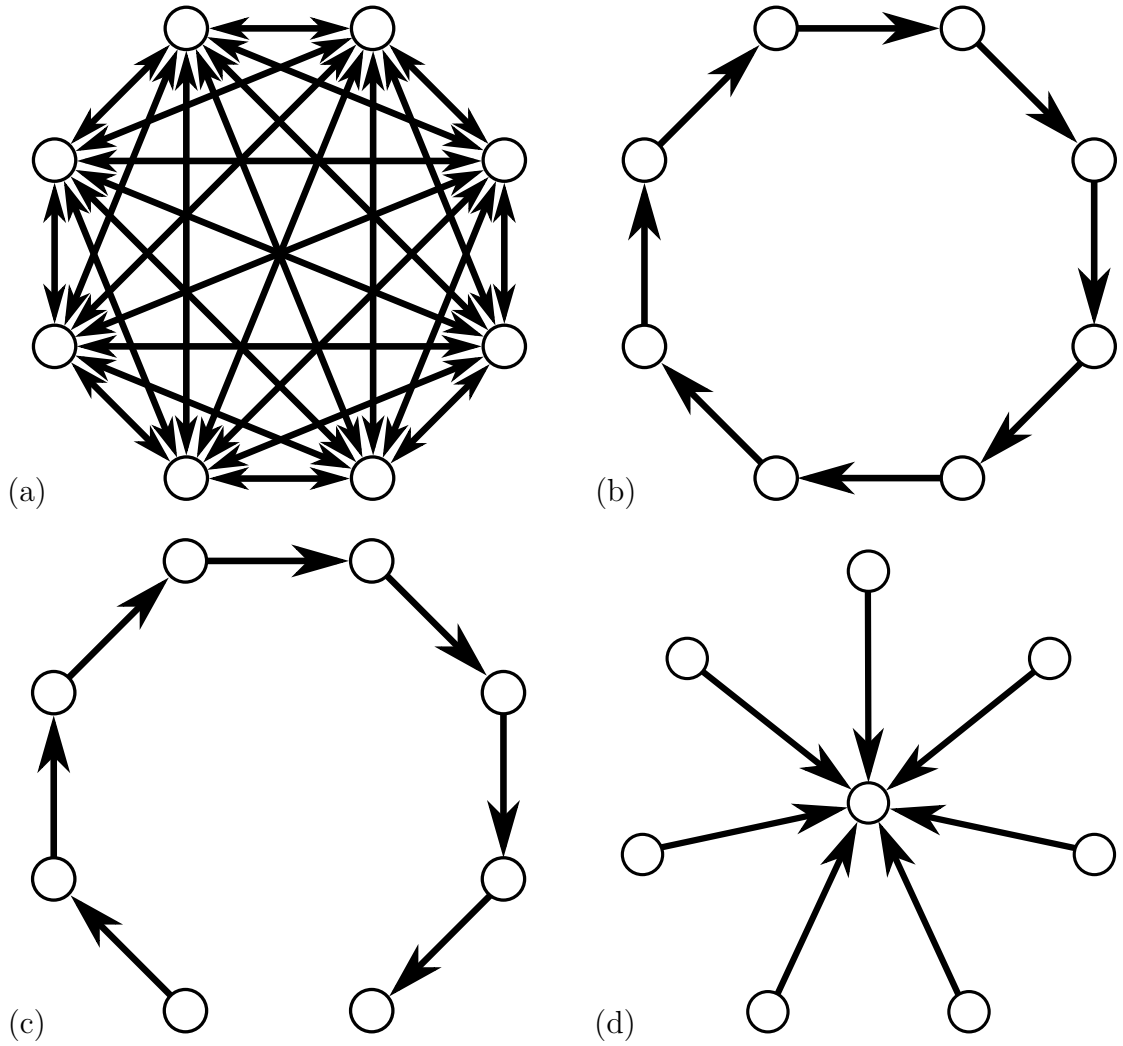

Figure 2.3: Examples of 8-node graphs from different graph families: (a) a complete graph, (b) a directed cycle graph, (c) a directed path graph and (d) a directed star graph.

ter equal to two. The underlying undirected graph of a directed star graph is an undirected star graph.

Some examples from the families of graphs described above are shown in Figure 2.3, and several examples of tree graphs are shown in §6.2.

Chapter 3

Robustness of Consensus to Noise

In this chapter we study the robustness of consensus to communication noise for directed communication topologies, which we show is naturally characterised by a $\mathcal{H}_2$ norm related to the system, as defined in §3.2 below. In this context, the $\mathcal{H}_2$ norm measures the expected steady-state dispersion of the agents under unit-intensity white noise. Thus systems with lower $\mathcal{H}_2$ norms will remain closer to consensus despite the presence of noise. It should be noted that we do not consider the "accuracy" of the final consensus value, merely how well it is maintained. We proceed to examine how this $\mathcal{H}_2$ norm depends on the eigenvalues of the graph Laplacian and another related matrix, providing cases where the $\mathcal{H}_2$ norm can be computed from the eigenvalues and others where it can be bounded by them. We conclude the chapter with an examination of the properties of several basic graph families. Some results in §3.1, §3.2.1, §3.2.2 and §3.3 have been published in [114].

3.1 Noisy Consensus Dynamics

In a consensus system, noise can be introduced to each agent through communication errors, spurious measurements, external disturbances and other means. As with any control system, the resulting dynamics depend on where the noise enters the system.

For example, measurement or communication errors could lead to uncertainty in the $x_j - x_i$ terms of (2.10). Alternatively, external disturbances, errors in control implementation (such as discretisation errors) or unmodelled dynamics can lead to uncertainty in either the $\dot{x}_i$ or u_i terms of (2.7). This type of uncertainty is particularly prevalent when the consensus variables x_i represent physical states of the agents, and it is this situation that we focus on in this dissertation. Then, to examine the effects of uncertainty on consensus, noise must be added to (2.11). In general, the amount of noise could vary between agents, either due to different environmental conditions or some agents being more "capable" than others (e.g. by having higher quality actuators, faster processing capacity or more robust internal control loops). However, if we assume that every agent is identical and they are all experiencing the same environment, it is reasonable to assume that all agents experience the same amount of noise. Therefore, we will assume that every agent is independently affected by additive white noise of the same intensity.

Following the discussion in §2.3, the addition of independent and identical noise to (2.11) leads to the following stochastic differential equation

$$d\mathbf{x}(t) = -L\mathbf{x}(t)\,dt\, + \sigma I_N\, d\mathbf{W}, \tag{3.1}$$

where $d\mathbf{W}$ represents an N-dimensional Wiener increment and σ is the intensity of the noise. For the rest of this dissertation, we will assume (unless otherwise specified) that the communication graph of the system is fixed, and hence that L does not change with time.

If we let $\boldsymbol{\mu}_{\mathbf{x}}(t) := E\left[\mathbf{x}(t)\right]$ and $\Sigma_{\mathbf{x}}(t) := E\left[\left(\mathbf{x}(t) - \boldsymbol{\mu}_{\mathbf{x}}(t)\right)\left(\mathbf{x}(t) - \boldsymbol{\mu}_{\mathbf{x}}(t)\right)^T\right]$ be the mean and covariance of $\mathbf{x}(t)$, we know from (2.20) and (2.21) that

$$\dot{\boldsymbol{\mu}}_{\mathbf{x}}(t) = -L\boldsymbol{\mu}_{\mathbf{x}}(t) \text{ and}$$

$$\dot{\Sigma}_{\mathbf{x}}(t) = -L\Sigma_{\mathbf{x}}(t) - \Sigma_{\mathbf{x}}(t)L^T + \sigma^2 I_N.$$

Therefore, following the discussion in §2.2, the mean of $\mathbf{x}(t)$ will be guaranteed to converge to consensus if and only if the communication graph is connected. However, we know that L has a zero eigenvalue and if we let $\mathbf{u}$ be the normalised left eigenvector of L with eigenvalue 0 (that is, $\mathbf{u}^T L = \mathbf{0}^T$ and $\mathbf{u}^T\mathbf{u} = 1$), we observe that

$$\frac{d}{dt}\left(\mathbf{u}^T\Sigma_{\mathbf{x}}(t)\mathbf{u}\right) = \sigma^2.$$

Thus the covariance matrix of $\mathbf{x}(t)$ grows unbounded with time.

Since L always has a zero eigenvector of $\mathbf{1}_N$, we can gain additional insight by examining (3.1) on the disagreement subspace, $\mathbf{1}_N^{\perp}$. To do this, we first define the average and disagreement of our system.

Definition 3.1. *Let* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T$ *be the state of a consensus system containing* N *agents. Then the* average *system state is*

$$x^{ave} := \frac{1}{N}\mathbf{1}_N^T\mathbf{x}, \tag{3.2}$$

and the disagreement *state is*

$$\mathbf{y} := Q\mathbf{x}, \tag{3.3}$$

where $Q \in \mathbb{R}^{(N-1)\times N}$ *is a matrix whose rows form an orthonormal basis for the disagreement subspace. Thus* Q *is a matrix satisfying (2.3).*

To understand why we call $\mathbf{y}$ the disagreement state, we can interpret (3.3) as stating that $\mathbf{y}$ is a coordinate vector of the orthogonal projection of $\mathbf{x}$ onto $\mathbf{1}_N^{\perp}$. Using (2.3) and (2.2), we observe that $Q^T\mathbf{y}$ (the disagreement state expressed in terms of the standard basis on $\mathbb{R}^N$) is orthogonal to $\mathbf{1}_N$,

$$\mathbf{x} = x^{\text{ave}}\mathbf{1}_N + Q^T\mathbf{y}, \tag{3.4}$$

and

$$\mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{x} = x^{\text{ave}}\mathbf{1}_N. \tag{3.5}$$

Next, we can define the reduced Laplacian matrix, as an $(N-1)\times(N-1)$ matrix whose action on the disagreement subspace is identical to that of L.

Definition 3.2. *The* reduced Laplacian *matrix, $\overline{L}$, of a graph, $\mathcal{G}$ is*

$$\overline{L} := QLQ^T, \tag{3.6}$$

where L is the Laplacian matrix of $\mathcal{G}$ and $Q \in \mathbb{R}^{(N-1)\times N}$ satisfies (2.3).

Then we can use (3.2), (3.3) and (3.4) to rewrite (3.1) as the system of stochastic differential equations

$$dx^{\text{ave}}(t) = -\frac{1}{N}\mathbf{1}_N^T LQ^T\mathbf{y}(t)\,dt + \frac{\sigma}{N}\mathbf{1}_N^T\,d\mathbf{W} \tag{3.7}$$

$$d\mathbf{y}(t) = -\overline{L}\mathbf{y}(t)\,dt + \sigma Q\,d\mathbf{W}\,. \tag{3.8}$$

In this way we observe that the dynamics of the disagreement state are decoupled from the dynamics of the average state (although for unbalanced graphs in which $\mathbf{1}_N^T L \neq \mathbf{0}$, the dynamics of the average state depend on the disagreement state).

Using the properties of $\mathbf{y}$ noted above, we can measure the distance of the system from consensus using the length of $\mathbf{y}$.

Definition 3.3. *Let* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T$ *be the state of a consensus system containing N agents, and $\mathbf{y}$ be the disagreement state as given by (3.3). The* dispersion *of the system, δ is*

$$\delta := \|\mathbf{y}\|_2 = \left(\mathbf{y}^T\mathbf{y}\right)^{\frac{1}{2}}. \tag{3.9}$$

Before proceeding further, we note that $\mathbf{y}$ and $\overline{L}$ are not unique, since we can compute them using any matrix Q that satisfies (2.3). However, if Q and Q' both

satisfy (2.3), we can define $W := Q'Q^T$. Then $Q' = WQ$ and W is orthogonal. Therefore, if $\mathbf{y}' := Q'\mathbf{x} = W\mathbf{y}$, we have that $\mathbf{y}'^T\mathbf{y}' = \mathbf{y}^T W^T W\mathbf{y} = \mathbf{y}^T\mathbf{y}$ and thus the dispersion is invariant to the choice of Q.

In order to examine the dynamics of the disagreement state, (3.8), we first determine how the eigenvalues of $\overline{L}$ are related to those of L.

Lemma 3.1. *$\overline{L}$ has the same eigenvalues as L except for a single zero eigenvalue.*

Proof. Let us define the matrix

$$V := \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^T \\ Q \end{bmatrix}.$$

Then V is orthogonal (i.e. $V^{-1} = V^T$), since

$$VV^T = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^T \\ Q \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & Q^T \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_{N-1} \end{bmatrix} = I_N, \text{ and}$$

$$V^TV = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & Q^T \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^T \\ Q \end{bmatrix} = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T + I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T = I_N.$$

Since the eigenvalues of a matrix are unchanged under a similarity transform [52, Corollary 1.3.4], the eigenvalues of L and VLV^T are the same. In particular, we have

$$VLV^T = \begin{bmatrix} QLQ^T & \emptyset \\ \frac{1}{\sqrt{N}}\mathbf{1}_N^T LQ^T & 0 \end{bmatrix} = \begin{bmatrix} \overline{L} & \emptyset \\ \frac{1}{\sqrt{N}}\mathbf{1}_N^T LQ^T & 0 \end{bmatrix}, \tag{3.10}$$

where we used the fact that $L\mathbf{1}_N = \mathbf{0}$. Since (3.10) is a block matrix, the eigenvalues of L are the solutions of

$$\lambda \det\left(\lambda I_{N-1} - \overline{L}\right) = 0.$$

We conclude that $\overline{L}$ has the same eigenvalues of L except for a single zero eigenvalue.

□

Corollary 3.1. *The reduced Laplacian matrix, $\overline{L}$, of a graph, $\mathcal{G}$, is invertible if and only if $\mathcal{G}$ is connected.*

Proof. We know that 0 is a simple eigenvalue of the Laplacian matrix L if and only if $\mathcal{G}$ is connected [1]. Thus by Lemma 3.1, $\overline{L}$ will have no 0 eigenvalues precisely when $\mathcal{G}$ is connected. □

Lemma 3.1 can be used to make the following observation about the disagreement state dynamics.

Proposition 3.1. *The disagreement state of a consensus system converges to a stationary distribution if and only if the communication graph is connected. In this case, the stationary distribution is Gaussian with zero mean and covariance matrix*

$$\Sigma_{stationary} := \lim_{t\to\infty} E\left[\mathbf{y}(t)\mathbf{y}(t)^T\right] =: \sigma^2\Sigma, \tag{3.11}$$

where Σ is the unique positive-definite solution of the Lyapunov equation

$$\overline{L}\Sigma + \Sigma\overline{L}^T = I_{N-1}. \tag{3.12}$$

Proof. We know that all the eigenvalues of L are either 0 or have positive real parts, and 0 is a simple eigenvalue of L if and only if the corresponding graph is connected [1]. Thus by Lemma 3.1, $-\overline{L}$ will be Hurwitz[‡] precisely when the communication graph is connected.

Now, if we let $\boldsymbol{\mu}_{\mathbf{y}}(t) := E\left[\mathbf{y}(t)\right]$ and $\Sigma_{\mathbf{y}}(t) := E\left[\left(\mathbf{y}(t) - \boldsymbol{\mu}_{\mathbf{y}}(t)\right)\left(\mathbf{y}(t) - \boldsymbol{\mu}_{\mathbf{y}}(t)\right)^T\right]$ be the mean and covariance of $\mathbf{y}(t)$, we know by [3, Theorem 8.2.10] that $\mathbf{y}(t)$ will be a Gaussian process with

$$\dot{\boldsymbol{\mu}}_{\mathbf{y}}(t) = -\overline{L}\boldsymbol{\mu}_{\mathbf{y}}(t) \text{ and} \tag{3.13}$$

[‡]A matrix is *Hurwitz* if every eigenvalue has negative real part.

$$\dot{\Sigma}_{\mathbf{y}}(t) = -\overline{L}\Sigma_{\mathbf{y}}(t) - \Sigma_{\mathbf{y}}(t)\overline{L}^T + \sigma^2 I_{N-1}. \tag{3.14}$$

Now, suppose that the communication graph is not connected, and so $-\overline{L}$ is not Hurwitz. Then $\overline{L}$ must have a zero eigenvalue, with corresponding normalised left eigenvector $\hat{\mathbf{u}}$. In this case, we can observe from (3.14) that

$$\frac{d}{dt}\left(\hat{\mathbf{u}}^T\Sigma_{\mathbf{y}}(t)\hat{\mathbf{u}}\right) = \sigma^2.$$

Thus that the covariance matrix of $\mathbf{y}(t)$ grows unbounded with time and hence cannot converge.

Next, suppose that the communication graph is connected, and so $-\overline{L}$ is Hurwitz. Since the solution of (3.13) is

$$\boldsymbol{\mu}_{\mathbf{y}}(t) = \mathrm{e}^{-\overline{L}t}\boldsymbol{\mu}_{\mathbf{y}}(0),$$

and $-\overline{L}$ is Hurwitz, we can say that

$$\lim_{t\to\infty} \boldsymbol{\mu}_{\mathbf{y}}(t) = \mathbf{0}. \tag{3.15}$$

Therefore, if the two limits exist, we observe that

$$\Sigma_{\text{stationary}} = \lim_{t\to\infty} \Sigma_{\mathbf{y}}(t).$$

Next, we can write the solution of (3.14) as

$$\Sigma_{\mathbf{y}}(t) = \sigma^2 \int_0^t \mathrm{e}^{-\overline{L}(t-\tau)}\mathrm{e}^{-\overline{L}^T(t-\tau)}\, d\tau + \mathrm{e}^{-\overline{L}t}\Sigma_{\mathbf{y}}(0)\mathrm{e}^{-\overline{L}^T t}.$$

Since $-\overline{L}$ is Hurwitz, we observe that

$$\lim_{t\to\infty} \mathrm{e}^{-\overline{L}t}\Sigma_{\mathbf{y}}(0)\mathrm{e}^{-\overline{L}^T t} = \emptyset\,,$$

and thus (after a change of variables in the integral),

$$\lim_{t\to\infty} \Sigma_{\mathbf{y}}(t) = \Sigma_{\text{stationary}} = \sigma^2 \int_0^\infty \mathrm{e}^{-\overline{L}t}\mathrm{e}^{-\overline{L}^T t}\,dt. \tag{3.16}$$

Using the fact that $-\overline{L}$ is Hurwitz once again, we know that this integral converges to the unique positive definite solution Σ to the Lyapunov equation (3.12) [37, Theorem 4.1].

Therefore, since $\mathbf{y}(t)$ is a Gaussian process with a mean and covariance matrix that converge to constant values, it must converge to a stationary Gaussian distribution with these parameters. □

Remark 3.1. *From the proof of Proposition 3.1, the rate of convergence of* $\mathbf{y}$ *to its stationary distribution for a connected graph will clearly be governed by the rate at which* $\mathrm{e}^{-\overline{L}t}$ *converges to* $\emptyset$*. This in turn depends on the real parts of the eigenvalues of* $-\overline{L}$ *and so will be limited by the eigenvalue whose exponential decays at the slowest rate - that is, the eigenvalue of* $\overline{L}$ *with smallest real part. By Lemma 3.1, this is the eigenvalue of* L *with second-smallest real part, or* $\lambda_2(L)$*. Furthermore, we know that when the graph is disconnected and no stationary distribution is reached,* $\lambda_2(L) = 0$*. Therefore, we will refer to the real part of* $\lambda_2(L)$ *as the* convergence speed *of the graph. Note that by (2.11), this also represents the speed at which a deterministic consensus system on the same graph converges to consensus. For undirected graphs, this corresponds to the standard definition of the* algebraic connectivity *of the graph. It is worth noting that the definition of algebraic connectivity for directed graphs, as given in Definition 3.6 below, does not correspond to the speed of convergence.*

Proposition 3.1 tells us that for noisy consensus systems with connected communication graphs, the system state will converge to a stationary distribution centred on $x^{\text{ave}}\mathbf{1}_N$, while (because the covariance of the system grows unbounded with time) $x^{\text{ave}}\mathbf{1}_N$ performs a random walk along the consensus subspace. From (3.7), we observe that when the communication graph is balanced, x^{ave} will undergo pure Brownian motion with noise intensity $\frac{\sigma}{N}$. However, when the graph is unbalanced, the disagreement state will bias the random walk of x^{ave}.

When the communication graph is disconnected, the proof of Proposition 3.1 demonstrates that the expected disagreement cannot be guaranteed to converge to $\mathbf{0}$ and furthermore that the covariance of the disagreement grows unbounded with time. Thus for disconnected communication graphs, noise will always tend to drive the system away from consensus.

3.2 Robustness and the $\mathcal{H}_2$ Norm

3.2.1 The $\mathcal{H}_2$ norm as a measure of robustness

We now seek to characterise the robustness of the consensus dynamics to noisy inputs. Since the dispersion of the system measures the distance to consensus, it is an appropriate measure of how well the system is able to filter noise and remain close to consensus. However, we saw in Proposition 3.1 that the distribution of the disagreement state depends on the intensity of the noise σ. Since σ is a property of the system or environment, rather than of the communication structure, we can measure the robustness of the communication graph by assuming that $\sigma = 1$. Then a system will be robust when it has a low value of dispersion and not robust when its dispersion is large. To avoid dependence on initial conditions, we will only examine dispersion in the limit as $t \to \infty$ (note that Proposition 3.1 implies that if $\mathbf{y}(0)$ is drawn from the limiting distribution of $\mathbf{y}$, then the distribution of $\mathbf{y}$ will be stationary for all values

of $t \geq 0$).

Definition 3.4. *For the consensus system (3.1) with $\sigma = 1$, disagreement state given by Definition 3.1 and dispersion given by Definition 3.3, let*

$$\eta := \lim_{t\to\infty} \left(E\left[\delta(t)^2\right]\right)^{\frac{1}{2}}, \tag{3.17}$$

when this limit exists and $\eta := \infty$ otherwise. Then the $\mathcal{H}_2$ robustness *of the system is given by the inverse of η (or 0 when $\eta = \infty$).*

Note that by our definition, η^2 corresponds to the steady-state mean-square deviation used in [111]. Furthermore, since we demonstrated above that $\delta(t)$ is independent of our choice of Q, η must also be independent of the choice of Q.

We use the term $\mathcal{H}_2$ in our definition of robustness since, as we can now show, η is the $\mathcal{H}_2$ norm of a system corresponding to the disagreement state dynamics.

Proposition 3.2. *The value of η from Definition 3.4 is precisely the $\mathcal{H}_2$ norm of the state-space system $\mathcal{P}$:*

$$\begin{aligned} \dot{\mathbf{y}} &= -\overline{L}\mathbf{y} + Q\mathbf{w} \\ \mathbf{z} &= I_{N-1}\mathbf{y}. \end{aligned} \tag{3.18}$$

Thus,

$$\eta = \begin{cases} \left[\mathrm{tr}\left(\Sigma\right)\right]^{\frac{1}{2}} & \text{if the graph is connected} \\ \infty & \text{otherwise,} \end{cases} \tag{3.19}$$

where Σ is the solution to the Lyapunov equation (3.12).

Proof. Since $\mathbf{y}^T(t)\mathbf{y}(t)$ is a scalar quantity, we know that $\mathbf{y}^T(t)\mathbf{y}(t) = \mathrm{tr}\left(\mathbf{y}^T(t)\mathbf{y}(t)\right) = \mathrm{tr}\left(\mathbf{y}(t)\mathbf{y}^T(t)\right)$. Thus

$$\delta(t) = \left[\mathrm{tr}\left(\mathbf{y}(t)\mathbf{y}^T(t)\right)\right]^{\frac{1}{2}},$$

and so

$$\eta = \left[\mathrm{tr}\left(\lim_{t\to\infty} E\left[\mathbf{y}(t)\mathbf{y}^T(t)\right]\right)\right]^{\frac{1}{2}},$$

which, by Proposition 3.1 and for $\sigma = 1$, means that η is given by (3.19).

Now, a standard definition [37, Chapter 6] of the $\mathcal{H}_2$ norm of the state-space system $\mathcal{S}$:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$
$$\mathbf{z} = C\mathbf{x},$$

with A Hurwitz, is

$$\|\mathcal{S}\|_2 = \left[\mathrm{tr}(CXC^T)\right]^{\frac{1}{2}},$$

where X is the unique positive-definite solution of the Lyapunov equation [37, Proposition 4.5]

$$AX + XA^T = -BB^T.$$

If A is not Hurwitz, then $\|\mathcal{S}\|_2 = \infty$.

Thus, η is precisely the $\mathcal{H}_2$ norm of $\mathcal{P}$ as defined in (3.18). □

Proposition 3.2 is unsurprising, since one common interpretation of the $\mathcal{H}_2$ norm is the steady-state standard deviation of the output of a system driven by unit-intensity white noise [37, §6.1]. In this case, (3.18) represents an ordinary differential equation interpretation of (3.8) (with white noise as the input signal) and the disagreement state as the output.

Since η depends only on $\overline{L}$, and hence only on the communication graph, throughout this dissertation we will refer to η as the $\mathcal{H}_2$ *norm of the graph*. Similarly, we will refer to the $\mathcal{H}_2$ robustness of a consensus system as the $\mathcal{H}_2$ *robustness of the graph*.

In accordance with our original motivation for using η to measure robustness, η can be thought of as a distance in $\mathbb{R}^N$ - i.e. how far away from a consensus state you could hope to find the system. Because of this, however, the value of η can be expected to be proportional to $\sqrt{N}$ in the same way that $\|\mathbf{1}_N\|_2 = \sqrt{N}$. Therefore, if

we want to compare systems with different numbers of agents (or graphs with different numbers of nodes), we must adjust accordingly.

Definition 3.5. *The* nodal $\mathcal{H}_2$ norm *of the consensus system (3.1) is given by* $\frac{\eta}{\sqrt{N}}$, *and the* nodal $\mathcal{H}_2$ robustness *by* $\frac{\sqrt{N}}{\eta}$ *(or* 0 *when* $\eta = \infty$*), where* η *is calculated as in (3.19).*

Thus the nodal $\mathcal{H}_2$ norm or nodal $\mathcal{H}_2$ robustness can be thought of as the average contributions from each node to the overall $\mathcal{H}_2$ norm or $\mathcal{H}_2$ robustness, respectively. Alternatively, since Σ is a covariance matrix with variances along its diagonal, the nodal $\mathcal{H}_2$ norm can be thought of as an average standard deviation of each node from consensus. This provides an interesting parallel to the *node certainty index* used in [83, 84].

3.2.2 The $\mathcal{H}_2$ norm for graphs with normal reduced Laplacian matrices

We can use Proposition 3.2 to derive a relationship between the eigenvalues of L and the $\mathcal{H}_2$ norm of the graph when the reduced Laplacian matrix $\overline{L}$ is normal. Normal matrices are particularly easy to work with since we know by the Spectral Theorem that they can be diagonalised using unitary matrices. We can use this fact to solve the Lyapunov equation (3.12) and thus obtain an expression for the $\mathcal{H}_2$ norm of the graph.

First, we seek to characterise those graphs with normal reduced Laplacians.

Lemma 3.2. *$\overline{L}$ is normal if and only if ΠL is normal, that is, if and only if $\Pi L L^T \Pi = L^T \Pi L$.*

Proof. By definition, ΠL is normal if $(\Pi L)(\Pi L)^T = (\Pi L)^T (\Pi L)$, that is, if $\Pi L L^T \Pi = L^T \Pi^2 L$ (as Π is symmetric). However, Π is a projection matrix, so $\Pi^2 = \Pi$. Thus

ΠL being normal is equivalent to

$$\Pi L L^T \Pi = L^T \Pi L. \tag{3.20}$$

Now, since $\overline{L} = QLQ^T$, $\overline{L}$ being normal is equivalent to

$$QLQ^T QL^T Q^T = QL^T Q^T QLQ^T,$$

which reduces to

$$QLL^T Q^T = QL^T \Pi L Q^T \tag{3.21}$$

since $Q^T Q = \Pi$ and $L\Pi = L$.

Suppose $\overline{L}$ is normal. Then, pre-multiplying (3.21) by Q^T and post-multiplying by Q gives us

$$\Pi L L^T \Pi = \Pi L^T \Pi L \Pi.$$

However, $L\Pi = L$ and $\Pi L^T = L^T$. Therefore, we obtain (3.20) and so ΠL is normal.

Suppose ΠL is normal. Then, pre-multiplying (3.20) by Q and post-multiplying by Q^T gives us

$$QQ^T QLL^T Q^T QQ^T = QL^T \Pi L Q^T.$$

But $QQ^T = I_{N-1}$, and so we obtain (3.21). Thus, $\overline{L}$ is normal. □

Although Lemma 3.2 gives us the most general condition on L for $\overline{L}$ to be normal, it is instructive to consider some special graphs that have normal reduced Laplacians. Incidentally, Lemma 3.2 also shows that the normality of $\overline{L}$ does not depend on a particular choice of matrix Q.

Lemma 3.3. *Let $\mathcal{G}$ be a connected graph with a normal Laplacian matrix. Then $\mathcal{G}$ is balanced.*

Proof. Suppose L is the Laplacian matrix of a connected graph $\mathcal{G}$ and that $LL^T =$

$L^T L$. Since $\mathcal{G}$ is connected, the 0 eigenvalue of L has multiplicity 1. Hence the only vectors $\mathbf{v}$ for which $L\mathbf{v} = \mathbf{0}$ are $\mathbf{v} = \beta \mathbf{1}_N, \beta \in \mathbb{R}$. Since $L^T L \mathbf{1}_N = \mathbf{0}$ we know by the normality of L that $LL^T \mathbf{1}_N = \mathbf{0}$. Therefore, we conclude that

$$L^T \mathbf{1}_N = \beta \mathbf{1}_N, \quad \beta \in \mathbb{R}. \tag{3.22}$$

Premultiplying both sides of (3.22) by $\mathbf{1}_N^T$ (and using the fact that $\mathbf{1}_N^T L^T = \mathbf{0}^T$), we obtain

$$N\beta = 0$$

$$\Rightarrow \beta = 0.$$

Thus (3.22) reduces to $L^T \mathbf{1}_N = \mathbf{0}$, and so the graph must be balanced. □

Note that the converse to Lemma 3.3 is not true - that is, a balanced digraph will not necessarily have a normal Laplacian matrix. For example, consider the graph shown in Figure 3.1. This graph has Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Then a straightforward calculation shows that this graph is balanced - that is, $\mathbf{1}_N^T L = \mathbf{0}^T$ - but L is not normal - that is, $LL^T \neq L^T L$. In fact, since balanced graphs may be formed through a weight-balancing algorithm (e.g., see [20]) that would be expected to result in many different edge weights used throughout the graph, it could be expected that "most" balanced digraphs encountered in practice will not have normal Laplacian matrices.

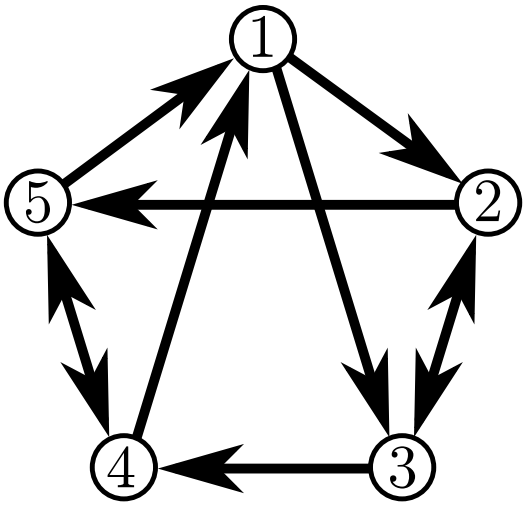

Figure 3.1: A balanced graph on 5 nodes with a Laplacian matrix that is not normal.

Lemma 3.4. *If L is a normal Laplacian matrix of a connected graph then $\overline{L}$ is also normal.*

Proof. Suppose L is the Laplacian matrix of a connected graph and that $LL^T = L^T L$. By Lemma 3.3 we know that $L^T \mathbf{1}_N = \mathbf{0}$, or equivalently, $\mathbf{1}_N^T L = \mathbf{0}^T$. Then

$$\Pi L = \left(I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T \right) L = L.$$

Thus ΠL is normal and so, by Lemma 3.2, $\overline{L}$ is normal as well. □

Any undirected graph will have a symmetric Laplacian matrix, which is trivially normal. Thus we know that $\overline{L}$ will be normal for all undirected graphs. In addition to this, there exist directed graphs with normal Laplacians, including all circulant graphs (for example, see §3.3.2), which by Lemma 3.4 have normal reduced Laplacians. Furthermore, some directed graphs do not have normal Laplacians but still satisfy the condition of Lemma 3.2 (for example, see §3.3.4).

Now we are able to derive a formula for the $\mathcal{H}_2$ norm of a graph with normal $\overline{L}$ in terms of the non-zero eigenvalues of the Laplacian matrix.

Proposition 3.3. *Suppose L is the Laplacian matrix of a connected graph with eigenvalues $\lambda_1 = 0 < \mathrm{Re}\{\lambda_2\} \leq \mathrm{Re}\{\lambda_3\} \leq \ldots \leq \mathrm{Re}\{\lambda_N\}$, and that ΠL is normal. Then the $\mathcal{H}_2$ norm of the graph is*

$$\eta = \left(\sum_{i=2}^{N} \frac{1}{2\,\mathrm{Re}\{\lambda_i\}} \right)^{\frac{1}{2}}. \tag{3.23}$$

Proof. By Lemma 3.2, we know that $\overline{L}$ is normal. Furthermore, by Lemma 3.1 the eigenvalues of $\overline{L}$ are $\lambda_2, \lambda_3, \ldots, \lambda_N$. Therefore, by the Spectral Theorem, we can find a unitary matrix $U \in \mathbb{C}^{(N-1)\times(N-1)}$ such that $\overline{L} = U\Lambda U^*$, where

$$\Lambda = \mathrm{diag}\left(\begin{bmatrix} \lambda_2 & \cdots & \lambda_N \end{bmatrix}^T \right).$$

Note that since $\overline{L}$ is real, $\overline{L}^T = \overline{L}^* = U\Lambda^* U^*$.

Now, to find the $\mathcal{H}_2$ norm, we must solve the Lyapunov equation

$$U\Lambda U^*\Sigma + \Sigma U\Lambda^* U^* = I_{N-1}.$$

Rearranging, we get

$$\Lambda U^*\Sigma U + U^*\Sigma U\Lambda^* = I_{N-1}.$$

Now, let $\Gamma := U^*\Sigma U$, and note that

$$\mathrm{tr}\,(\Sigma) = \mathrm{tr}\,(U\Gamma U^*) = \mathrm{tr}\,(U^*U\Gamma) = \mathrm{tr}\,(\Gamma).$$

Thus, $\eta = [\mathrm{tr}(\Gamma)]^{\frac{1}{2}}$, where

$$\Lambda\Gamma + \Gamma\Lambda^* = I_{N-1}.$$

Since Λ and I_{N-1} are both diagonal matrices, Γ must be a diagonal matrix as well.

Thus Γ and Λ commute and we can write

$$\Gamma\Lambda + \Gamma\Lambda^* = \Gamma\left(2\,\mathrm{Re}\{\Lambda\}\right) = I_{N-1},$$

which implies

$$\Gamma = \left(2\,\mathrm{Re}\{\Lambda\}\right)^{-1}.$$

Thus Γ is the diagonal matrix with entries $\frac{1}{2\,\mathrm{Re}\{\lambda_2\}}, \ldots, \frac{1}{2\,\mathrm{Re}\{\lambda_N\}}$ and we conclude that

$$\eta = \left(\sum_{i=2}^{N} \frac{1}{2\,\mathrm{Re}\{\lambda_i\}}\right)^{\frac{1}{2}}.$$

□

One important property of undirected graphs is the *effective resistance*, which defines a distance function between nodes of the graph [56]. The sum of all distinct effective resistances is known as the *Kirchhoff index* of the graph, and is denoted by K_f. This is related to the eigenvalues of the graph Laplacian by the formula [21, 112]

$$K_f = N \sum_{j=2}^{N} \frac{1}{\lambda_j}. \tag{3.24}$$

A comparison between (3.24) and (3.23) reveals that for undirected graphs,

$$\eta = \left(\frac{K_f}{2N}\right)^{\frac{1}{2}}, \tag{3.25}$$

which implies that for undirected graphs with equal numbers of nodes, the Kirchhoff index is equivalent to the $\mathcal{H}_2$ norm in that one can be computed from the other, and any ordering induced by one measure is the same as the ordering induced by the other. Further implications of this relationship will be examined in more detail in Chapters 4, 5 and 6.

3.2.3 Bounds on the $\mathcal{H}_2$ norm of any graph

Equation (3.23) does not hold for every graph (e.g., see §3.3.3). In fact, the $\mathcal{H}_2$ norm of a graph cannot depend only on the Laplacian eigenvalues since there exist graphs with the same set of Laplacian eigenvalues but different $\mathcal{H}_2$ norms (e.g., see §3.3.3 and §3.3.4). Instead, we will show that the formula for the $\mathcal{H}_2$ norm for normal graphs provides a lower bound to the $\mathcal{H}_2$ norm of any graph. To do this, we will use the fact that the solution to the Lyapunov equation (3.12) can be written as [37, Theorem 4.1]

$$\Sigma = \int_0^\infty \mathrm{e}^{-\overline{L}t}\mathrm{e}^{-\overline{L}^T t}\, dt \tag{3.26}$$

and therefore, by (3.19), we can express the $\mathcal{H}_2$ norm as

$$\eta^2 = \int_0^\infty \operatorname{tr}\left(\mathrm{e}^{-\overline{L}t}\mathrm{e}^{-\overline{L}^T t}\right) dt. \tag{3.27}$$

Proposition 3.4. *Suppose L is the Laplacian matrix of a connected graph with eigenvalues $\lambda_1 = 0 < \operatorname{Re}\{\lambda_2\} \leq \operatorname{Re}\{\lambda_3\} \leq \ldots \leq \operatorname{Re}\{\lambda_N\}$. Then*

$$\eta \geq \left(\sum_{i=2}^{N} \frac{1}{2\operatorname{Re}\{\lambda_i\}}\right)^{\frac{1}{2}} \tag{3.28}$$

where η is the $\mathcal{H}_2$ norm of system (3.8).

Proof. We start with an inequality due to Weyl [105]. For any square matrix $A \in \mathbb{R}^{k\times k}$,

$$\sum_{i=1}^{k} \sigma_i^2(A) \geq \sum_{i=1}^{k} |\lambda_i(A)|^2.$$

In particular, we have that

$$\sum_{i=1}^{k} \sigma_i^2\left(\mathrm{e}^A\right) \geq \sum_{i=1}^{k} \left|\lambda_i\left(\mathrm{e}^A\right)\right|^2.$$

However, $\left|\lambda_i\left(\mathrm{e}^{A}\right)\right| = \mathrm{e}^{\mathrm{Re}\{\lambda_i(A)\}}$ and by the definition of singular values [52, §7.3], $\sum_{i=1}^{k}\sigma_i^2\left(\mathrm{e}^{A}\right) = \mathrm{tr}\left(\mathrm{e}^{A}\mathrm{e}^{A^T}\right)$. Therefore, we can say

$$\mathrm{tr}\left(\mathrm{e}^{A}\mathrm{e}^{A^T}\right) \geq \sum_{i=1}^{k}\mathrm{e}^{2\,\mathrm{Re}\{\lambda_i(A)\}}.$$

Letting $A = -\overline{L}t$, we obtain

$$\mathrm{tr}\left(\mathrm{e}^{-\overline{L}t}\mathrm{e}^{-\overline{L}^T t}\right) \geq \sum_{i=1}^{N-1}\mathrm{e}^{-2\,\mathrm{Re}\{\lambda_i(\overline{L})\}t}.$$

This inequality will still hold when we integrate with respect to t, since both sides are positive for all values of t. Thus

$$\int_0^\infty \mathrm{tr}\left(\mathrm{e}^{-\overline{L}t}\mathrm{e}^{-\overline{L}^T t}\right)dt \geq \int_0^\infty \sum_{i=1}^{N-1}\mathrm{e}^{-2\,\mathrm{Re}\{\lambda_i(\overline{L})\}t}\,dt.$$

Evaluating the integral on the right hand side and substituting from (3.27) for the left hand side gives us

$$\eta^2 \geq \sum_{i=1}^{N-1}\frac{1}{2\,\mathrm{Re}\left\{\lambda_i\left(\overline{L}\right)\right\}},$$

and hence (by Lemma 3.1),

$$\eta \geq \left(\sum_{i=2}^{N}\frac{1}{2\,\mathrm{Re}\{\lambda_i\}}\right)^{\frac{1}{2}}.$$

□

In order to derive an upper bound for the $\mathcal{H}_2$ norm, we must first introduce the definition of *algebraic connectivity* for directed graphs [108].

Definition 3.6. *The algebraic connectivity of a graph is the real number $\alpha(\mathcal{G})$, computed as*

$$\alpha(\mathcal{G}) = \min_i \lambda_i\left(\frac{1}{2}Q\left(L + L^T\right)Q^T\right) = \min_i \lambda_i\left(\frac{1}{2}\left(\overline{L} + \overline{L}^T\right)\right) \tag{3.29}$$

Using this definition, it can be shown that a graph that is not connected will have a non-positive algebraic connectivity. Furthermore, any undirected graph that is connected will have a positive algebraic connectivity. However, there do exist connected directed graphs (even strongly connected graphs) that have negative algebraic connectivities [108]. Our upper bound on the $\mathcal{H}_2$ norm of a graph will only apply to those graphs with positive algebraic connectivity.

Proposition 3.5. *Suppose L is the Laplacian matrix of a connected graph with positive algebraic connectivity. Then*

$$\eta \leq \left(\sum_{i=1}^{N-1} \frac{1}{2\lambda_i \left(\overline{L}_{\text{sym}} \right)} \right)^{\frac{1}{2}} \tag{3.30}$$

where η is the $\mathcal{H}_2$ norm of the graph and $\overline{L}_{\text{sym}} = \frac{1}{2}\left(\overline{L} + \overline{L}^T \right)$.

Proof. We start with an inequality due to Bernstein [10]. For any matrix A,

$$\text{tr}\left(\text{e}^{A} \text{e}^{A^T} \right) \leq \text{tr}\left(\text{e}^{A + A^T} \right).$$

Letting $A = -\overline{L}t$, we obtain

$$\text{tr}\left(\text{e}^{-\overline{L}t} \text{e}^{-\overline{L}^T t} \right) \leq \text{tr}\left(\text{e}^{-\left(\overline{L} + \overline{L}^T \right) t} \right) = \text{tr}\left(\text{e}^{-2\overline{L}_{\text{sym}} t} \right).$$

This inequality will still hold when we integrate with respect to t, since both sides are positive for all values of t. Thus

$$\int_0^\infty \text{tr}\left(\text{e}^{-\overline{L}t} \text{e}^{-\overline{L}^T t} \right) dt \leq \int_0^\infty \text{tr}\left(\text{e}^{-2\overline{L}_{\text{sym}} t} \right) dt.$$

Exchanging the trace and the integral on the right hand side and substituting from

(3.27) for the left hand side gives us

$$\eta^2 \leq \operatorname{tr}\left(\int_0^\infty \mathrm{e}^{-2\overline{L}_{\mathrm{sym}}t}\,dt\right).$$

Now, the integral on the right hand side will be finite precisely when $-\overline{L}_{\mathrm{sym}}$ is Hurwitz [37, §4.1]. Since $\overline{L}_{\mathrm{sym}}$ is symmetric, this is equivalent to all eigenvalues of $\overline{L}_{\mathrm{sym}}$ being positive. However, the smallest eigenvalue of $\overline{L}_{\mathrm{sym}}$ is the algebraic connectivity of the graph. Therefore, when the algebraic connectivity is positive, we can say

$$\begin{aligned}\eta^2 &\leq \operatorname{tr}\left(\frac{1}{2}\overline{L}_{\mathrm{sym}}^{-1}\right)\\ &\leq \sum_{i=1}^{N-1}\frac{1}{2\lambda_i\left(\overline{L}_{\mathrm{sym}}\right)},\end{aligned}$$

and so

$$\eta \leq \left(\sum_{i=1}^{N-1}\frac{1}{2\lambda_i\left(\overline{L}_{\mathrm{sym}}\right)}\right)^{\frac{1}{2}}.$$

□

We can see that the bounds derived in Propositions 3.4 and 3.5 are both tight for graphs with normal reduced Laplacians. By Proposition 3.3, (3.28) becomes an equality when $\overline{L}$ is normal. In addition, when $\overline{L}$ is normal, it can be written as

$$\overline{L} = U\Lambda U^*,$$

where U is a unitary matrix and Λ is the diagonal matrix containing the eigenvalues of $\overline{L}$ [52, Theorem 2.5.4]. Then (since $\overline{L}$ is real), $\overline{L}^T = \overline{L}^*$ and so

$$\overline{L}_{\mathrm{sym}} = \frac{1}{2}U\left(\Lambda + \Lambda^*\right)U^*.$$

Thus the eigenvalues of $\overline{L}_{\mathrm{sym}}$ are the real parts of the eigenvalues of $\overline{L}$ and so by

Proposition 3.3, (3.30) also becomes an equality.

Finally, we can derive an absolute lower bound for the $\mathcal{H}_2$ norm of any graph, provided that its Laplacian matrix is normalised (note that for unnormalised Laplacians, the $\mathcal{H}_2$ norm of any connected graph can be made arbitrarily small by simply scaling the whole matrix).

Proposition 3.6. *Suppose a graph $\mathcal{G}$ on N nodes has a normalised Laplacian matrix L. Then*

$$\eta \geq \frac{N-1}{\sqrt{2N}}, \tag{3.31}$$

where η is the $\mathcal{H}_2$ norm of $\mathcal{G}$, computed as in (3.19).

Proof. If $\mathcal{G}$ is disconnected, $\eta = \infty$ by (3.19) and so (3.31) trivially holds. Therefore, in the following we will assume that $\mathcal{G}$ is connected. Now suppose that $\mathcal{G}$ has a normalised Laplacian matrix L. Then the out-degree of any node in $\mathcal{G}$ is either 0 or 1 (see §2.4). Thus the diagonal entries of L are all either 0 or 1 and so

$$\operatorname{tr}(L) \leq N.$$

But the trace of a matrix is equal to the sum of its eigenvalues [52, Theorem 1.2.12], and thus we must have

$$\sum_{i=1}^{N} \lambda_i(L) \leq N. \tag{3.32}$$

Furthermore, we know that $\lambda_1(L) = 0$ and that all eigenvalues of L must be either real or appear as complex conjugate pairs [52, §1.2]. Thus, if we let

$$\mu_i := \operatorname{Re}\{\lambda_i(L)\},$$

we can rewrite (3.32) as

$$\sum_{i=2}^{N} \mu_i \leq N, \tag{3.33}$$

where $\mu_i \geq 0 \ \forall i$ since L is a Laplacian matrix.

In order to derive (3.31), we seek to find the smallest possible value of the lower bound (3.28) under the constraint (3.33). Since the right hand side of (3.28) is always positive, minimising this expression is equivalent to minimising its square. Thus, we seek to minimise

$$\sum_{i=2}^{N} \frac{1}{2\mu_i},$$

subject to (3.33).

Following the approach of Kuhn and Tucker [57], we let $\boldsymbol{\mu} := \begin{bmatrix} \mu_2 & \cdots & \mu_N \end{bmatrix}^T$, and form the Lagrangian function

$$\phi\left(\boldsymbol{\mu}, \rho\right) := -\sum_{i=2}^{N} \frac{1}{2\mu_i} + \rho\left(N - \sum_{i=2}^{N} \mu_i\right). \tag{3.34}$$

Then, for our choice of ϕ and according to the Kuhn-Tucker conditions [57], $\hat{\boldsymbol{\mu}} \in \mathbb{R}^{N-1}$ will solve our original problem if and only if $\exists\, \hat{\rho} \in \mathbb{R}$ such that

$$\left.\frac{\partial \phi}{\partial \mu_i}\right|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}, \rho=\hat{\rho}} \leq 0 \ \forall\, i, \quad \left(\left.\frac{\partial \phi}{\partial \boldsymbol{\mu}}\right|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}, \rho=\hat{\rho}}\right) \hat{\boldsymbol{\mu}} = 0, \quad \hat{\mu}_i \geq 0 \ \forall\, i, \text{ and} \tag{3.35}$$

$$\left.\frac{\partial \phi}{\partial \rho}\right|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}, \rho=\hat{\rho}} \geq 0, \quad \left(\left.\frac{\partial \phi}{\partial \rho}\right|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}, \rho=\hat{\rho}}\right) \hat{\rho} = 0, \quad \hat{\rho} \geq 0. \tag{3.36}$$

Now,

$$\frac{\partial \phi}{\partial \mu_i} = \frac{1}{2\mu_i^2} - \rho, \quad i \in \{2, 3, \ldots, N\},$$

and so to satisfy (3.35), we must have

$$\hat{\mu}_i = \frac{1}{\sqrt{2\hat{\rho}}}, \quad i \in \{2, 3, \ldots, N\}. \tag{3.37}$$

Then $\hat{\rho}$ cannot equal 0, and since

$$\frac{\partial \phi}{\partial \rho} = N - \sum_{i=2}^{N} \mu_i,$$

to satisfy (3.36) we must have

$$\sum_{i=2}^{N} \hat{\mu}_i = N. \tag{3.38}$$

Substituting (3.37) into (3.38), we find the solution

$$\begin{aligned} \hat{\rho} &= \frac{(N-1)^2}{2N^2}, \\ \hat{\mu}_i &= \frac{N}{N-1}, \quad i \in \{2, 3, \ldots, N\}. \end{aligned} \tag{3.39}$$

Thus for any normalised Laplacian matrix, we can say that

$$\left(\sum_{i=2}^{N} \frac{1}{2\,\mathrm{Re}\{\lambda_i\}} \right)^{\frac{1}{2}} \geq \frac{N-1}{\sqrt{2N}},$$

which, along with Proposition 3.4, implies (3.31). □

We will see in §3.3.1 that the bound stated in Proposition 3.6 is also tight, since it is equal to the $\mathcal{H}_2$ norm of the complete graph.

3.3 Properties of Families of Graphs

In addition to the $\mathcal{H}_2$ norm, the speed of convergence to consensus is an important performance measure for a consensus system [75, 110]. A graph with "good" robustness will have a small value of the $\mathcal{H}_2$ norm, while a graph with "good" convergence speed will have a large value for the real part of the second-smallest eigenvalue. Based on Propositions 3.3 and 3.4, these two features are not necessarily incompatible, although they cannot be guaranteed to be complementary. In this section we

compare the properties of families of directed and undirected graphs to investigate the behaviour of these two measures of performance.

3.3.1 Complete graphs

The complete graph on N nodes contains an edge connecting every pair of nodes (e.g., see Figure 2.3(a)). In its most common form, every edge has unit weight and so its Laplacian matrix would equal $NI - \mathbf{1}_N\mathbf{1}_N^T = N\Pi$ (by (2.2)). In its normalised form, every edge has a weight of $\frac{1}{N-1}$ and thus the complete graph has Laplacian matrix

$$L_N^{\text{complete}} = \frac{N}{N-1}\Pi.$$

The complete graph is undirected and so L_N^{complete} is symmetric, and hence normal. Now, since Π is the orthogonal projection matrix onto $\mathbf{1}_N^{\perp}$, we know that

$$\Pi\mathbf{1}_N = \mathbf{0} \text{ and } \Pi\mathbf{v} = \mathbf{v}$$

for any vector $\mathbf{v} \in \mathbf{1}_N^{\perp}$. Thus

$$L_N^{\text{complete}}\mathbf{1}_N = \mathbf{0} \text{ and } L_N^{\text{complete}}\mathbf{v} = \frac{N}{N-1}\mathbf{v}$$

for any vector $\mathbf{v} \in \mathbf{1}_N^{\perp}$, and so the eigenvalues of L_N^{complete} are

$$\lambda_i\left(L_N^{\text{complete}}\right) = \begin{cases} 0 & i = 1 \\ \dfrac{N}{N-1} & 2 \leq i \leq N. \end{cases} \tag{3.40}$$

Therefore, by substituting (3.40) into (3.23), we see that the complete graph has $\mathcal{H}_2$ norm

$$\eta^{\text{complete}} = \frac{N-1}{\sqrt{2N}},$$

and so by Definition 3.5, the nodal $\mathcal{H}_2$ norm is

$$\frac{N-1}{N\sqrt{2}},$$

which converges to $\frac{1}{\sqrt{2}}$ as $N \to \infty$. Furthermore, by Remark 3.1, the complete graph has convergence speed

$$\lambda_2\left(L_N^{\text{complete}}\right) = \frac{N}{N-1}. \tag{3.41}$$

Upon comparison with the lower bound derived in Proposition 3.6, the complete graph has the minimum possible $\mathcal{H}_2$ norm for any graph with a normalised Laplacian matrix. It is also straightforward to observe that the convergence speed of the complete graph is the maximum possible for any graph with a normalised Laplacian.

Lemma 3.5. *Suppose a graph $\mathcal{G}$ on N nodes has a normalised Laplacian matrix L. Then*

$$\operatorname{Re}\{\lambda_2(L)\} \leq \frac{N}{N-1}. \tag{3.42}$$

Proof. For the same reasons given in the proof of Proposition 3.6, we know that any normalised Laplacian matrix will have a trace of at most N, and the trace of a real Laplacian matrix is given by

$$\operatorname{tr}(L) = \sum_{i=2}^{N} \operatorname{Re}\{\lambda_i(L)\}$$

Then, since $\operatorname{Re}\{\lambda_3(L)\}, \ldots, \operatorname{Re}\{\lambda_N(L)\}$ are all greater than or equal to $\operatorname{Re}\{\lambda_2(L)\}$, we observe that

$$(N-1)\operatorname{Re}\{\lambda_2(L)\} \leq \sum_{i=2}^{N} \operatorname{Re}\{\lambda_i(L)\} \leq N,$$

from which (3.42) follows. □

3.3.2 Cycle graphs

The directed cycle graph on N nodes consists of a closed "chain" of nodes with each node connected to the next node in the chain (e.g., see Figure 2.3(b)). Since every node has a single out-going edge, in its normalised form the directed cycle graph will have a unit weight on every edge. If we traverse the cycle in the reverse direction of the edges and label the nodes in the order they appear, the Laplacian matrix of the directed cycle graph will be

$$L_N^{\text{cycle}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}. \tag{3.43}$$

Now, L_N^{cycle} is a circulant matrix, and all circulant matrices are normal [46, §3.1]. Furthermore, we can derive an analytical expression for the eigenvalues of L_N^{cycle}.

Lemma 3.6. *Let L_N^{cycle} be the Laplacian matrix of the directed cycle graph on N nodes, as given by (3.43). Then L_N^{cycle} has eigenvalues*

$$\lambda_k\left(L_N^{cycle}\right) = 1 + \begin{cases} \mathrm{e}^{\mathrm{i}\pi\left(1-\frac{k}{N}\right)} & k \text{ even} \\ \mathrm{e}^{\mathrm{i}\pi\left(1+\frac{k-1}{N}\right)} & k \text{ odd}, \end{cases} \tag{3.44}$$

for $k = 1, 2, \ldots, N$.

Proof. The proof is given in Appendix A.1.1. □

Since L_N^{cycle} is a normal matrix, we know from Lemmas 3.4 and 3.2 that ΠL_N^{cycle} is normal. Thus we can write the eigenvalues in the form given in (A.2) and use

Proposition 3.3 to obtain

$$\begin{aligned}\eta^{\text{cycle}} &= \left[\sum_{j=1}^{N-1} \frac{1}{2\,\mathrm{Re}\left\{1+\mathrm{e}^{\mathrm{i}\pi\left(1-\frac{2j}{N}\right)}\right\}}\right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\left[\sum_{j=1}^{N-1} \frac{1}{1+\cos\left(\pi\left(1-\frac{2j}{N}\right)\right)}\right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\left[\sum_{j=1}^{N-1} \frac{1}{2\sin^2\left(\frac{j\pi}{N}\right)}\right]^{\frac{1}{2}} \\ &= \frac{1}{2}\left[\sum_{j=1}^{N-1} \csc^2\left(\frac{j\pi}{N}\right)\right]^{\frac{1}{2}}.\end{aligned}$$

But by [50, Equation 24.1.2],

$$\sum_{j=1}^{N-1} \csc^2\left(\frac{j\pi}{N}\right) = \frac{N^2-1}{3},$$

and so we conclude that

$$\eta^{\text{cycle}} = \sqrt{\frac{N^2-1}{12}}.$$

Therefore, by Definition 3.5, the nodal $\mathcal{H}_2$ norm is

$$\sqrt{\frac{N^2-1}{12N}},$$

which grows as $\sqrt{\frac{N}{12}}$ for large N.

By Remark 3.1, the convergence speed of the directed cycle is

$$\begin{aligned}\mathrm{Re}\left\{\lambda_2\left(L_N^{\text{cycle}}\right)\right\} &= \mathrm{Re}\left\{1+\mathrm{e}^{\mathrm{i}\pi\left(1-\frac{2}{N}\right)}\right\} \\ &= 1+\cos\left(\pi-\frac{2\pi}{N}\right) \\ &= 2\sin^2\left(\frac{\pi}{N}\right).\end{aligned}$$

We can also consider the undirected form of the cycle graph. As outlined in §2.4, the adjacency matrix of the undirected cycle is the symmetric part of the adjacency matrix of the directed cycle. Thus, the undirected cycle will have Laplacian matrix

$$L_N^{\text{u cycle}} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}. \tag{3.45}$$

Now $L_N^{\text{u cycle}}$ is symmetric, so Proposition 3.3 applies. Furthermore, we can make the following observation.

Lemma 3.7. *The eigenvalues of the Laplacian matrix of the undirected cycle graph are equal to the real parts of the eigenvalues of the Laplacian of the directed cycle.*

Proof. The proof is given in Appendix A.1.2. □

Thus the undirected cycle has the same $\mathcal{H}_2$ norm, nodal $\mathcal{H}_2$ norm and convergence speed as the directed cycle.

3.3.3 Path graphs

The directed path graph on N nodes consists of an open "chain" of nodes with each node connected to the next. It is equivalent to the directed cycle graph with one edge removed (e.g., see Figure 2.3(c)). Since every node has at most one out-going edge, in its normalised form the directed path graph will have a unit weight on every edge. If we traverse the path in the reverse direction of the edges and label the nodes in

the order they appear, the Laplacian matrix of the directed path graph will be

$$L_N^{\text{path}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}. \tag{3.46}$$

In this case it is clear that L_N^{path} is not balanced, and so by Lemma 3.3, we know that L_N^{path} is not normal. Furthermore, we can write L_N^{path} as

$$L_N^{\text{path}} = I_N - \mathbf{e}_N^{(1)}\mathbf{e}_N^{(1)T} - \sum_{j=2}^{N} \mathbf{e}_N^{(j)}\mathbf{e}_N^{(j-1)T},$$

which (using (2.2)) means that

$$\Pi L_N^{\text{path}} = I_N - \mathbf{e}_N^{(1)}\mathbf{e}_N^{(1)T} - \sum_{j=2}^{N} \mathbf{e}_N^{(j)}\mathbf{e}_N^{(j-1)T} + \frac{1}{N}\mathbf{1}_N\mathbf{e}_N^{(1)T} - \frac{1}{N}\mathbf{1}_N\mathbf{e}_N^{(N)T}.$$

It is then straightforward to verify that ΠL_N^{path} is not normal. Therefore, in order to compute its $\mathcal{H}_2$ norm, we cannot use Proposition 3.3. Instead, we will show in Chapter 5 that the $\mathcal{H}_2$ norm of the directed path is equal to that of the underlying undirected path.

Since L_N^{path} is lower triangular, its diagonal entries are its eigenvalues [52, §1.2] and so

$$\lambda_i\left(L_N^{\text{path}}\right) = \begin{cases} 0 & i = 1 \\ 1 & 2 \leq i \leq N. \end{cases} \tag{3.47}$$

Thus the convergence speed of the path is

$$\lambda_2\left(L_N^{\text{path}}\right) = 1.$$

We can consider the undirected path graph by finding the symmetric part of the adjacency matrix of the directed path. This produces the following Laplacian matrix

$$L_N^{\text{u path}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{3.48}$$

Now we can compute the eigenvalues of $L_N^{\text{u path}}$.

Lemma 3.8. *Let $L_N^{u\ path}$ be the Laplacian matrix of the undirected path graph on N nodes, as given by (3.48). Then $L_N^{u\ path}$ has eigenvalues*

$$\lambda_k\left(L_N^{u\ path}\right) = 2\sin^2\left(\frac{(k-1)\pi}{2N}\right) \tag{3.49}$$

for $k = 1, 2, \ldots, N$.

Proof. The proof is given in Appendix A.2.1. □

Since $L_N^{\text{u path}}$ is symmetric and hence normal, we can apply Proposition 3.3. Then, we can let $j = k - 1$ in (3.49) and use the facts that

$$\sum_{j=1}^{N-1} \csc^2\left(\frac{j\pi}{2N}\right) = \frac{1}{2}\left(\sum_{j=1}^{2N-1} \csc^2\left(\frac{j\pi}{2N}\right) - 1\right)$$

(since $\sin\left(\frac{j\pi}{2N}\right) = \sin\left(\pi - \frac{j\pi}{2N}\right)$) and [50, Equation 24.1.2]

$$\sum_{j=1}^{2N-1} \csc^2\left(\frac{j\pi}{2N}\right) = \frac{4N^2 - 1}{3},$$

to find that the undirected path has $\mathcal{H}_2$ norm

$$\eta^{\text{u path}} = \sqrt{\frac{N^2-1}{6}},$$

and thus nodal $\mathcal{H}_2$ norm

$$\sqrt{\frac{N^2-1}{6N}},$$

which grows as $\sqrt{\frac{N}{6}}$ for large N. Furthermore, the convergence speed of the undirected path is

$$\lambda_2\left(L_N^{\text{u path}}\right) = 2\sin^2\frac{\pi}{2N}.$$

3.3.4 Star graphs

The directed star graph on N nodes consists of a "central" node with every other node connected to this central one (e.g., see Figure 2.3(d)). Since every node has at most one out-going edge, in its normalised form the directed path graph will have a unit weight on every edge. If we label the central node as node 1, the Laplacian matrix of the directed star graph will be

$$L_N^{\text{star}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.50}$$

Now, we can write L_N^{star} as

$$L_N^{\text{star}} = I_N - \mathbf{1}_N \mathbf{e}_N^{(1)T}.$$

Then

$$\Pi L_N^{\text{star}} = \Pi\left(I_N - \mathbf{1}_N \mathbf{e}_N^{(1)T}\right) = \Pi,$$

since $\Pi \mathbf{1}_N = \mathbf{0}$. As Π is symmetric and hence normal, ΠL_N^{star} is also normal and so Proposition 3.3 can be applied. Since L_N^{star} is lower triangular, its diagonal entries are its eigenvalues and so

$$\lambda_i \left(L_N^{\text{star}} \right) = \begin{cases} 0 & i = 1 \\ 1 & 2 \leq i \leq N. \end{cases} \tag{3.51}$$

Therefore by (3.23), the $\mathcal{H}_2$ norm of the star graph is

$$\eta^{\text{star}} = \sqrt{\frac{N-1}{2}},$$

and thus the nodal $\mathcal{H}_2$ norm is

$$\sqrt{\frac{N-1}{2N}},$$

which converges to $\frac{1}{\sqrt{2}}$ as $N \to \infty$. Furthermore, the convergence speed of the star graph is

$$\lambda_2 \left(L_N^{\text{star}} \right) = 1.$$

Once again we can consider the undirected form of this graph by finding the symmetric part of the adjacency matrix and then forming a new Laplacian. For the undirected star graph, we find that the Laplacian matrix is

$$L_N^{\text{u star}} = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & 0 & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & \frac{N-1}{2} \end{bmatrix}. \tag{3.52}$$

We can then compute the eigenvalues of $L_N^{\text{u star}}$.

Lemma 3.9. *Let $L_N^{u\ star}$ be the Laplacian matrix of the undirected star graph on N nodes, as given by (3.52). Then $L_N^{u\ star}$ has eigenvalues*

$$\lambda_k\left(L_N^{u\ star}\right) = \begin{cases} 0 & k = 1 \\ \frac{1}{2} & 2 \leq k \leq N-1 \\ \frac{N}{2} & k = N. \end{cases} \tag{3.53}$$

Proof. The proof is given in Appendix A.3.1. □

Since $L_N^{\mathrm{u\ star}}$ is symmetric and hence normal, we can apply Proposition 3.3 to find its $\mathcal{H}_2$ norm. Thus the undirected star has $\mathcal{H}_2$ norm

$$\eta^{\mathrm{u\ star}} = \frac{N-1}{\sqrt{N}},$$

and thus nodal $\mathcal{H}_2$ norm

$$\frac{N-1}{N},$$

which converges to 1 as $N \to \infty$. Furthermore, the convergence speed of the undirected star is

$$\lambda_2\left(L_N^{\mathrm{u\ star}}\right) = \frac{1}{2}.$$

3.3.5 Comparison of complete, cycle, path and star graphs

Figure 3.2 shows the nodal $\mathcal{H}_2$ norm and convergence speed of the complete, cycle, path and star graphs. The complete graph has the best performance in both categories among these families. However, as the number of nodes increases, the directed path and directed star approach the complete graph in speed, and the directed star approaches it in robustness. This shows that the performance of the complete graph (which requires $(N-1)^2$ directed edges for N nodes) can be almost matched by certain graphs with many fewer edges (e.g., the star graph with $N-1$ edges).

We can observe a trend in Figure 3.2 that within this set of examples, graphs with higher convergence speeds mostly have smaller (i.e., better) $\mathcal{H}_2$ norms. This is to be expected from Proposition 3.3, which implies that for graphs with normal reduced Laplacians, $\lambda_2(L)$ makes the largest contribution to the $\mathcal{H}_2$ norm. However, even under the conditions of Proposition 3.3, it is not guaranteed that improving the convergence speed of a graph will also improve its $\mathcal{H}_2$ norm. Furthermore, we can see from the directed path (the only graph in this section that does not have a normal reduced Laplacian) that this trend does not carry over to all graphs. In fact, the directed path has a convergence speed that approaches the maximum possible speed for a normalised graph (by Lemma 3.5), while its $\mathcal{H}_2$ norm is the worst of the graphs considered here. Therefore, for general directed graphs, the $\mathcal{H}_2$ norm cannot be guaranteed to be small when the convergence speed is high.

These results also provide additional motivation for considering directed graphs for consensus problems. As well as the practical problems with maintaining undirected communication links in physical networks, undirected graphs can sometimes be outperformed in both speed and robustness by their directed counterparts. The directed and undirected path graphs have the same $\mathcal{H}_2$ robustness, but the directed graph produces much higher convergence speeds. In addition, the directed star graph outperforms the undirected star in terms of both speed and robustness.

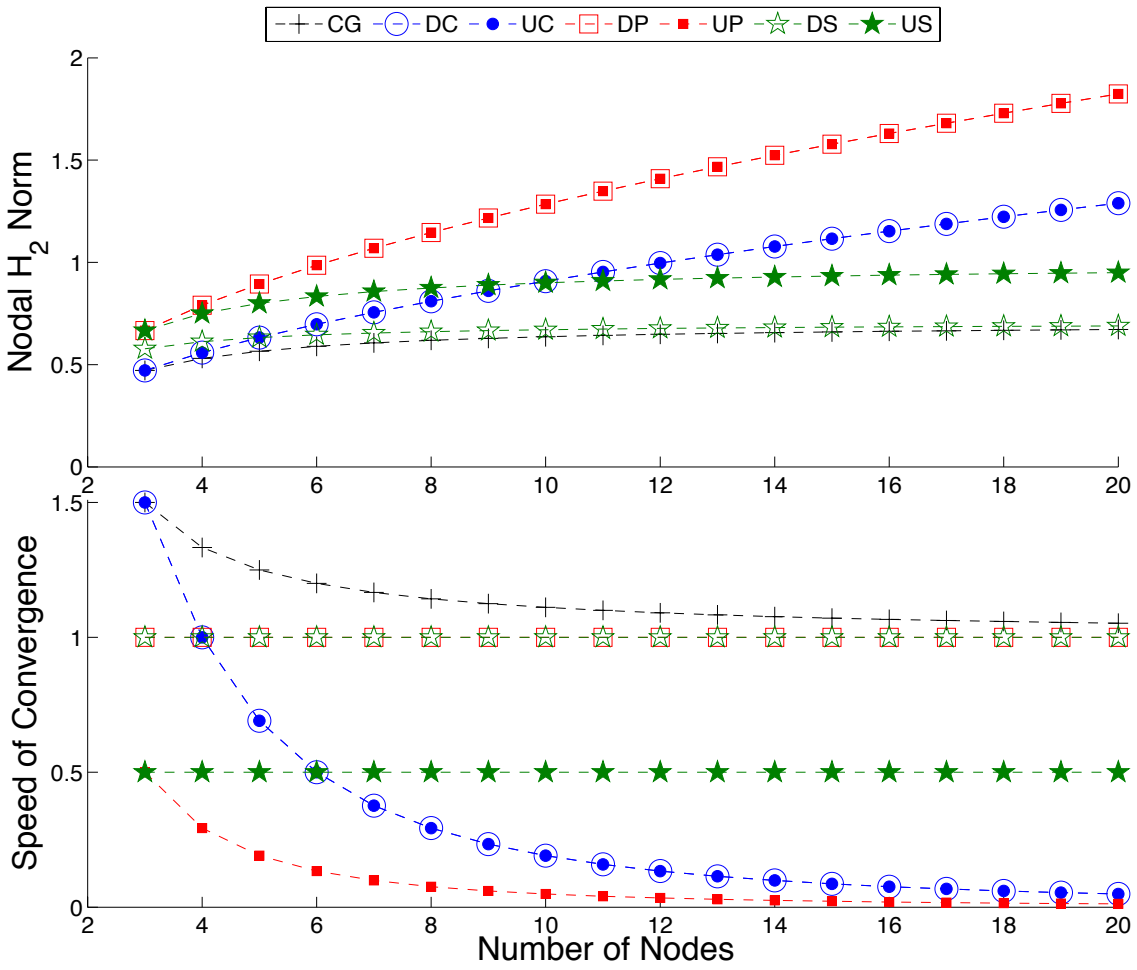

Figure 3.2: Nodal $\mathcal{H}_2$ norm and convergence speed for certain directed and undirected graphs. The graphs shown are: CG - complete graph, DC - directed cycle, UC - undirected cycle, DP - directed path, UP - undirected path, DS - directed star and US - undirected star.

Chapter 4

Effective Resistance for Directed Graphs

In this chapter we propose a generalised definition of effective resistance for any graph, constructed in such a way that it preserves the connection between effective resistance and networked control and decision systems (see §3.2.2 or [84, 114, 115]). This new definition produces a well-defined pairwise property of nodes that depends only on the connections between the nodes. Although it is no longer always a metric on the nodes of a graph, our notion of effective resistance does allow for the construction of a resistance-based metric for any graph. This is in contrast with a (perhaps) more intuitive generalisation based on the use of pseudoinverses, which does not yield a resistance-based metric in the general case. Further, this suggests that our construction should prove to be useful for applications other than those we have presented here. The results presented in this chapter have been submitted for publication in [117]. In Chapter 5, we explore some of the implications of our new approach by computing effective resistances in several canonical directed graphs.

4.1 An Extended Definition of Effective Resistance

We now proceed to examine the derivation of effective resistance for undirected graphs and compare the matrices involved to those that arise in control-theoretic applications. Using this comparison, we propose a generalisation of effective resistance to directed graphs that preserves key control-theoretic properties related to consensus-type dynamics.

A complete derivation of the standard notion of effective resistance is given in [56], in which the effective resistance between two nodes in an undirected graph can be calculated by appropriately applying Kirchhoff's voltage and current laws. This calculation relies on what the authors call the "generalised inverse" of the Laplacian matrix, a matrix X which satisfies

$$\begin{aligned} XL = LX = \Pi \text{ and} \\ X\Pi = \Pi X = X. \end{aligned} \tag{4.1}$$

Then, if we let $X = [x_{i,j}]$, the effective resistance is given by

$$r_{k,j} = \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T X \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right) = x_{k,k} + x_{j,j} - 2x_{k,j}. \tag{4.2}$$

Although (4.1) are not the defining equations for the Moore-Penrose generalised inverse [52, §7.3, Problem 7], it is easy to show that for a symmetric Laplacian matrix L, any solution to (4.1) will indeed be the Moore-Penrose generalised inverse of L (as well as the group inverse of L). In fact, it is standard practice to define the effective resistance in terms of the Moore-Penrose generalised inverse [112].

In [56], the authors describe X in the following way (with notation changed to match this dissertation):

Definition 4.1 (Klein and Randić, [56])**.** *X is equal on $\mathbf{1}_N^\perp$ to the inverse of L and is otherwise $\mathbb{0}$.*

This definition means that $XL\mathbf{v} = LX\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathbf{1}_N^\perp$ and $X\mathbf{1}_N = \mathbf{0}$. It is therefore instructive to characterise the action of L restricted to the subspace $\mathbf{1}_N^\perp$. By the discussion in §2.1, on $\mathbf{1}_N^\perp$ the Laplacian matrix is equivalent to

$$\overline{L} = QLQ^T,$$

which is the reduced Laplacian matrix, as defined in Definition 3.2. We can see that $\overline{L}$ is a symmetric matrix if the graph is undirected. By Corollary 3.1, $\overline{L}$ is invertible for a connected graph. For undirected graphs, this allows us to give an explicit construction for X as

$$X = Q^T\overline{L}^{-1}Q, \tag{4.3}$$

which satisfies Definition 4.1 since $X\mathbf{1}_N = \mathbf{0}$ and $\overline{X\mathbf{v}} = \overline{L}^{-1}\overline{\mathbf{v}}$ for any $\mathbf{v} \in \mathbb{R}^N$. Furthermore, we can use (2.3) and the fact that $L = L\Pi = \Pi L$ for undirected graphs to show that (4.3) satisfies (4.1) when the graph is undirected.

It should be noted that $\overline{L}$ is not unique, since it depends on the choice of Q. However, if Q and Q' both satisfy (2.3), we can define $W := Q'Q^T$. Then $Q' = WQ$ and W is orthogonal. Hence $X' := Q'^T\left(Q'LQ'^T\right)^{-1}Q' = Q^TW^T\left(WQLQ^TW^T\right)^{-1}WQ = X$ and thus the computation of X in (4.3) is independent of the choice of Q.

These multiple ways (the Moore-Penrose generalised inverse, Definition 4.1, Equation (4.1) and Equation (4.3)) to describe the matrix X no longer agree when the graph is directed. While (4.3) still satisfies Definition 4.1, it does not satisfy Equation (4.1) (specifically, LX no longer equals Π). Furthermore, the Moore-Penrose generalised inverse satisfies neither (4.1) nor (4.3) for non-symmetric Laplacian matrices. Thus, instead of seeking to extend the notion of effective resistance to directed graphs using one of the above descriptions (which all arose through an analysis of electrical networks that were, by definition, undirected), we draw inspiration from a different context that is not as fundamentally tied to electrical networks.

In Chapter 3, as well as in previous work on evidence-accumulation for decision-making [83, 84], effective resistances arose due to a correspondence between covariance matrices and the matrix X as described above (in the case of undirected graphs). Both applications involved stochastic systems evolving on graphs with dynamics driven by the Laplacian matrix, and covariance matrices were sought to describe the distribution of node values. For general (i.e. directed or undirected) graphs, these covariance matrices were computed using integrals of the form [84]

$$\Sigma_1 = \int_0^\infty Q^T \mathrm{e}^{-\overline{L}t} \mathrm{e}^{-\overline{L}^T t} Q \, dt \tag{4.4}$$

and (as in (3.26))

$$\Sigma = \int_0^\infty \mathrm{e}^{-\overline{L}t} \mathrm{e}^{-\overline{L}^T t} \, dt. \tag{4.5}$$

Now, we can observe that $\Sigma_1 = Q^T \Sigma Q$, and that Σ can also be expressed as the solution to the Lyapunov equation (3.12), that is,

$$\overline{L}\Sigma + \Sigma \overline{L}^T = I_{N-1}.$$

It should be noted that (3.12) has a unique positive definite solution when all the eigenvalues of $\overline{L}$ have positive real part (i.e. when the graph is connected) [37, Proposition 4.2]. It is then clear that for undirected graphs (where $\overline{L}$ is symmetric),

$$\Sigma = \frac{1}{2} \overline{L}^{-1}, \tag{4.6}$$

and so (using (4.3)),

$$\Sigma_1 = \frac{1}{2} X.$$

It is this relationship that links these covariance matrices to the generalised inverse X, and hence to effective resistances. Since these covariance matrices arise naturally

from directed graphs as well as undirected graphs, we use their solutions to define effective resistances on directed graphs. Thus, for any connected digraph, we let Σ be the unique solution to the Lyapunov equation (3.12). Then, we let

$$X := 2Q^T \Sigma Q, \tag{4.7}$$

and notice that X will be symmetric for any graph because Σ is always symmetric. Finally we can use (4.2) to define the effective resistance between any two nodes in the graph.

Definition 4.2. *Let $\mathcal{G}$ be a connected graph with N nodes and Laplacian matrix L. Then the* effective resistance between nodes k and j in $\mathcal{G}$ *is defined as*

$$r_{k,j} = \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T X \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right) = x_{k,k} + x_{j,j} - 2x_{k,j}, \tag{4.8}$$

where

$$\begin{aligned} X &= 2Q^T \Sigma Q, \\ \overline{L}\Sigma + \Sigma \overline{L}^T &= I_{N-1}, \\ \overline{L} &= QLQ^T, \end{aligned} \tag{4.9}$$

and Q is a matrix satisfying (2.3).

By summing all distinct effective resistances in a graph, we extend the definition of the *Kirchhoff index* of the graph.

Definition 4.3. *The Kirchhoff index, K_f, of a connected (directed) graph is*

$$K_f := \sum_{k<j} r_{k,j}. \tag{4.10}$$

Our definition of K_f matches the Kirchhoff index defined for undirected graphs [112], but can now be computed for directed graphs as well.

In Chapter 3, we noticed that the $\mathcal{H}_2$ norm η of an undirected graph can be expressed in terms of the Kirchhoff index as

$$\eta = \left(\frac{K_f}{2N}\right)^{\frac{1}{2}}.$$

Using Definition 4.2, this relationship can be shown to extend to directed graphs as well.

Proposition 4.1. *The $\mathcal{H}_2$ norm of a connected directed graph, as defined in Definition 3.4, is related to the Kirchhoff index, as defined in Definition 4.3, by*

$$\eta = \left(\frac{K_f}{2N}\right)^{\frac{1}{2}}. \tag{4.11}$$

Proof. The matrix X, as defined in (4.9), is symmetric because Σ is a positive definite (and hence symmetric) matrix. Therefore, $x_{k,j} = x_{j,k}$ and so by (4.8),

$$r_{k,j} = r_{j,k}.$$

Furthermore, by (4.8), $r_{k,k} = 0$. We can therefore rewrite (4.10) as

$$\begin{aligned} K_f &= \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} r_{k,j} \\ &= \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N} x_{k,k} + x_{j,j} - 2x_{k,j} \\ &= N\,\mathrm{tr}\,(X) - \sum_{k=1}^{N}\sum_{j=1}^{N} x_{k,j}. \end{aligned}$$

But the row sums of X are all zero, since $X\mathbf{1}_N = \mathbf{0}$ by (2.3). This gives us

$$K_f = N\,\mathrm{tr}\,(X). \tag{4.12}$$

Now, from (4.9), we have $X = 2Q^T \Sigma Q$, and so we can say

$$\begin{aligned}\operatorname{tr}(X) &= 2\operatorname{tr}\left(Q^T \Sigma Q\right) \\ &= 2\operatorname{tr}\left(QQ^T \Sigma\right) \text{ (by the properties of the trace function [52, §1.2, Problem 2])} \\ &= 2\operatorname{tr}(\Sigma) \text{ (by (2.3)).}\end{aligned}$$

Then using (3.19), we have

$$\eta = \sqrt{\frac{\operatorname{tr}(X)}{2}}. \tag{4.13}$$

Together, (4.12) and (4.13) give us (4.11). □

Thus our definition of effective resistance immediately connects to the robustness to noise of linear consensus on directed graphs. In a similar fashion, the variance of each node in a balanced network of stochastic decision-makers can be computed (in part) using the diagonal entries of the matrix Σ_1 from (4.4). Then the variance of any particular node can be found using this definition of effective resistance [84].

4.2 Basic Properties of our Definition

Although Definition 4.2 ensures that effective resistance maintains our desired relationship with the $\mathcal{H}_2$ norm of a graph, by itself it remains an algebraic construction that yields little insight into the ways in which effective resistance (and hence robustness) depends on the graph structure. We now proceed to analyse our definition to understand some of its fundamental properties. In §4.2.1 we verify that Definition 4.2 results in a well-defined property of pairs of nodes in a connected digraph. In §4.2.2 we investigate how effective resistances depend on connections in the graph and extend Definition 4.2 further to apply to disconnected graphs. Finally in §4.2.3 we determine that effective resistance is a distance-like function and explore the limitations of the triangle inequality for effective resistances in directed graphs.

4.2.1 Effective resistance is well-defined

By construction, (4.8) will yield the regular effective resistance for any undirected graph. However, we must confirm that our concept of effective resistance for directed graphs is well-defined. This is achieved by the following two lemmas.

Lemma 4.1. *The value of the effective resistance between two nodes in a connected digraph is independent of the choice of Q.*

Proof. Let Q and Q' be two matrices that satisfy (2.3), and let $r_{k,j}$ and $r'_{k,j}$ be the corresponding effective resistances between nodes k and j, computed by using Q and Q' in (4.8), respectively. Furthermore, let $W := Q'Q^T$. Then by (2.3), $Q' = WQ$ and $WW^T = W^TW = I_{N-1}$. Now, we can use (3.6) and the properties of W to write

$$\overline{L}' = W\overline{L}W^T.$$

Next, substituting this expression into (3.12) for $\overline{L}'$, we see that

$$\Sigma' = W\Sigma W^T.$$

Finally, we observe that

$$X' = 2Q'^T\Sigma'Q' = X,$$

and hence, $r'_{k,j} = r_{k,j}$. □

From Lemma 4.1 we see that no matter how it is computed, the effective resistance between two nodes will be the same unique number. Next, we will show in Lemma 4.2 that the effective resistance is a property of a pair of nodes, irrespective of the way in which they are labelled.

Lemma 4.2. *The value of the effective resistance between two nodes in a connected digraph is independent of the labelling of the nodes.*

Proof. Any two labelings of the nodes in a graph can be related via a permutation. Suppose L and L' are two Laplacian matrices associated with the same graph, but with different labelings of the nodes. Then L' can be found from L by permuting its rows and columns. That is, there exists an $N \times N$ permutation matrix P such that

$$L' = PLP^T.$$

Note that as a permutation matrix, there is exactly one 1 in every row and column of P with every other entry equal to 0. Furthermore, $P^{-1} = P^T$, $P\mathbf{1}_N = \mathbf{1}_N$ and $\mathbf{1}_N^T P = \mathbf{1}_N^T$ [52, §0.9.5]. Thus we can observe that

$$QP = \overline{P}Q, \quad PQ^T = Q^T\overline{P}$$

and

$$\overline{P}^{-1} = \overline{P}^T,$$

where, as usual, $\overline{P} = QPQ^T$.

Now, we can use (3.6) and the properties of P to write

$$\overline{L}' = \overline{P}\,\overline{L}\,\overline{P}^T.$$

Then the solution to the Lyapunov equation associated with $\overline{L}'$ becomes

$$\Sigma' = \overline{P}\Sigma\overline{P}^T.$$

Hence, we observe that $X' = PXP^T$.

Thus if P permutes node k to node m and node j to node ℓ, we find that

$$r'_{m,\ell} = r_{k,j}. \qquad \square$$

4.2.2 Effective resistance depends on connections between nodes

Next we consider which features of a digraph will affect the effective resistance between a given pair of nodes. For undirected graphs, we know that effective resistances depend on every path between a pair of nodes [56]. The situation becomes more complicated with directed graphs since there can exist pairs of nodes in a connected digraph with no path between them. Instead of looking at paths between nodes, we therefore have to consider connections. To incorporate all of the connections between two nodes, we examine the *reachable subgraph*, an example of which is shown in Figure 4.1.

Definition 4.4. *The* reachable subgraph, denoted $\mathcal{R}_{\mathcal{G}}(k, j)$, of nodes k and j in the graph $\mathcal{G}$ *is the graph formed by every node in* $\mathcal{G}$ *that is reachable from node* k *or node* j *and every edge in* $\mathcal{G}$ *between these nodes.*

As we demonstrate in the following lemma, if $\mathcal{G}$ is connected, the reachable subgraph of nodes k and j is precisely the subgraph formed by every connection between them.

Lemma 4.3. *If* $\mathcal{G}$ *is connected, then for any pair of nodes* k *and* j*,*

(i) $\mathcal{R}_{\mathcal{G}}(k, j)$ *is connected,*

(ii) Every node in $\mathcal{R}_{\mathcal{G}}(k, j)$ *is part of a connection between nodes* k *and* j*,*

(iii) Every edge in $\mathcal{R}_{\mathcal{G}}(k, j)$ *is part of a connection between nodes* k *and* j *and*

(iv) Every connection in $\mathcal{G}$ *between nodes* k *and* j *is contained in* $\mathcal{R}_{\mathcal{G}}(k, j)$*.*

Proof. (i) Since $\mathcal{G}$ is connected, there is a node, ℓ, in $\mathcal{G}$ which is reachable from every other node. Since ℓ is reachable from nodes k and j, it is also in $\mathcal{R}_{\mathcal{G}}(k, j)$. Now, suppose that m is a node in $\mathcal{R}_{\mathcal{G}}(k, j)$. Then there is a path in $\mathcal{G}$ from

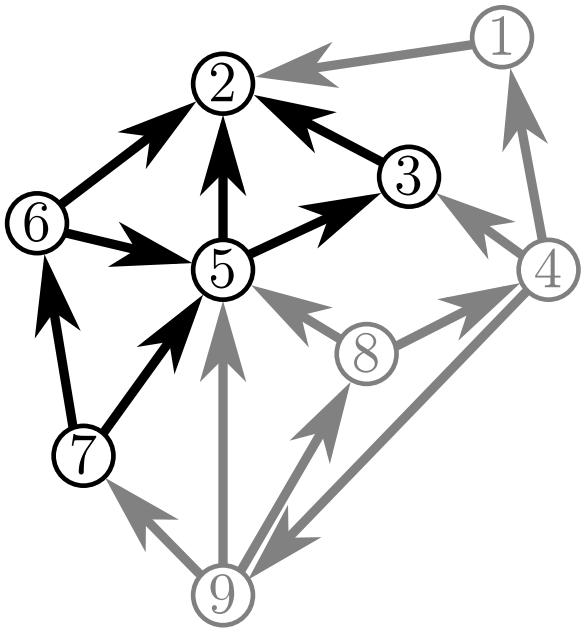

Figure 4.1: A directed graph on 9 nodes, with the reachable subgraph of nodes 3 and 7 highlighted.

m to ℓ. Since m is reachable from either k or j, every node along this path is as well. Thus, this path is contained in $\mathcal{R}_{\mathcal{G}}(k, j)$ and so ℓ is reachable (in $\mathcal{R}_{\mathcal{G}}(k, j)$) from every node in $\mathcal{R}_{\mathcal{G}}(k, j)$.

(ii) Let m be a node in $\mathcal{R}_{\mathcal{G}}(k, j)$. Then m must be reachable from either k or j. Without loss of generality, suppose that m is reachable from k. Then (as we saw in part (i)) there must be a path in $\mathcal{R}_{\mathcal{G}}(k, j)$ from m to the globally reachable node ℓ as well as a path from j to ℓ. Thus m is part of a connection between k and j.

(iii) Let (m, n) be an edge in $\mathcal{R}_{\mathcal{G}}(k, j)$. Without loss of generality, suppose that m is reachable from k. Then n is also reachable from k. Then (as we saw in part (i)) there must be a path in $\mathcal{R}_{\mathcal{G}}(k, j)$ from n to the globally reachable node ℓ as well as a path from j to ℓ. Thus (m, n) is part of a connection between k and j.

(iv) Every node along a path is reachable from the node where the path started. Thus, every node in a connection between k and j is reachable from either k or j and hence in $\mathcal{R}_{\mathcal{G}}(k,j)$. Since $\mathcal{R}_{\mathcal{G}}(k,j)$ contains every edge in $\mathcal{G}$ between its nodes, every edge in the connection must also be in $\mathcal{R}_{\mathcal{G}}(k,j)$.

□

Next we proceed to show in Theorem 4.1 that the effective resistance between two nodes in a connected digraph can only depend on the connections between them. The proof relies on the following lemma, which describes sufficient conditions under which effective resistances in a subgraph will be equal to those in the original graph.

Lemma 4.4. *Suppose that $\mathcal{G}_1$ is a connected subgraph (containing N_1 nodes and with Laplacian matrix L_1) of a connected graph $\mathcal{G}$ (containing N nodes and with Laplacian matrix L) and suppose that the nodes in $\mathcal{G}_1$ are labelled 1 through N_1. Let $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ be a matrix satisfying (2.3) and suppose there is a $Q \in \mathbb{R}^{(N-1)\times N}$ satisfying (2.3) that can be written as*

$$Q = \begin{bmatrix} Q_1 & \emptyset \\ \alpha \mathbf{1}_{N_1}^T & \mathbf{r}^T \\ \emptyset & S \end{bmatrix}$$

for some $\alpha \in \mathbb{R}$, $\mathbf{r} \in \mathbb{R}^{N-N_1}$ and $S \in \mathbb{R}^{(N-N_1-1)\times(N-N_1)}$. If the solution to (3.12) for $\mathcal{G}$ (with $\overline{L} = QLQ^T$) can be written as

$$\Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{t} & U \\ \mathbf{t}^T & v & \mathbf{w}^T \\ U^T & \mathbf{w} & Y \end{bmatrix},$$

for some $\mathbf{t} \in \mathbb{R}^{N_1-1}$, $U \in \mathbb{R}^{(N_1-1)\times(N-N_1-1)}$, $v \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^{N-N_1-1}$ and $Y \in \mathbb{R}^{(N-N_1-1)\times(N-N_1-1)}$ with $Y = Y^T$ and where $\Sigma_1 \in \mathbb{R}^{(N_1-1)\times(N_1-1)}$ is the solution

to (3.12) for $\mathcal{G}_1$ (with $\overline{L}_1 = Q_1 L_1 Q_1^T$), then for any $k, j \leq N_1$, the effective resistance between nodes k and j in $\mathcal{G}$ is equal to the effective resistance between the same two nodes in $\mathcal{G}_1$.

Proof. Effective resistances in $\mathcal{G}_1$ can be found from $X_1 = 2Q_1^T \Sigma_1 Q_1$ as

$$r_{1\,k,j} = x_{1\,k,k} + x_{1\,j,j} - 2x_{1\,k,j}.$$

To compute effective resistances in $\mathcal{G}$, we must examine $X = 2Q^T \Sigma Q$. Using the matrices given in the statement of the lemma, we obtain

$$X = \begin{bmatrix} 2Q_1^T \Sigma_1 Q_1 + 2\alpha \mathbf{1}_{N_1} \mathbf{t}^T Q_1 + 2\alpha Q_1^T \mathbf{t} \mathbf{1}_{N_1}^T + 2\alpha^2 v \mathbf{1}_{N_1} \mathbf{1}_{N_1}^T & \star \\ \star & \star \end{bmatrix}.$$

If we let $\mathbf{p} := 2\alpha Q_1^T \mathbf{t} = [p_i]$, we can write

$$X = \begin{bmatrix} X_1 + \mathbf{1}_{N_1} \mathbf{p}^T + \mathbf{p} \mathbf{1}_{N_1}^T + 2\alpha^2 v \mathbf{1}_{N_1} \mathbf{1}_{N_1}^T & \star \\ \star & \star \end{bmatrix}.$$

Finally, since nodes k and j are both in $\mathcal{G}_1$, we obtain

$$\begin{aligned} r_{k,j} &= x_{1\,k,k} + 2p_k + 2\alpha^2 v + x_{1\,j,j} + 2p_j + 2\alpha^2 v - 2x_{1\,k,j} - 2(p_k + p_j) - 4\alpha^2 v \\ &= x_{1\,k,k} + x_{1\,j,j} - 2x_{1\,k,j} \\ &= r_{1\,k,j}. \end{aligned}$$

Note that the same calculation applies if $N = N_1 + 1$, in which case the $\emptyset_{(N-N_1-1)\times N_1}$, S, U, $\mathbf{w}$ and Y blocks of Q and Σ are all empty. □

Now we can state the first main result of the chapter, that the effective resistance between two nodes in a connected digraph can only depend on the connections between them.

Theorem 4.1. *The effective resistance between nodes k and j in a connected graph $\mathcal{G}$ is equal to the effective resistance between nodes k and j in $\mathcal{R}_{\mathcal{G}}(k,j)$.*

Proof. Let $\mathcal{G}_1 = \mathcal{R}_{\mathcal{G}}(k,j)$. Let N_1, A_1, D_1 and L_1 be the number of nodes, the adjacency matrix, the matrix of node out-degrees and the Laplacian matrix of $\mathcal{G}_1$, respectively. Let $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ satisfy (2.3). Since we know by Lemma 4.3 that $\mathcal{G}_1$ is connected, we can find matrices $\overline{L}_1$, Σ_1 and X_1 from (4.9) for $\mathcal{G}_1$.

Let $\mathcal{G}_2$ be the subgraph of $\mathcal{G}$ formed by every node in $\mathcal{G}$ which is not in $\mathcal{G}_1$ and every edge in $\mathcal{G}$ between these nodes. Then $\mathcal{G}_2$ will contain N_2 nodes and have associated matrices A_2, D_2, L_2, Q_2 and $\overline{L}_2$. Note that $\mathcal{G}_2$ may or may not be connected.

Now, if there was an edge (m,n) in $\mathcal{G}$ from a node m in $\mathcal{G}_1$ to a node n in $\mathcal{G}_2$, then n would be reachable from either k or j (as m is reachable from one of these nodes). Thus there are no edges in $\mathcal{G}$ leading from nodes in $\mathcal{G}_1$ to nodes in $\mathcal{G}_2$. Therefore, if we order the nodes in $\mathcal{G}$ with the nodes in $\mathcal{G}_1$ listed first, followed by the nodes in $\mathcal{G}_2$, then the adjacency matrix of $\mathcal{G}$ can be written as

$$A = \begin{bmatrix} A_1 & \emptyset \\ A_{2,1} & A_2 \end{bmatrix},$$

where $A_{2,1} \in \mathbb{R}^{N_2\times N_1}$ contains the edge weights for all edges in $\mathcal{G}$ leading from nodes in $\mathcal{G}_2$ to nodes in $\mathcal{G}_1$. Similarly, we can write the matrix of node out-degrees as

$$D = \begin{bmatrix} D_1 & \emptyset \\ \emptyset & D_2 + D_{2,1} \end{bmatrix},$$

where $D_{2,1}$ is the diagonal matrix containing the row sums of $A_{2,1}$ along its diagonal. Utilising these two expressions, we find that the Laplacian matrix of $\mathcal{G}$ can be written as

$$L = \begin{bmatrix} L_1 & \emptyset \\ -A_{2,1} & L_2 + D_{2,1} \end{bmatrix}.$$

Now, let

$$Q = \begin{bmatrix} Q_1 & \emptyset \\ \alpha \mathbf{1}_{N_1}^T & -\beta \mathbf{1}_{N_2}^T \\ \emptyset & Q_2 \end{bmatrix},$$

where $\alpha = \sqrt{\dfrac{N_2}{N_1 (N_1 + N_2)}}$ and $\beta = \sqrt{\dfrac{N_1}{N_2 (N_1 + N_2)}}$. Then Q satisfies (2.3) (note that $N = N_1 + N_2$). Substituting this matrix into (3.6) gives us

$$\overline{L} = \begin{bmatrix} \overline{L}_1 & \mathbf{0} & \emptyset \\ \alpha \mathbf{1}_{N_1}^T L_1 Q_1^T + \beta \mathbf{1}_{N_2}^T A_{2,1} Q_1^T & \beta (\alpha + \beta) \mathbf{1}_{N_2}^T \mathbf{d}_{2,1} & -\beta \mathbf{1}_{N_2}^T L_2 Q_2^T - \beta \mathbf{1}_{N_2}^T D_{2,1} Q_2^T \\ -Q_2 A_{2,1} Q_1^T & -(\alpha + \beta) Q_2 \mathbf{d}_{2,1} & \overline{L}_2 + Q_2 D_{2,1} Q_2^T \end{bmatrix},$$

where $\mathbf{d}_{2,1} := D_{2,1} \mathbf{1}_{N_2} = A_{2,1} \mathbf{1}_{N_1}$.

In order to compute effective resistances in $\mathcal{G}$, we must find the matrix Σ which solves (3.12). Since we have partitioned $\overline{L}$ into a 3×3 block matrix, we will do the same for Σ. Let

$$\Sigma = \begin{bmatrix} S & \mathbf{t} & U \\ \mathbf{t}^T & v & \mathbf{w}^T \\ U^T & \mathbf{w} & Y \end{bmatrix},$$

where $S \in \mathbb{R}^{(N_1 - 1) \times (N_1 - 1)}$ and $Y \in \mathbb{R}^{(N_2 - 1) \times (N_2 - 1)}$ are symmetric matrices, $U \in \mathbb{R}^{(N_1 - 1) \times (N_2 - 1)}$, $\mathbf{t} \in \mathbb{R}^{N_1 - 1}$, $\mathbf{w} \in \mathbb{R}^{N_2 - 1}$ and $v \in \mathbb{R}$. Then when we multiply out the matrices of (3.12) and equate the $(1, 1)$ blocks, we find

$$\overline{L}_1 S + S \overline{L}_1^T = I_{N_1 - 1},$$

which implies that

$$S = \Sigma_1.$$

Thus, by Lemma 4.4, the effective resistance between two nodes in $\mathcal{G}_1$ is equal to the effective resistance between the same two nodes in $\mathcal{G}$. □

We can use Theorem 4.1 to partially extend the definition of effective resistance to disconnected digraphs. To do this, we will first define *connection subgraphs*.

Definition 4.5. *A* connection subgraph between nodes k and j in the graph $\mathcal{G}$ *is a maximal connected subgraph of $\mathcal{G}$ in which every node and edge form part of a connection between nodes k and j in $\mathcal{G}$. That is, a connection subgraph is formed from the union of connections between nodes k and j, and the addition of any other connections would make the subgraph disconnected. If only one connection subgraph exists in $\mathcal{G}$ between nodes k and j, it is referred to as* the *connection subgraph and is denoted by $\mathcal{C}_{\mathcal{G}}(k, j)$.*

From Lemma 4.3 we know that $\mathcal{C}_{\mathcal{G}}(k, j) = \mathcal{R}_{\mathcal{G}}(k, j)$ if $\mathcal{G}$ is connected. However, a disconnected graph may contain 0, 1 or more connection subgraphs between a pair of nodes. There will be no connection subgraphs precisely when there are no connections between nodes k and j in $\mathcal{G}$. However, there may also be multiple connections between a pair of nodes that lead to multiple connection subgraphs. A simple example of connection subgraphs in a disconnected graph is shown in Figure 4.2.

By definition, whenever it exists, $\mathcal{C}_{\mathcal{G}}(k, j)$ will be connected, and we can thus compute effective resistances between its nodes. We can now define effective resistances between some node pairs in any digraph, whether or not it is connected.

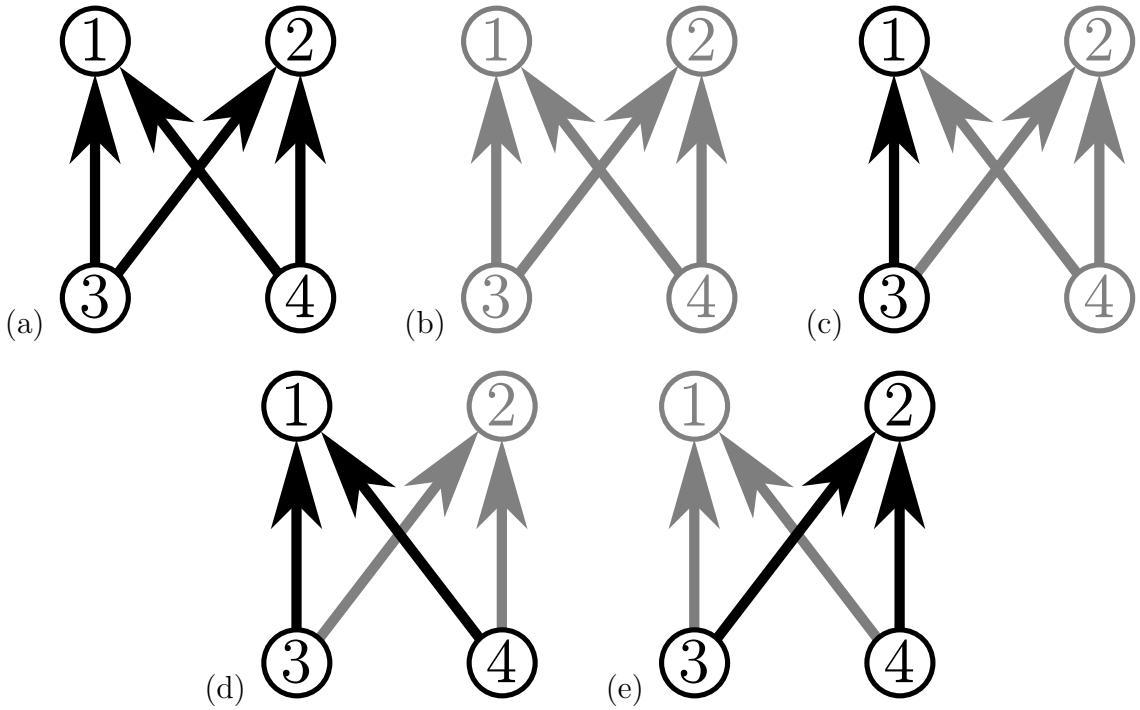

Figure 4.2: (a) A disconnected graph on 4 nodes. (b) There are no connection subgraphs between nodes 1 and 2. (c) The connection subgraph between nodes 1 and 3 is highlighted. (d) One connection subgraph between nodes 3 and 4 is highlighted. (e) A second connection subgraph between nodes 3 and 4 is highlighted. In this example, the effective resistance between nodes 3 and 4 is undefined.

Definition 4.6. *The effective resistance between nodes k and j in a graph $\mathcal{G}$ is*

$$r_{k,j} = \begin{cases} \infty & \text{if there are no connections between nodes } k \text{ and } j \\ r_{k,j} \text{ in } \mathcal{C}_{\mathcal{G}}(k,j) & \text{if } \mathcal{C}_{\mathcal{G}}(k,j) \text{ exists (computed using (4.8))} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

By Theorem 4.1, this new definition specialises to our original definition of effective resistance for connected graphs. For certain applications, there may be an appropriate way to handle pairs of nodes with multiple connection subgraphs, but that falls outside the scope of the present work.

In undirected graphs, we know that the effective resistance between two nodes does not depend on edges that do not lie on any simple path between the nodes [56].

Unfortunately, the situation is not as straightforward for directed graphs. Consider the 4-node graphs shown in Figure 4.3. In $\mathcal{G}_4^{\text{path}}$, we observe that the Laplacian matrix is

$$L_4^{\text{path}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Then, if we let

$$Q_4 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{\sqrt{3}}{2} \end{bmatrix},$$

we can see that Q_4 satisfies (2.3) and so by (3.6),

$$\overline{L}_4^{\text{path}} = Q_4 L_4^{\text{path}} Q_4^T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{\sqrt{3}} & 1 & 0 \\ -\frac{1}{2\sqrt{6}} & -\frac{3}{2\sqrt{2}} & 1 \end{bmatrix}.$$

Then following Definition 4.2, we find that

$$\Sigma_4^{\text{path}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & \frac{5}{6} & \frac{5}{6\sqrt{2}} \\ \frac{1}{2\sqrt{6}} & \frac{5}{6\sqrt{2}} & \frac{7}{6} \end{bmatrix},$$

$$X_4^{\text{path}} = 2Q_4^T \Sigma_4^{\text{path}} Q_4 = \begin{bmatrix} \frac{7}{4} & \frac{1}{4} & -\frac{3}{4} & -\frac{5}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{5}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{7}{4} \end{bmatrix},$$

and so

$$r_{3,4}^{\text{path}} = x_{4\,3,3}^{\text{path}} + x_{4\,4,4}^{\text{path}} - 2x_{4\,3,4}^{\text{path}} = 2.$$

However in $\mathcal{G}^{\text{line}}$, we observe that the Laplacian matrix is

$$L_4^{\text{line}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Using the same matrix Q_4 as above, we find by (3.6) that

$$\overline{L}_4^{\text{line}} = Q_4 L_4^{\text{line}} Q_4^T = \begin{bmatrix} 2 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 1 & 0 \\ \frac{1}{2\sqrt{6}} & -\frac{3}{2\sqrt{2}} & 1 \end{bmatrix}.$$

Then following Definition 4.2, we find that

$$\Sigma_4^{\text{line}} = \begin{bmatrix} \frac{1}{4} & \frac{1}{12\sqrt{3}} & 0 \\ \frac{1}{12\sqrt{3}} & \frac{19}{36} & \frac{7}{18\sqrt{2}} \\ 0 & \frac{7}{18\sqrt{2}} & \frac{19}{24} \end{bmatrix},$$

$$X_4^{\text{line}} = 2Q_4^T \Sigma_4^{\text{line}} Q_4 = \begin{bmatrix} \frac{107}{144} & \frac{3}{16} & -\frac{49}{144} & -\frac{85}{144} \\ \frac{3}{16} & \frac{91}{144} & -\frac{11}{48} & -\frac{85}{144} \\ -\frac{49}{144} & -\frac{11}{48} & \frac{83}{144} & -\frac{1}{144} \\ -\frac{85}{144} & -\frac{85}{144} & -\frac{1}{144} & \frac{19}{16} \end{bmatrix},$$

and so

$$r_{3,4}^{\text{line}} = x_{4\,3,3}^{\text{line}} + x_{4\,4,4}^{\text{line}} - 2x_{4\,3,4}^{\text{line}} = \frac{16}{9}.$$

Therefore, even though no new simple connections were introduced between nodes 3 and 4 in $\mathcal{G}_4^{\text{line}}$, the effective resistance between these two nodes changed. This demonstrates that effective resistance in general directed graphs cannot be thought of in terms of an electrical analogy. Instead, for a consensus system (for example) the effective resistance between two nodes measures how "close" the states of each agent can be expected to be to each other. Thus the additional edge in $\mathcal{G}_4^{\text{line}}$ should bring nodes 1 and 2 "closer" together, but also reduce their variance. Then nodes 3 and 4, which are following node 2, are able to get "closer" to node 2 and hence "closer" to each other as well.

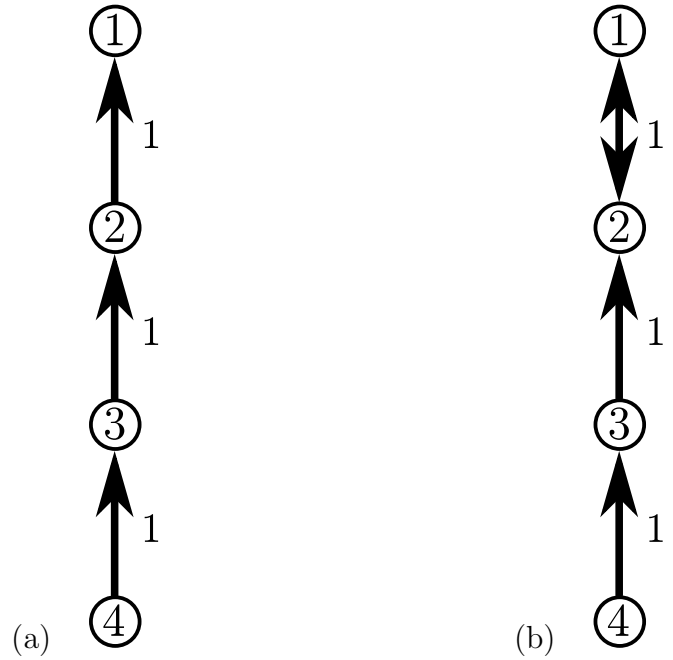

Figure 4.3: Two simple 4-node graphs: (a) $\mathcal{G}_4^{\text{path}}$, a 4-node directed path graph with unit edge weights, and (b) $\mathcal{G}^{\text{line}}$, similar to $\mathcal{G}_4^{\text{path}}$ but with the directed edge $(2,1)$ replaced by an undirected edge.

Despite this (perhaps) unexpected behaviour, we are able to show below that certain parts of the connection subgraph do not affect the effective resistance between two nodes. The proof relies on the following lemma, which provides a solution to Lyapunov equations with a certain structure.

Lemma 4.5. *Suppose that $L_1 \in \mathbb{R}^{N_1 \times N_1}$ is the Laplacian matrix of a connected graph satisfying $L_1^T \mathbf{e}_{N_1}^{(1)} = \mathbf{0}$ and $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ satisfies (2.3). Let $\alpha = \dfrac{1}{\sqrt{N_1(N_1+1)}}$, $\beta = \dfrac{N_1}{\sqrt{N_1(N_1+1)}}$ and $a > 0$. If $\overline{L}_1 = Q_1 L_1 Q_1^T$, then a solution to the Lyapunov equation*

$$\begin{bmatrix} \overline{L}_1 + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T & a(\alpha+\beta)Q_1\mathbf{e}_{N_1}^{(1)} \\ \alpha\mathbf{1}_{N_1}^T L_1 Q_1^T + a\alpha\mathbf{e}_{N_1}^{(1)T}Q_1^T & \frac{a}{N_1} \end{bmatrix} \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix}$$
$$+ \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix} \begin{bmatrix} \overline{L}_1^T + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T & \alpha Q_1 L_1^T \mathbf{1}_{N_1} + a\alpha Q_1 \mathbf{e}_{N_1}^{(1)} \\ a(\alpha+\beta)\mathbf{e}_{N_1}^{(1)T}Q_1^T & \frac{a}{N_1} \end{bmatrix} = \begin{bmatrix} I_{N_1-1} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

is

$$\begin{aligned} S &= \Sigma_1, \\ \mathbf{t} &= -N_1\alpha\Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} \text{ and} \\ u &= \frac{N_1}{2a} + \frac{N_1^2\alpha^2}{a}\left(\mathbf{1}_{N_1}^T L_1 + a\mathbf{e}_{N_1}^{(1)T}\right) Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)}, \end{aligned} \tag{4.14}$$

where Σ_1 is the solution to (3.12) for $\overline{L}_1$.

Proof. First we note that Σ_1 exists since L_1 is the Laplacian of a connected graph. Next, equating blocks of the given matrix equation gives us

$$\overline{L}_1 S + S\overline{L}_1^T + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T S + aSQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T$$
$$+ a(\alpha+\beta)\left(Q_1\mathbf{e}_{N_1}^{(1)}\mathbf{t}^T + \mathbf{t}\mathbf{e}_{N_1}^{(1)T}Q_1^T\right) = I_{N_1-1}, \tag{4.15}$$

$$\alpha SQ_1\left(L_1^T\mathbf{1}_{N_1} + a\mathbf{e}_{N_1}^{(1)}\right) + au(\alpha+\beta)Q_1\mathbf{e}_{N_1}^{(1)}$$
$$+ \left(\frac{a}{N_1}I_{N_1-1} + \overline{L}_1 + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T\right)\mathbf{t} = \mathbf{0} \text{ and} \tag{4.16}$$

$$\frac{2au}{N_1} + 2\alpha\left(\mathbf{1}_{N_1}^T L_1 + a\mathbf{e}_{N_1}^{(1)T}\right)Q_1^T\mathbf{t} = 1. \tag{4.17}$$

By directly substituting (4.14) into (4.15) and (4.17) (and noting that $N_1\alpha(\alpha+\beta) = 1$), we observe that (4.14) satisfies (4.15) and (4.17). Hence, we now focus our

attention on (4.16). Substituting (4.14) into the left-hand side of (4.16) gives us

$$LHS = \alpha\Sigma_1 Q_1 L_1^T \mathbf{1}_{N_1} - N_1\alpha\overline{L}_1\Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} + \frac{1}{2\alpha} Q_1 \mathbf{e}_{N_1}^{(1)} + N_1\alpha Q_1 \mathbf{e}_{N_1}^{(1)} \mathbf{1}_{N_1}^T L_1 Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)}. \tag{4.18}$$

By (3.12), we know that $\overline{L}_1\Sigma_1 = I_{N_1-1} - \Sigma_1\overline{L}_1^T$. Therefore, using (2.3) and our assumption that $L_1^T\mathbf{e}_{N_1}^{(1)} = \mathbf{0}$, we have

$$\overline{L}_1\Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} = Q_1 \mathbf{e}_{N_1}^{(1)} + \frac{1}{N_1}\Sigma_1 Q_1 L_1^T \mathbf{1}_{N_1}.$$

Thus $N_1\alpha\overline{L}_1\Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} = N_1\alpha Q_1 \mathbf{e}_{N_1}^{(1)} + \alpha\Sigma_1 Q_1 L_1^T \mathbf{1}_{N_1}$, and (4.18) becomes

$$LHS = \left(\frac{1}{2\alpha} - N_1\alpha + N_1\alpha \mathbf{1}_{N_1}^T L_1 Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)}\right) Q_1 \mathbf{e}_{N_1}^{(1)}. \tag{4.19}$$

Next, if we define V to be the matrix $V := L_1 Q_1^T \Sigma_1 Q_1 + Q_1^T \Sigma_1 Q_1 L_1^T$, we have that $V = V^T$ and (since $\mathbf{1}_{N_1}^T Q_1^T = \mathbf{0}^T$ by (2.3)),

$$\begin{aligned}
\mathbf{1}_{N_1}^T V \mathbf{e}_{N_1}^{(1)} &= \mathbf{1}_{N_1}^T L_1 Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} + \mathbf{1}_{N_1}^T Q_1^T \Sigma_1 Q_1 L_1^T \mathbf{e}_{N_1}^{(1)} \\
&= \mathbf{1}_{N_1}^T L_1 Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)}.
\end{aligned}$$

But pre- and post-multiplying V by $\Pi = Q_1^T Q_1$ and using (3.12) gives us

$$\begin{aligned}
\Pi V \Pi &= Q_1^T Q_1 L_1 Q_1^T \Sigma_1 Q_1 Q_1^T Q_1 + Q_1^T Q_1 Q_1^T \Sigma_1 Q_1 L_1^T Q_1^T Q_1 \\
&= Q_1^T \left(\overline{L}_1\Sigma_1 + \Sigma_1\overline{L}_1^T\right) Q_1 \\
&= Q_1^T I_{N_1-1} Q_1 \\
&= \Pi,
\end{aligned}$$

and then by pre- and post-multiplying by $\mathbf{e}_{N_1}^{(1)T}$ and $\mathbf{e}_{N_1}^{(1)}$, we find

$$\mathbf{e}_{N_1}^{(1)T} V \mathbf{e}_{N_1}^{(1)} - \frac{2}{N_1} \mathbf{1}_{N_1}^T V \mathbf{e}_{N_1}^{(1)} + \frac{1}{N_1^2} \mathbf{1}_{N_1}^T V \mathbf{1}_{N_1} = \frac{N_1 - 1}{N_1} \text{ (since } V \text{ is symmetric).}$$

But since $L_1^T \mathbf{e}_{N_1}^{(1)} = \mathbf{0}$ and $Q_1 \mathbf{e}_{N_1}^{(1)} = \mathbf{0}$, we know that both $\mathbf{e}_{N_1}^{(1)T} V \mathbf{e}_{N_1}^{(1)} = 0$ and $\mathbf{1}_{N_1}^T V \mathbf{1}_{N_1} = 0$. Thus

$$\mathbf{1}_{N_1}^T V \mathbf{e}_{N_1}^{(1)} = \frac{1 - N_1}{2} \Rightarrow \mathbf{1}_{N_1}^T L_1 Q_1^T \Sigma_1 Q_1 \mathbf{e}_{N_1}^{(1)} = \frac{1 - N_1}{2},$$

and so (4.19) becomes $LHS = \mathbf{0}$. Thus (4.14) also satisfies (4.16) and is therefore a solution to the given matrix equation. □

We can now proceed to state our next main result, that a globally reachable node that is connected to the rest of the graph through a single directed edge does not affect the effective resistance between any pair of other nodes in the graph.

Theorem 4.2. *Suppose $\mathcal{G}_1$ is a connected graph containing only one globally reachable node, and let $\mathcal{G}$ be the graph formed by connecting the globally reachable node in $\mathcal{G}_1$ to an additional node via a directed edge of arbitrary weight. Then the effective resistance between any two nodes in $\mathcal{G}_1$ is equal to the effective resistance between them in $\mathcal{G}$.*

Proof. Let N_1, A_1, D_1 and L_1 be the number of nodes, the adjacency matrix, the matrix of node out-degrees and the Laplacian matrix of $\mathcal{G}_1$, respectively. Let $Q_1 \in \mathbb{R}^{(N_1-1)\times N_1}$ satisfy (2.3). Using Q_1, we can compute $\overline{L}_1$ from (3.6) and since $\mathcal{G}_1$ is connected, we can find matrices Σ_1 and X_1 from (4.9). Without loss of generality, suppose that the globally reachable node in $\mathcal{G}_1$ is node 1. Since this is the only globally reachable node in $\mathcal{G}_1$, no other nodes can be reached from node 1 and hence $d_1^{out} = 0$. Thus

$$L_1^T \mathbf{e}_{N_1}^{(1)} = \mathbf{0}. \tag{4.20}$$

Let the additional node in $\mathcal{G}$ be node $N = N_1 + 1$. We can see that since node 1

is globally reachable in $\mathcal{G}_1$ and node N is reachable from node 1, node N is globally reachable in $\mathcal{G}$. Thus $\mathcal{G}$ is connected.

Now, we can write the adjacency matrix of $\mathcal{G}$ as

$$A = \begin{bmatrix} A_1 & a\mathbf{e}_{N_1}^{(1)} \\ \mathbf{0}^T & 0 \end{bmatrix},$$

where $a > 0$ is the weight on edge $(1, N)$ in $\mathcal{G}$. Similarly, we can write the matrix of node out-degrees as

$$D = \begin{bmatrix} D_1 + a\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}.$$

Utilizing these two expressions, we find that the Laplacian matrix of $\mathcal{G}$ can be written as

$$L = \begin{bmatrix} L_1 + a\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T} & -a\mathbf{e}_{N_1}^{(1)} \\ \mathbf{0}^T & 0 \end{bmatrix}.$$

Now, let

$$Q = \begin{bmatrix} Q_1 & \mathbf{0} \\ \alpha\mathbf{1}_{N_1}^T & -\beta \end{bmatrix},$$

where $\alpha = \dfrac{1}{\sqrt{N_1\,(N_1+1)}}$ and $\beta = \dfrac{N_1}{\sqrt{N_1\,(N_1+1)}}$. Then Q satisfies (2.3). Substituting this matrix into (3.6) gives us

$$\overline{L} = \begin{bmatrix} \overline{L}_1 + aQ_1\mathbf{e}_{N_1}^{(1)}\mathbf{e}_{N_1}^{(1)T}Q_1^T & a(\alpha+\beta)Q_1\mathbf{e}_{N_1}^{(1)} \\ \alpha\mathbf{1}_{N_1}^T L_1 Q_1^T + a\alpha\mathbf{e}_{N_1}^{(1)T}Q_1^T & \frac{a}{N_1} \end{bmatrix}.$$

In order to compute effective resistances in $\mathcal{G}$, we must find the matrix Σ which solves (3.12). Since we have partitioned $\overline{L}$ into a 2×2 block matrix, we will do the

same for Σ. Let

$$\Sigma = \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix},$$

where $S \in \mathbb{R}^{(N_1-1)\times(N_1-1)}$ is a symmetric matrix, $\mathbf{t} \in \mathbb{R}^{N_1-1}$ and $u \in \mathbb{R}$. Then Lemma 4.5 gives a solution to (3.12) using $\overline{L}$ and our desired form of Σ. However, since $\mathcal{G}$ is connected, we know that there must be a unique solution. Thus (4.14) from Lemma 4.5 is the unique solution to the Lyapunov equation, and in particular,

$$S = \Sigma_1.$$

Thus, by Lemma 4.4, the effective resistance between two nodes in $\mathcal{G}_1$ is equal to the effective resistance between the same two nodes in $\mathcal{G}$. □

Corollary 4.1. *Suppose $\mathcal{C}_{\mathcal{G}}(k,j)$ consists of a subgraph $\mathcal{C}'_{\mathcal{G}}(k,j)$ that is connected via a single edge of arbitrary weight to the leaf node of a directed path. Then the effective resistance between nodes k and j in the graph $\mathcal{G}$ is equal to the effective resistance between nodes k and j in $\mathcal{C}'_{\mathcal{G}}(k,j)$.*

Proof. This follows by simply applying Theorem 4.2 repeatedly to "prune" away the nodes in the directed path. □

We can see from the graphs shown in Figure 4.3 that this "pruning" operation can only be applied to directed edges in general. There may, however, be other graphical structures that also do not affect the effective resistance between two nodes.

4.2.3 Effective resistance is a distance-like function and its square root is a metric

One useful property of effective resistance in undirected graphs is that it is a metric on the nodes of the graph [56]. This allows effective resistance to substitute for the shortest-path distance in various graphical indices and analyses, as well as offering an alternative interpretation of effective resistance that does not rely on an electrical analogy. Although many of the requirements of a metric follow from the algebraic construction of the effective resistance, the triangle inequality depends on Kirchhoff's laws [56]. Consequently, we shall see that effective resistance does not satisfy the triangle inequality on general digraphs.

Importantly, however, the *square root* of the effective resistance *is* a metric. Therefore, if a true metric on digraphs is sought which incorporates information about all connections between two nodes, the square root of the effective resistance is a valid option. In contrast, if effective resistance had been generalised using the Moore-Penrose generalised inverse instead of our definition, then neither would it be a metric nor would its square root be a metric.

We note that the only difference in the conditions for a metric between a function and its square root lies in the triangle inequality. Furthermore, if a function $d(\cdot,\cdot)$ is a metric, then $\sqrt{d(\cdot,\cdot)}$ is by necessity a metric too.

Theorem 4.3. *The square root of the effective resistance is a metric on the nodes of any connected directed graph. That is,*

$$r_{k,j} \geq 0 \ \ \forall \ \textit{nodes } k \textit{ and } j, \tag{4.21}$$

$$r_{k,j} = 0 \ \Leftrightarrow \ k = j, \tag{4.22}$$

$$r_{k,j} = r_{j,k}, \ \textit{and} \tag{4.23}$$

$$\sqrt{r_{k,\ell}} + \sqrt{r_{\ell,j}} \geq \sqrt{r_{k,j}} \ \ \forall \ \textit{nodes } k, \ j \ \textit{and } \ell. \tag{4.24}$$

Furthermore, the effective resistance itself is not a metric since it fails to satisfy the triangle inequality.

Proof. From (4.2), we know that the effective resistance can be computed as

$$r_{k,j} = \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right)^T X \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right),$$

where $X = 2Q^T \Sigma Q$ and Σ is a positive definite matrix. Now, by (2.3), we know that the matrix

$$V := \begin{bmatrix} \frac{1}{\sqrt{N}} \mathbf{1}_N^T \\ Q \end{bmatrix}$$

is orthogonal and thus X is similar to

$$VXV^T = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & 2\Sigma \end{bmatrix}.$$

Hence X has a single 0 eigenvalue and its remaining eigenvalues are twice those of Σ. Furthermore, $X\mathbf{1}_N = \mathbf{0}$ since $Q\mathbf{1}_N = \mathbf{0}$. Thus X is positive semi-definite with null space given by the span of $\mathbf{1}_N$.

Since X is positive semi-definite, we can find a matrix $Y \in \mathbb{R}^{N \times N}$ such that $X = Y^T Y$ (e.g. by the Cholesky decomposition or the positive semi-definite square root [52, §7.2]). This means that we can write effective resistances as

$$r_{k,j} = \left\| Y \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right) \right\|_2^2,$$

and therefore

$$\sqrt{r_{k,j}} = \left\| Y \left(\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}\right) \right\|_2,$$

where $\|\cdot\|_2$ denotes the regular 2-norm on $\mathbb{R}^N$. Therefore, if we associate each node

k of $\mathcal{G}$ to the point

$$\mathbf{p}_k := Y\mathbf{e}_N^{(k)} \in \mathbb{R}^N,$$

we observe that $\sqrt{r_{k,j}}$ is equal to the Euclidean distance in $\mathbb{R}^N$ between $\mathbf{p}_k$ and $\mathbf{p}_j$. Since $\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}$ is perpendicular to $\mathbf{1}_N$ for any $k \neq j$, $\mathbf{e}_N^{(k)} - \mathbf{e}_N^{(j)}$ is not in the null space of Y and so $\mathbf{p}_k \neq \mathbf{p}_j$ for $k \neq j$. Hence $\sqrt{r_{k,j}}$ is a metric on the nodes of the graph.

Finally, to show that $r_{k,j}$ is not a metric, we consider the 3-node graph shown in Figure 4.4. In this case, we find that $r_{1,3} = 20$, $r_{1,2} = \frac{131}{21} \approx 6.24$ and $r_{2,3} = \frac{37}{7} \approx 5.29$. Thus $r_{1,3} > r_{1,2} + r_{2,3}$ and the triangle inequality does not hold. □

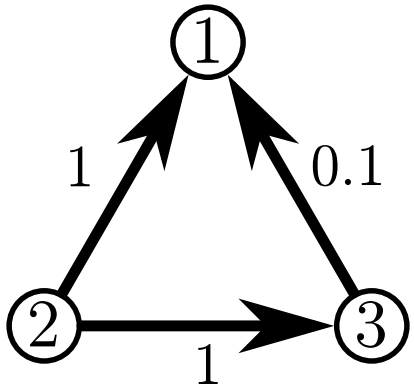

Figure 4.4: A simple 3-node directed graph, $\mathcal{G}^{\text{triangle}}$,where the triangle inequality fails.

Remark 4.1. *We can observe that if effective resistance was defined using $X = L^\dagger$ (the Moore-Penrose generalised inverse), then for the graph shown in Figure 4.4 the triangle inequality would fail for both $r_{k,j}$ and $\sqrt{r_{k,j}}$. Indeed, the counterexample in the proof of Theorem 4.3 also demonstrates why the triangle inequality could be expected to fail for any effective resistance definition that respects edge direction. We can observe that there should be a "low" effective resistance between nodes 1 and 2 due to the connecting edge with unit weight. Likewise, nodes 2 and 3 should have a "low" effective resistance between them for the same reason. But node 2 does not belong to $\mathcal{C}_{\mathcal{G}^{triangle}}(1,3)$ and so there should be a "high" effective resistance between nodes 1 and*

3 *due to their only connection being an edge with low weight. Thus, the sum of the effective resistances between nodes* 1 *and* 2 *and between nodes* 2 *and* 3 *should be lower than the effective resistance between nodes* 1 *and* 3.

4.3 Conclusions

We have generalised the concept of effective resistance to directed graphs in a way that maintains the connection between effective resistances and the $\mathcal{H}_2$ norm, as well as some other control-theoretic properties relating to consensus-type dynamics. Despite the algebraic nature of our generalisation, we have shown that effective resistances in directed graphs bear a fundamental relationship to the structure of the connections between nodes. Moreover, the square root of effective resistance provides a well-defined metric on connected directed graphs, allowing for a notion of distance between nodes, even in cases where neither node is reachable from the other.

Although it may have been tempting to use the Moore-Penrose generalised inverse of a directed graph's Laplacian matrix to define effective resistance, we have shown that not only would this approach ignore the complexity of the derivation of effective resistance for undirected graphs, but also fail to lead to a distance function for directed graphs. Instead, our generalisation derives from an analysis of applications of effective resistance in which directed graphs arise naturally. We believe that this approach will allow for the application of this directed version of effective resistance in other situations than those examined in this dissertation.

In the following chapter, we demonstrate how to compute effective resistances in certain prototypical classes of graphs and we find cases where effective resistances in directed graphs behave analogously to effective resistances in undirected graphs as well as cases where they behave in unexpected ways.

Chapter 5

Computing Effective Resistances in Directed Graphs

In Chapter 4, we presented a generalisation of the concept of effective resistance to directed graphs. This extension was constructed algebraically to preserve the relationships for directed graphs, as they exist for undirected graphs, between effective resistance and control-theoretic properties, including robustness of linear consensus to noise (see §3.2.2 or or [114, 115]), and node certainty in networks of stochastic decision-makers [84]. Further applications of this concept to directed graphs should be possible in formation control [7], distributed estimation [8, 9] and optimal leader selection in networked control systems [22, 41, 43, 82].

Effective resistances have proved to be important in the study of networked systems because they relate global network properties to the individual connections between nodes, and they relate local network changes (e.g. the addition or deletion of an edge, or the change of an edge weight) to global properties without the need to re-compute these properties for the entire network (since only resistances that depend on the edge in question will change). Accordingly, the concept of effective resistance for directed graphs will be most useful if the resistance of any given con-

nection within a graph can be computed, and if it is understood how to combine resistances from multiple connections. Computation and combination of resistances are possible for undirected graphs using the familiar rules for combining resistors in series and parallel.

In this chapter, we address the problems of computing and combining effective resistances for directed graphs. In §5.1 we develop some theory to identify directed graphs that have the same resistances as an equivalent undirected graph. We use some of these results in §5.2 to recover the series-resistance formula [33, §3.3] for nodes connected by one directed path and the parallel-resistance formula [33, §3.4] for nodes connected by two directed paths in the form of a directed cycle. In §5.3 we examine nodes connected by a directed tree and derive a resistance formula that has no analogue from undirected graphs. The results presented in this chapter have been submitted for publication in [118].

5.1 Directed and Undirected Graphs with Equal Effective Resistances

In this section we prove three propositions, each of which provides a set of sufficient conditions for the resistances in a directed graph to be the same as the resistances in an equivalent undirected graph.

Our first result bears an interesting parallel to Proposition 3.3 (although the conditions of Proposition 5.1 are stronger) since it shows that directed graphs that satisfy a normality condition on their Laplacian matrices are much "easier" to work with. In fact, Proposition 5.1 implies that directed graphs with normal Laplacian matrices can be thought of as equivalent to undirected graphs in many essential respects.

Proposition 5.1. *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is a connected graph on N vertices with Laplacian matrix L. Let $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u, A_u)$ be the underlying undirected graph of $\mathcal{G}$. If L is normal, then the effective resistance between any two nodes in $\mathcal{G}$ is equal to the effective resistance between the same two nodes in $\mathcal{G}_u$.*

Proof. Since L is normal by assumption, we know by Lemma 3.3 that $\mathcal{G}$ is balanced, and so $\mathbf{1}_N^T L = \mathbf{0}^T$. Next, we claim that the Laplacian matrix of $\mathcal{G}_u$ is given by

$$L_u := \frac{1}{2}\left(L + L^T\right). \tag{5.1}$$

To see this, first note that $L_u \mathbf{1}_N = \mathbf{0}$ since $L\mathbf{1}_N = \mathbf{0}$ and $L^T\mathbf{1}_N = \mathbf{0}$. Then, we can rewrite L_u as

$$\begin{aligned} L_u &= \frac{1}{2}\left(D - A + D - A^T\right) \\ &= D - A_u, \end{aligned}$$

where D is the diagonal matrix of node out-degrees of $\mathcal{G}$ and $A_u = \frac{1}{2}\left(A + A^T\right)$ (see §2.4). Therefore, L_u is equal to a diagonal matrix minus A_u and has zero row sums. Hence D must also be the diagonal matrix of the row sums of A_u, i.e. the matrix of node degrees of $\mathcal{G}_u$.

Now, let $Q \in \mathbb{R}^{(N-1)\times N}$ be a matrix that satisfies (2.3). Then, since L_u is symmetric, we know that $\overline{L}_u = QL_uQ^T$ must be symmetric as well. Since every edge in $\mathcal{G}$ exists in $\mathcal{G}_u$ and $\mathcal{G}$ is connected, we know that $\mathcal{G}_u$ must be connected as well and so $\overline{L}_u$ is invertible (by Corollary 3.1). Hence (3.12) for $\mathcal{G}_u$ reduces to

$$\overline{L}_u\Sigma_u + \Sigma_u\overline{L}_u = I_{N-1},$$

which clearly has solution

$$\Sigma_u = \frac{1}{2}\overline{L}_u^{-1}. \tag{5.2}$$

We will now show that Σ_u also solves (3.12) for $\mathcal{G}$.

First, let $\overline{L} = QLQ^T$ and note that substituting (5.1) into (5.2) gives us

$$\Sigma_u = \left(\overline{L} + \overline{L}^T\right)^{-1}. \tag{5.3}$$

Next, we know that $\overline{L}$ is normal (by Lemma 3.4) and invertible (by Corollary 3.1, since $\mathcal{G}$ is connected). This means that we can say

$$\overline{L}\,\overline{L}^T = \overline{L}^T\overline{L} \Rightarrow \overline{L}^{-T}\overline{L} = \overline{L}\,\overline{L}^{-T}, \tag{5.4}$$

To show that Σ_u also solves (3.12) for $\mathcal{G}$, we consider the term $\overline{L}\Sigma_u + \Sigma_u\overline{L}^T$, which based on (5.3) and the results above gives

$$\begin{aligned}
\overline{L}\Sigma_u + \Sigma_u\overline{L}^T &= \overline{L}\left(\overline{L} + \overline{L}^T\right)^{-1} + \left(\overline{L} + \overline{L}^T\right)^{-1}\overline{L}^T \\
&= \left(I_{N-1} + \overline{L}^T\overline{L}^{-1}\right)^{-1} + \left(I_{N-1} + \overline{L}^{-T}\overline{L}\right)^{-1} \\
&= I_{N-1} - \left(I_{N-1} + \overline{L}\,\overline{L}^{-T}\right)^{-1} + \left(I_{N-1} + \overline{L}^{-T}\overline{L}\right)^{-1},
\end{aligned}$$

where the Matrix Inversion Lemma (2.1) was used. But by (5.4), the final two terms are identical with opposite signs. Hence, Σ_u solves (3.12) for $\mathcal{G}$ as well as $\mathcal{G}_u$. This implies that $\Sigma = \Sigma_u$, $X = X_u$ and $r_{k,j} = r_{u\,k,j}$ for all nodes k and j in $\mathcal{V}$. □

The next proposition analyses the case when the adjacency matrix of a graph is related to its matrix of node out-degrees through a permutation. Although the assumption for the proposition may seem relatively general, it is straightforward to show that this can only apply to directed path and cycle graphs. The proof of Proposition 5.2 relies on two lemmas, which we give first.

Recall that a *permutation matrix* is a square matrix containing precisely one entry of 1 in each row and column with every other entry being 0.

Lemma 5.1. *Let $P \in \mathbb{R}^{n\times n}$ be a permutation matrix. Then P has the following properties*

(i) $P^{-1} = P^T$, (5.5)

(ii) $P\Pi = \Pi P$, *and* (5.6)

(iii) $(P - I_n)\,\Pi = \Pi\,(P - I_n) = P - I_n$, (5.7)

where Π is the matrix defined in (2.2).

Proof. The proof is given in Appendix B.1.1. □

Since P^T also satisfies the requirements of a permutation matrix, the results of Lemma 5.1 apply to P^T as well (this can also be seen by simply transposing equations (5.5), (5.6) and (5.7)).

Lemma 5.2. *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix and P be a permutation matrix of the same dimension as A. Suppose that AP is diagonal. Then*

(i) *PA is also diagonal,*

(ii) *$A\,(P - I_n) + A^T\left(P^T - I_n\right)$ is symmetric, that is*

$$A\,(P - I_n) + A^T\left(P^T - I_n\right) = (P - I_n)\,A + \left(P^T - I_n\right)A^T, \text{ and} \tag{5.8}$$

(iii) $\overline{(P - I_n)}^T\overline{A}^T\overline{A}\,\overline{(P - I_n)} = \overline{(P - I_n)}\,\overline{A}\,\overline{A}^T\overline{(P - I_n)}^T$. (5.9)

Proof. The proof is given in Appendix B.1.2. □

We are now ready to prove our next set of sufficient conditions.

Proposition 5.2. *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is a connected (directed) graph on N nodes with matrix of node out-degrees D. Furthermore, suppose there is a permutation matrix P such that $D = AP$. Let $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u, A_u)$ be the underlying undirected graph of $\mathcal{G}$. Then the effective resistance between any two nodes in $\mathcal{G}$ is equal to the effective resistance between the same two nodes in $\mathcal{G}_u$.*

Proof. The Laplacian matrix L of $\mathcal{G}$ is given by $L = D - A = A(P - I_N)$. Thus $\overline{L}$ is given by $\overline{L} = QA(P - I_N)Q^T$, which can be rewritten (using (5.7)) as

$$\overline{L} = \overline{A}\,\overline{(P - I_N)}.$$

Furthermore, since $\mathcal{G}$ is connected, $\overline{L}$ is invertible by Corollary 3.1.

Next, we claim that the Laplacian matrix of $\mathcal{G}_u$ is given by

$$L_u := \frac{1}{2}\left[A(P - I_N) + A^T\left(P^T - I_N\right)\right].$$

To see this, we first note that we can rewrite L_u as

$$L_u = \frac{1}{2}\left(D + A^T P^T\right) - A_u.$$

But by part (i) of Lemma 5.2, PA is diagonal and therefore so is $A^T P^T$. Hence $D_u := \frac{1}{2}\left(D + A^T P^T\right)$ is a diagonal matrix. Furthermore,

$$L_u \mathbf{1}_N = \frac{1}{2}\left[A(P\mathbf{1}_N - \mathbf{1}_N) + A^T\left(P^T\mathbf{1}_N - \mathbf{1}_N\right)\right].$$

But since P is a permutation matrix, $P\mathbf{1}_N = \mathbf{1}_N$ and $P^T\mathbf{1}_N = \mathbf{1}_N$, and so $L_u\mathbf{1}_N = \mathbf{0}$. Therefore, L_u is equal to a diagonal matrix minus A_u and has zero row sums. Hence D_u must be the diagonal matrix of the row sums of A_u, i.e. the matrix of out-degrees of $\mathcal{G}_u$.

Since $\mathcal{G}_u$ contains every edge in $\mathcal{G}$ (in addition to the reversal of each edge) and $\mathcal{G}$ is connected, $\mathcal{G}_u$ must also be connected. Thus

$$\Sigma_u = \frac{1}{2}\overline{L}_u^{-1}$$

is the solution to the Lyapunov equation (3.12) for $\mathcal{G}_u$. Using our expression for L_u and (5.7), we can write

$$\Sigma_u = \left[\overline{A}\,\overline{(P - I_N)} + \overline{A}^T\,\overline{(P - I_N)}^T\right]^{-1}.$$

Since Σ_u is symmetric, we can also write

$$\Sigma_u = \left[\overline{(P - I_N)}\,\overline{A} + \overline{(P - I_N)}^T\,\overline{A}^T\right]^{-1}.$$

Now, substituting Σ_u into the left hand side of equation (3.12) for $\mathcal{G}$, gives

$$\overline{L}\Sigma_u + \Sigma_u\overline{L}^T =$$
$$\left[I_{N-1} + \overline{A}^T\,\overline{(P - I_N)}^T\Big(\overline{A}\,\overline{(P - I_N)}\Big)^{-1}\right]^{-1} + \left[I_{N-1} + \Big(\overline{(P - I_N)}^T\,\overline{A}^T\Big)^{-1}\,\overline{(P - I_N)}\,\overline{A}\right]^{-1}.$$

Using the Matrix Inversion Lemma, (2.1), applied to the first term, we can rewrite this as

$$\overline{L}\Sigma_u + \Sigma_u\overline{L}^T = I_{N-1} - \left[I_{N-1} + \overline{A}\,\overline{(P - I_N)}\left(\overline{A}^T\overline{(P - I_N)}^T\right)^{-1}\right]^{-1}$$
$$+ \left[I_{N-1} + \Big(\overline{(P - I_N)}^T\overline{A}^T\Big)^{-1}\,\overline{(P - I_N)}\,\overline{A}\right]^{-1}. \quad (5.10)$$

But by (5.9),

$$\overline{(P - I_N)}^T\overline{A}^T\overline{A}\,\overline{(P - I_N)} = \overline{(P - I_N)}\,\overline{A}\,\overline{A}^T\overline{(P - I_N)}^T,$$

so

$$\overline{A}\,\overline{(P-I_N)}\left(\overline{A}^T\overline{(P-I_N)}^T\right)^{-1}=\left(\overline{(P-I_N)}^T\overline{A}^T\right)^{-1}\overline{(P-I_N)}\,\overline{A},$$

and thus the final two terms in (5.10) are equal (with opposite signs). Thus

$$\overline{L}\Sigma_u+\Sigma_u\overline{L}^T=I_{N-1},$$

and so Σ_u solves (3.12) for $\mathcal{G}$. This implies that $\Sigma=\Sigma_u$, $X=X_u$ and $r_{k,j}=r_{u\,k,j}$ for all nodes k and j in $\mathcal{V}$. □

Our final result in this section provides conditions under which resistances in a directed graph are equal to those in an undirected graph that is not necessarily the underlying undirected graph of the first. The first condition requires that reversing the labels of the nodes in the directed graph is equivalent to reversing the direction of every directed edge (with no restrictions on undirected edges). The second condition adds an additional symmetry requirement. The proof of Proposition 5.3 relies on the following three lemmas.

Let $J\in\mathbb{R}^{n\times n}$ be the matrix with ones along the counterdiagonal and zeros everywhere else. That is, $J=[\hat{\jmath}_{i,j}]$ where $\hat{\jmath}_{i,j}=\delta_{n+1-i,j}$ and δ is the *Kronecker delta*[‡].

Lemma 5.3. *The matrix J has the following properties*

(i) $J^T=J,$ (5.11)

(ii) $J^{-1}=J,$ (5.12)

(iii) $J\Pi=\Pi J,$ (5.13)

(iv) $\overline{J}^T=\overline{J},$ *and* (5.14)

(v) $\overline{J}^{-1}=\overline{J}.$ (5.15)

[‡]The Kronecker delta function of integers i and j, written as $\delta_{i,j}$, is equal to 1 when $i=j$ and 0 when $i\neq j$.

Proof. The proof is given in Appendix B.1.3. □

Lemma 5.4. *Let L be the Laplacian matrix of a connected graph on N nodes, and suppose that $JL^T\Pi L = L^T\Pi LJ$. Then*

$$I_{N-1} + \overline{L}\,\overline{J}\,\overline{L}^{-1}\,\overline{J} = I_{N-1} + \overline{L}^{-T}\,\overline{J}\,\overline{L}^{T}\,\overline{J}. \tag{5.16}$$

Proof. The proof is given in Appendix B.1.4. □

Lemma 5.5. *Let L be the Laplacian matrix of a graph. Suppose $L_u = \frac{1}{2}(L + JLJ)$. Then*

$$\overline{L}_u = \frac{1}{2}\left(\overline{L} + \overline{J}\,\overline{L}\,\overline{J}\right).$$

Proof. The proof is given in Appendix B.1.5. □

Proposition 5.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a connected (directed) graph on N nodes with Laplacian matrix L, and $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u, A_u)$ be a connected undirected graph with Laplacian matrix L_u. Suppose that $L_u = \frac{1}{2}(L + JLJ)$ and $JL^T\Pi L = L^T\Pi LJ$. Then the effective resistance between any two nodes in $\mathcal{G}$ is equal to the effective resistance between the same two nodes in $\mathcal{G}_u$.*

Proof. First we note that since $\mathcal{G}$ is connected, $\overline{L}$ is invertible by Corollary 3.1. Now, as L_u is the Laplacian of an undirected graph, $L_u^T = L_u$ and $\overline{L}_u^T = \overline{L}_u$. Thus $\Sigma_u = \frac{1}{2}\overline{L}_u^{-1}$ is the solution to the Lyapunov equation (3.12) for $\mathcal{G}_u$. Now, using Lemma 5.5, we can write

$$\Sigma_u = \left(\overline{L} + \overline{J}\,\overline{L}\,\overline{J}\right)^{-1}$$

Furthermore, since Σ_u and $\overline{J}$ are symmetric, we can also write

$$\Sigma_u = \left(\overline{L}^T + \overline{J}\,\overline{L}^T\,\overline{J}\right)^{-1}.$$

Now, when we substitute Σ_u into the left hand side of equation (3.12) for $\mathcal{G}$, we obtain

$$\begin{aligned}\overline{L}\Sigma_u + \Sigma_u\overline{L}^T &= \overline{L}\left(\overline{L} + \overline{J}\,\overline{L}\,\overline{J}\right)^{-1} + \left(\overline{L}^T + \overline{J}\,\overline{L}^T\,\overline{J}\right)^{-1}\overline{L}^T \\ &= \left(I_{N-1} + \overline{J}\,\overline{L}\,\overline{J}\,\overline{L}^{-1}\right)^{-1} + \left(I_{N-1} + \overline{L}^{-T}\,\overline{J}\,\overline{L}^T\,\overline{J}\right)^{-1}.\end{aligned}$$

Using the Matrix Inversion Lemma (2.1) applied to the first term (and taking $\overline{J}^{-1} = \overline{J}$ from Lemma 5.3), we can rewrite this as

$$\overline{L}\Sigma_u + \Sigma_u\overline{L}^T = I_{N-1} - \left(I_{N-1} + \overline{L}\,\overline{J}\,\overline{L}^{-1}\,\overline{J}\right)^{-1} + \left(I_{N-1} + \overline{L}^{-T}\,\overline{J}\,\overline{L}^T\,\overline{J}\right)^{-1}.$$

But by Lemma 5.4, the final two terms are equal (with opposite signs). Thus

$$\overline{L}\Sigma_u + \Sigma_u\overline{L}^T = I_{N-1},$$

and so Σ_u solves (3.12) for $\mathcal{G}$. This implies that $\Sigma = \Sigma_u$, $X = X_u$ and $r_{k,j} = r_{u\,k,j}$ for all nodes k and j in $\mathcal{V}$. □

5.2 Effective Resistances from Direct Connections

In this section we compute the resistance in directed graphs between a pair of nodes that are only connected through a single direct connection, or two direct connections in opposite directions (i.e. the connection subgraph consists of either a directed path or a directed cycle). These two scenarios are analogous (in undirected graphs) to combining multiple resistances in series and combining two resistances in parallel. At present, we do not have general rules for combining resistances from multiple direct connections.

The most basic connection is a single directed edge. Intuitively, since an undirected edge with a given weight is equivalent to two directed edges (in opposite directions)

with the same weight, one would expect that the resistance of a directed edge should be twice that of an undirected edge with the same weight. The following lemma shows that this is indeed true.

Lemma 5.6. *If $\mathcal{C}_{\mathcal{G}}(k,j)$ consists of a single directed edge from node k to node j with weight $a_{k,j}$, then*

$$r_{k,j} = \frac{2}{a_{k,j}}. \tag{5.17}$$

Proof. If we take node j to be the first node in $\mathcal{C}_{\mathcal{G}}(k,j)$ and node k to be the second, then $\mathcal{C}_{\mathcal{G}}(k,j)$ has Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 \\ -a_{k,j} & a_{k,j} \end{bmatrix}.$$

In this case, there is only one matrix Q (up to a choice of sign) which satisfies (2.3), namely

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then we have $\overline{L} = QLQ^T = a_{k,j}$, and hence $\Sigma = \frac{1}{2a_{k,j}}$. Thus

$$X = 2Q^T \Sigma Q = \begin{bmatrix} \frac{1}{2a_{k,j}} & -\frac{1}{2a_{k,j}} \\ -\frac{1}{2a_{k,j}} & \frac{1}{2a_{k,j}} \end{bmatrix},$$

and finally,

$$r_{k,j} = \left(\mathbf{e}_2^{(1)} - \mathbf{e}_2^{(2)}\right)^T X \left(\mathbf{e}_2^{(1)} - \mathbf{e}_2^{(2)}\right) = \frac{2}{a_{k,j}}.$$

□

As a result of Lemma 5.6, when we refer to the effective resistance of a single (directed) edge, we mean twice the inverse of the edge weight. Our next two results extend to some directed graphs the familiar rules from undirected graphs for combin-

ing resistances in series and parallel. These cover the cases when a pair of nodes is connected only by either a directed path or cycle.

Theorem 5.1. *Suppose $\mathcal{C}_{\mathcal{G}}(k,j)$ consists of a single directed path. Then $r_{k,j}$ is given by the sum of the resistances of each edge in the path between the two nodes (where the resistance of each edge is computed as in Lemma 5.6).*

Proof. Suppose we label the nodes in $\mathcal{C}_{\mathcal{G}}(k,j)$ from 1 to N in the order in which they appear along the path, starting with the root and moving in the direction opposite the edges. Then we can write the adjacency matrix of $\mathcal{C}_{\mathcal{G}}(k,j)$ as

$$A = \operatorname{diag}^{(-1)}\left(\begin{bmatrix} a_1 & a_2 & \cdots & a_{N-1} \end{bmatrix}\right),$$

and the matrix of node out-degrees as

$$D = \operatorname{diag}\left(\begin{bmatrix} 0 & a_1 & \cdots & a_{N-1} \end{bmatrix}\right).$$

If we let P be the permutation matrix containing ones above the main diagonal and in the lower left corner, that is

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{5.18}$$

we can observe that $D = AP$. Therefore, by Proposition 5.2, the resistance between any two nodes in $\mathcal{C}_{\mathcal{G}}(k,j)$ is equal to the resistance between the same two nodes in the underlying undirected path graph with adjacency matrix $A_u = \frac{1}{2}\left(A + A^T\right)$.

Now, A_u is the adjacency matrix of an undirected path with weights of $\dfrac{1}{2}a_i$ on

each edge. But the resistance of an edge in an undirected graph is the inverse of the edge weight and so each edge has resistance $\frac{2}{a_i}$. Thus the edge resistances in this undirected path graph match those in the original directed path graph (computed according to Lemma 5.6). Furthermore, the resistance between two nodes connected by an undirected path is simply the sum of the resistances of the edges between them. Thus the same is true for two nodes connected by a directed path. □

Theorem 5.2. *Suppose $\mathcal{C}_{\mathcal{G}}(k,j)$ consists of a single directed cycle. Then $r_{k,j}$ is given by the inverse of the sum of the inverses of the resistances of each path connecting nodes k and j (where the resistance of each path is computed as in Theorem 5.1).*

Proof. Suppose we label the nodes in $\mathcal{C}_{\mathcal{G}}(k,j)$ from 1 to N in the reverse of the order in which they appear around the cycle, starting with any node. Then we can write the adjacency matrix of $\mathcal{C}_{\mathcal{G}}(k,j)$ as

$$A = \text{diag}^{(N-1)}\left(\begin{bmatrix} a_1 \end{bmatrix}\right) + \text{diag}^{(-1)}\left(\begin{bmatrix} a_2 & a_3 & \cdots & a_N \end{bmatrix}\right)$$

and the matrix of node out-degrees as

$$D = \text{diag}\left(\begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}\right).$$

If we let P be the permutation matrix containing ones above the main diagonal and in the lower left corner (as in (5.18)), we can observe that $D = AP$. Therefore, by Proposition 5.2, the resistance between any two nodes in $\mathcal{C}_{\mathcal{G}}(k,j)$ is equal to the resistance between the same two nodes in the underlying undirected cycle graph with adjacency matrix $A_u = \frac{1}{2}\left(A + A^T\right)$.

Now, A_u is the adjacency matrix of an undirected cycle with weights of $\frac{1}{2}a_i$ on each edge. But the resistance of an edge in an undirected graph is the inverse of the edge weight, so each edge has resistance $\frac{2}{a_i}$. Thus the edge resistances in this undi-

rected graph match those in the original directed cycle graph (computed according to Lemma 5.6). Furthermore, the resistance between nodes k and j connected by an undirected cycle is given by

$$r_{u\,k,j} = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}},$$

where r_1 is the resistance of one path between nodes k and j and r_2 is the resistance of the other path. Thus the same is true for two nodes connected by a directed cycle, where (by Theorem 5.1) r_1 and r_2 are equal to the resistances of the two directed paths between nodes k and j. □

5.3 Effective Resistances from Indirect Connections

Lemma 5.6 and Theorems 5.1 and 5.2 suggest a very intuitive interpretation of effective resistance for directed graphs. A directed edge can be thought of as "half" of an undirected edge - either by noting that a directed edge allows half of the interaction to take place that occurs through an undirected edge, or by viewing an undirected edge as consisting of two directed edges with equal weights but in opposite directions. Thus, the resistance of a directed edge is twice the resistance of an undirected edge with the same weight. Then, in path and cycle graphs, resistances combine in exactly the ways (i.e., in series and in parallel) we are used to. However, connections in directed graphs can be more complicated than these. In particular, two nodes in a directed graph may be connected even if neither node is reachable from the other. This will occur when the only connections between the nodes consist of two non-zero length paths which meet at a distinct node. In Theorem 5.3 we prove an explicit expression for resistances in the case when $\mathcal{C}_{\mathcal{G}}(k,j)$ is a directed tree with unit edge weights. Before doing so we prove two lemmas on the correspondence between resistances and the matrix X from (4.7), and two lemmas on the resistance between two leaves in a directed tree. We also rely on the finite series expressions given and proved in Appendix B.3.

Lemma 5.7. *There is a one-to-one relationship between the effective resistances between nodes in a graph and the entries of the matrix X from (4.7). In particular,*

$$r_{k,j} = x_{k,k} + x_{j,j} - 2x_{k,j}, \text{ and} \tag{5.19}$$

$$x_{k,j} = \frac{1}{2N}\sum_{i=1}^{N} r_{k,i} + \frac{1}{2N}\sum_{i=1}^{N} r_{j,i} - \frac{1}{N^2}\sum_{i=1}^{N-1}\sum_{\ell=i+1}^{N} r_{i,\ell} - \frac{1}{2}r_{k,j}. \tag{5.20}$$

Proof. (5.19) is simply the definition of $r_{k,j}$. To derive (5.20), we first note that from (2.3) and (4.7), X has the property that $X\mathbf{1}_N = \mathbf{0}$ and $\mathbf{1}_N^T X = \mathbf{0}^T$. That is, X has zero row- and column-sums.

Now, using (5.19), we can write

$$r_{k,i} = x_{k,k} + x_{i,i} - 2x_{k,i} \text{ for any } 1 \le i \le N.$$

Then, by summing this equation over i, we obtain

$$\sum_{i=1}^{N} r_{k,i} = Nx_{k,k} + \operatorname{tr}(X)$$

since X has zero row-sums. Next, by summing again over k, we find that

$$\operatorname{tr}(X) = \frac{1}{2N}\sum_{k=1}^{N}\sum_{i=1}^{N} r_{k,i}.$$

But $r_{i,i} = 0\ \forall i$ and $r_{i,k} = r_{k,i}$ (by Theorem 4.3). Thus we can say that

$$\operatorname{tr}(X) = \frac{1}{N}\sum_{i=1}^{N-1}\sum_{\ell=i+1}^{N} r_{i,\ell}. \tag{5.21}$$

Combining (5.21) with our expression for $\sum_{i=1}^{N} r_{k,i}$ gives us

$$x_{k,k} = \frac{1}{N}\sum_{i=1}^{N} r_{k,i} - \frac{1}{N^2}\sum_{i=1}^{N-1}\sum_{\ell=i+1}^{N} r_{i,\ell}. \tag{5.22}$$

Substituting the expression from (5.22) for $x_{k,k}$ and $x_{j,j}$ into (5.19) and rearranging produces (5.20). □

Lemma 5.8. *Suppose $\mathcal{G}$ is a directed path on N nodes with unit edge weights, in which the nodes are labelled from 1 to N in the order in which they appear along the path, starting with the root. Let X be the corresponding matrix from (4.7). Then the entries of X are given by*

$$x_{k,j} = \frac{2N^2 + 3N + 1 + 3k^2 + 3j^2 - 3(N+1)k - 3(N+1)j}{3N} - |k-j|. \tag{5.23}$$

Proof. Suppose $k, j \in \{1, 2, \ldots, N\}$. Then by Theorem 5.1, we know that the resistance between nodes k and j in our directed path is equal to 2 (the resistance of each edge) times the number of edges between them. Since the nodes are labelled in order along the path, this gives us

$$r_{k,j} = 2|k-j|.$$

Therefore, from Lemma 5.7, we know that

$$x_{k,j} = \frac{1}{N}\sum_{i=1}^{N}|k-i| + \frac{1}{N}\sum_{i=1}^{N}|j-i| - \frac{2}{N^2}\sum_{i=1}^{N-1}\sum_{\ell=i+1}^{N}|i-\ell| - |k-j|. \tag{5.24}$$

We now proceed by examining each summation in turn. The first sum can be broken into two parts (for $i < k$ and $i > k$) and then simplified using (B.26) to obtain

$$\sum_{i=1}^{N}|k-i| = \frac{2k^2 - 2(N+1)k + (N+1)N}{2}. \tag{5.25}$$

By replacing k with j in the previous expression, we observe that

$$\sum_{i=1}^{N} |j-i| = \frac{2j^2 - 2\,(N+1)\,j + (N+1)\,N}{2}. \tag{5.26}$$

In the third sum in (5.24), we observe that $\ell > i$ for every term. Thus $|i-\ell| = \ell - i$, and we can use (B.26) and (B.27) to obtain

$$\sum_{i=1}^{N-1} \sum_{\ell=i+1}^{N} |i-\ell| = \frac{(N^2-1)\,N}{6}. \tag{5.27}$$

Finally, (5.23) follows from substituting (5.25), (5.26) and (5.27) into (5.24). □

The following results are needed to prove Theorem 5.3. In them, we examine the resistance between the leaves of a tree containing two branches that meet at its root and with unit weights on every edge, $\mathcal{G}^{\text{tree}}_{n,m}$, as shown in Figure 5.1(b). The effective resistance between the two leaves of $\mathcal{G}^{\text{tree}}_{n,m}$ will be denoted by $r(n,m)$.

Lemma 5.9. *The effective resistance between the two leaves of $\mathcal{G}^{tree}_{n,1}$ is given by*

$$r(n,1) = 2(n-1) + 2^{2-n}. \tag{5.28}$$

Proof. The number of nodes in $\mathcal{G}^{\text{tree}}_{n,1}$ is $N = n+2$. Let us label the nodes in $\mathcal{G}^{\text{tree}}_{n,1}$ from 1 to $n+1$, in the reverse order of the edges, along the branch of length n, starting with the root (thus the leaf of this branch is node $n+1$). Then the other leaf (with an edge connecting it to the root) will be node $N = n+2$. Thus the resistance we seek to find is $r(n,1) = r_{n+1,n+2}$.

Let $A^{\text{path}}_{N_p}$, $D^{\text{path}}_{N_p}$ and $L^{\text{path}}_{N_p}$ denote the adjacency matrix, matrix of out-degrees and Laplacian matrix of a directed path containing N_p nodes and unit weights on every edge. Let the nodes in this path be labelled from 1 to N_p in the reverse of the order

in which they appear, starting with the root. Thus

$$A_{N_p}^{\text{path}} = \text{diag}^{(-1)}\left(\mathbf{1}_{N_p-1}\right),$$

$$D_{N_p}^{\text{path}} = \text{diag}\left(\begin{bmatrix} 0 & \mathbf{1}_{N_p-1}^T \end{bmatrix}\right) \text{ and}$$

$$L_{N_p}^{\text{path}} = \text{diag}\left(\begin{bmatrix} 0 & \mathbf{1}_{N_p-1}^T \end{bmatrix}\right) - \text{diag}^{(-1)}\left(\mathbf{1}_{N_p-1}\right).$$

From these, we can observe that

$$\mathbf{1}_{N_p}^T L_{N_p}^{\text{path}} = \mathbf{e}_{N_p}^{(N_p)T} - \mathbf{e}_{N_p}^{(1)T}, \text{ and} \tag{5.29}$$

$$\mathbf{e}_{N_p}^{(i)T} L_{N_p}^{\text{path}} = \begin{cases} \mathbf{e}_{N_p}^{(i)T} - \mathbf{e}_{N_p}^{(i-1)T} & \text{if } 1 < i \leq N_p, \\ \mathbf{0}^T & \text{if } i = 1. \end{cases} \tag{5.30}$$

Next, we will let Q_{N_p} be a $(N_p - 1) \times N_p$ matrix which satisfies (2.3), and $\overline{L}_{N_p}^{\text{path}}$ and $\Sigma_{N_p}^{\text{path}}$ be derived from (3.6) and (3.12) using $L_{N_p}^{\text{path}}$ and Q_{N_p}. Let $X_{N_p}^{\text{path}} = 2Q_{N_p}^T \Sigma_{N_p}^{\text{path}} Q_{N_p}$, according to (4.7). Then, by Lemma 5.8, the entries of $X_{N_p}^{\text{path}}$ are given by (5.23).

Now, we can write the adjacency matrix of $\mathcal{G}_{n,1}^{\text{tree}}$ as

$$A = \begin{bmatrix} A_{n+1}^{\text{path}} & \mathbf{0} \\ \mathbf{e}_{n+1}^{(1)T} & 0 \end{bmatrix}.$$

In a similar fashion, we can write the matrix of node out-degrees as

$$D = \begin{bmatrix} D_{n+1}^{\text{path}} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

and the Laplacian matrix as

$$L = \begin{bmatrix} L_{n+1}^{\text{path}} & \mathbf{0} \\ -\mathbf{e}_{n+1}^{(1)T} & 1 \end{bmatrix}.$$

Next, let

$$Q = \begin{bmatrix} Q_{n+1} & \mathbf{0} \\ \alpha \mathbf{1}_{n+1}^T & -\beta \end{bmatrix},$$

where $\alpha = \frac{1}{\sqrt{(n+1)(n+2)}}$ and $\beta = \sqrt{\frac{n+1}{n+2}}$. Then Q satisfies (2.3). We can use (3.6), (5.29) and the facts that $L_{n+1}^{\text{path}} \mathbf{1}_{n+1} = \mathbf{0}_{n+1}$ and $\beta(\alpha + \beta) = 1$ to express $\overline{L}$ as

$$\overline{L} = \begin{bmatrix} \overline{L}_{n+1}^{\text{path}} & \mathbf{0} \\ (\beta - \alpha)\, \mathbf{e}_{n+1}^{(1)T} Q_{n+1}^T + \alpha \mathbf{e}_{n+1}^{(n+1)T} Q_{n+1}^T & 1 \end{bmatrix}.$$

In order to compute resistances in $\mathcal{G}_{n,1}^{\text{tree}}$, we must find the matrix Σ which solves (3.12). Since we have partitioned $\overline{L}$ into a 2×2 block matrix, we will do the same for Σ. Let

$$\Sigma = \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix},$$

where $S \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $\mathbf{t} \in \mathbb{R}^n$ and $u \in \mathbb{R}$. Then multiplying out the matrices in (3.12) and equating blocks in this matrix equation gives us

$$\overline{L}_{n+1}^{\text{path}} S + S \overline{L}_{n+1}^{\text{path}\,T} = I_n, \tag{5.31}$$

$$\overline{L}_{n+1}^{\text{path}} \mathbf{t} + \mathbf{t} + (\beta - \alpha)\, S Q_{n+1} \mathbf{e}_{n+1}^{(1)} + \alpha S Q_{n+1} \mathbf{e}_{n+1}^{(n+1)} = \mathbf{0}, \text{ and} \tag{5.32}$$

$$2u + 2(\beta - \alpha)\, \mathbf{e}_{n+1}^{(1)T} Q_{n+1}^T \mathbf{t} + 2\alpha \mathbf{e}_{n+1}^{(n+1)T} Q_{n+1}^T \mathbf{t} = 1. \tag{5.33}$$

From (5.31), it is clear that $S = \Sigma_{n+1}^{\text{path}}$. In addition, we can rewrite (5.33) as

$$u = \frac{1}{2} - (\beta - \alpha)\, \mathbf{e}_{n+1}^{(1)T} Q_{n+1}^T \mathbf{t} - \alpha \mathbf{e}_{n+1}^{(n+1)T} Q_{n+1}^T \mathbf{t}. \tag{5.34}$$

In order to find a complete solution for Σ, we must solve (5.32) for $\mathbf{t}$. However, resistances are computed from X, which, if we let $\mathbf{v} := Q_{n+1}^T\mathbf{t} = [v_i]$ and use (4.7), can be written as

$$X = \begin{bmatrix} X_{n+1}^{\text{path}} + 2\alpha\mathbf{v}\mathbf{1}_{n+1}^T + 2\alpha\mathbf{1}_{n+1}\mathbf{v}^T + 2\alpha^2 u\mathbf{1}_{n+1}\mathbf{1}_{n+1}^T & -2\beta\mathbf{v} - 2\alpha\beta u\mathbf{1}_{n+1} \\ -2\beta\mathbf{v}^T - 2\alpha\beta u\mathbf{1}_{n+1}^T & 2\beta^2 u \end{bmatrix}.$$

Hence, to compute resistances in $\mathcal{G}_{n,1}^{\text{tree}}$, we need only compute $\mathbf{v}$, not $\mathbf{t}$. We can also note that as X does not depend on our choice of Q (by Lemma 4.1), neither does $\mathbf{v}$. In fact, we can write (5.34) as

$$u = \frac{1}{2} + (\alpha - \beta)\, v_1 - \alpha v_{n+1},$$

and the resistance we seek as

$$r(n,1) = x_{n+1\,n+1,n+1}^{\text{path}} + (\alpha+\beta)^2 + 2\,(\alpha+\beta)^2\,(\alpha-\beta)\,v_1 \\ + 2\,(\alpha+\beta)\,[2 - \alpha(\alpha+\beta)]\,v_{n+1}. \quad (5.35)$$

Thus we only need to find v_1 and v_{n+1} in order to compute $r(n,1)$.

Now, $v_i = \mathbf{e}_{n+1}^{(i)T}\mathbf{v} = \mathbf{e}_{n+1}^{(i)T}Q_{n+1}^T\mathbf{t}$. We will therefore proceed by left-multiplying (5.32) by $\mathbf{e}_{n+1}^{(i)T}Q_{n+1}^T$. Using the fact that $S = \Sigma_{n+1}^{\text{path}}$, we obtain

$$\mathbf{e}_{n+1}^{(i)T}Q_{n+1}^TQ_{n+1}L_{n+1}^{\text{path}}\mathbf{v} + v_i + \frac{\beta-\alpha}{2}\mathbf{e}_{n+1}^{(i)T}X_{n+1}^{\text{path}}\mathbf{e}_{n+1}^{(1)} + \frac{\alpha}{2}\mathbf{e}_{n+1}^{(i)T}X_{n+1}^{\text{path}}\mathbf{e}_{n+1}^{(n+1)} = 0. \quad (5.36)$$

But $\mathbf{e}_{n+1}^{(i)T}Q_{n+1}^TQ_{n+1} = \mathbf{e}_{n+1}^{(i)T}\left(I_{n+1} - \frac{1}{n+1}\mathbf{1}_{n+1}\mathbf{1}_{n+1}^T\right) = \mathbf{e}_{n+1}^{(i)T} - \frac{1}{n+1}\mathbf{1}_{n+1}^T$ by (2.3), and so

by (5.29) and (5.30), we find

$$\mathbf{e}_{n+1}^{(i)T} Q_{n+1}^T Q_{n+1} L_{n+1}^{\text{path}} \mathbf{v} = \begin{cases} \frac{1}{n+1} v_1 + v_i - v_{i-1} - \frac{1}{n+1} v_{n+1} & \text{if } 1 < i \leq n+1, \\ \frac{1}{n+1} v_1 - \frac{1}{n+1} v_{n+1} & \text{if } i = 1. \end{cases}$$

Furthermore, using (5.23), we observe that

$$\mathbf{e}_{n+1}^{(i)T} X_{n+1}^{\text{path}} \mathbf{e}_{n+1}^{(1)} = x_{n+1\,i,1}^{\text{path}} = \frac{(2n+3)(n+2)}{3(n+1)} + \frac{i(i-2n-3)}{n+1}, \text{ and}$$

$$\mathbf{e}_{n+1}^{(i)T} X_{n+1}^{\text{path}} \mathbf{e}_{n+1}^{(n+1)} = x_{n+1\,i,n+1}^{\text{path}} = -\frac{n(n+2)}{3(n+1)} + \frac{i(i-1)}{n+1}.$$

Substituting these expressions into (5.36) gives us

$$v_i = \frac{1}{2} v_{i-1} - \frac{1}{2(n+1)} v_1 + \frac{1}{2(n+1)} v_{n+1} + f + g(i) \text{ if } 1 < i \leq n+1 \tag{5.37}$$

$$v_1 = \frac{1}{n+2} v_{n+1} + h, \tag{5.38}$$

where

$$f = \frac{[(3\alpha - 2\beta)n + 3(\alpha - \beta)]\,(n+2)}{12(n+1)},$$

$$g(i) = \frac{i\,[-\beta i + 2(\beta - \alpha)n - 2\alpha + 3\beta]}{4(n+1)} \text{ and}$$

$$h = \frac{\alpha n}{6} + \frac{(\alpha - \beta)n(2n+1)}{6(n+2)}.$$

We can now recursively apply (5.37) n times, starting with $i = n+1$, to find

$$v_{n+1} = 2^{-n} v_1 - \frac{v_1}{n+1} \sum_{k=1}^{n} 2^{-k} + \frac{v_{n+1}}{n+1} \sum_{k=1}^{n} 2^{-k} + 2f \sum_{k=1}^{n} 2^{-k} + 2 \sum_{k=1}^{n} g(n+2-k) 2^{-k}. \tag{5.39}$$

But we can write $g(n+2-k) = g_1 + g_2 k + g_3 k^2$, where

$$g_1 = \frac{(\beta - 2\alpha)(n+2)}{4},$$

$$g_2 = \frac{2\alpha n + 2\alpha + \beta}{4(n+1)} \text{ and}$$

$$g_3 = \frac{-\beta}{4(n+1)}.$$

Therefore, by (B.28), (B.29) and (B.30), the sum involving $g(n+2-k)$ is

$$\sum_{k=1}^{n} g(n+2-k)2^{-k} = g_1\left(1-2^{-n}\right) + g_2\left[2-(n+2)\,2^{-n}\right] + g_3\left[6-\left(n^2+4n+6\right)2^{-n}\right].$$

Using this result and (B.28), (5.39) becomes

$$\frac{n+2^{-n}}{n+1}v_{n+1} = \frac{(n+2)\,2^{-n}-1}{n+1}v_1 + 2f\left(1-2^{-n}\right) + 2g_1\left(1-2^{-n}\right) + 2g_2\left[2-(n+2)\,2^{-n}\right] + 2g_3\left[6-\left(n^2+4n+6\right)2^{-n}\right]. \quad (5.40)$$

But now (5.40) and (5.38) form a pair of linear equations in v_1 and v_{n+1}. Using the expressions for f, g_1, g_2, g_3 and h, along with the definitions of α and β, their solution is given by

$$\begin{aligned} v_1 &= \frac{\alpha\left[-2n^2+5n-6+6.2^{-n}\right]}{6} \\ v_{n+1} &= \frac{\alpha\left[n^2+2n-12+(6n+12)2^{-n}\right]}{6}. \end{aligned} \quad (5.41)$$

Finally, using (5.23) and (5.41) in (5.35), along with the expressions for α and β, gives us (5.28). □

Lemma 5.10. *For positive integers n and ℓ, the resistance between the two leaves of $\mathcal{G}^{tree}_{n,\ell+1}$ satisfies the recurrence relation*

$$r(n,\ell{+}1) = \frac{-3n^2+3\ell^2-2n\ell-n+5\ell+2}{2(n+\ell+1)^2}+\frac{\ell^2+2n\ell+2n+3\ell}{n+\ell+1}2^{-n}+\frac{n^2+n+2}{2(n+\ell+1)}2^{-\ell}$$
$$+\frac{1}{4(n+\ell+1)}\sum_{k=1}^{\ell}\left(4-\tfrac{2}{n+\ell+1}-2^{k-\ell}\right)r(n,k)-\frac{n+\ell+2}{2(n+\ell+1)}\sum_{k=1}^{n}\left(\tfrac{1}{n+\ell+1}-2^{k-n}\right)r(k,\ell)$$
$$-\frac{1}{4(n+\ell+1)}\sum_{k=1}^{n}\sum_{j=1}^{\ell}\left(2^{1+k-n}-2^{j-\ell}\right)r(k,j). \quad (5.42)$$

The proof of Lemma 5.10 relies on similar ideas to the proof of Lemma 5.9, and is given in Appendix B.2. We now proceed to solve the recurrence relation given by Lemmas 5.9 and 5.10 using several finite series results given in Appendix B.3.

Theorem 5.3. *Suppose $\mathcal{C}_{\mathcal{G}}(k,j)$ consists of a directed tree with unit weights on every edge. Then $r_{k,j}$ is given by*

$$r_{k,j} = 2\,(n-m) + 2^{3-n-m}\sum_{i=1}^{\left\lfloor\frac{m+1}{2}\right\rfloor} i\binom{n+m+2}{n+2i+1}, \quad (5.43)$$

where n is the length of the shortest path from node k to a mutually reachable node and m is the length of the shortest path from node j to a mutually reachable node.

Proof. Since every node in $\mathcal{C}_{\mathcal{G}}(k,j)$ is reachable from either node k or node j, if $\mathcal{C}_{\mathcal{G}}(k,j)$ is a tree then only nodes k and j can be leaves. But every tree has at least one leaf, so suppose that node k is a leaf. If node j is not a leaf, then $\mathcal{C}_{\mathcal{G}}(k,j)$ must be a directed path and node j is the closest mutually reachable node to both nodes k and j. Then $m=0$, n is the path length from k to j and (5.43) reduces to

$$r_{k,j} = 2n,$$

which follows from Theorem 5.1. Conversely, if node j is a leaf but node k is not,

$\mathcal{C}_{\mathcal{G}}(k,j)$ must be a directed path and node k is the closest mutually reachable node to both nodes k and j. Then $n=0$, m is the path length from j to k and (5.43) reduces to

$$r_{k,j} = -2m + 2^{3-m} \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} i \binom{m+2}{2i+1}.$$

But by (B.40) and (B.38) from Lemma B.3,

$$\sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} i \binom{m+2}{2i+1} = m2^{m-1},$$

and so (5.43) becomes

$$r_{k,j} = 2m,$$

which follows from Theorem 5.1.

Now, if both node k and node j are leaves, then $\mathcal{C}_{\mathcal{G}}(k,j)$ must be a directed tree with exactly two branches. Thus $\mathcal{C}_{\mathcal{G}}(k,j)$ must correspond to the tree shown in Figure 5.1(a) and n and m are the path lengths from nodes k and j, respectively, to the point where the two branches meet. Furthermore, both n and m are at least 1.

By Corollary 4.1, we observe that the resistance between nodes k and j remains the same as we remove all the nodes of $\mathcal{C}_{\mathcal{G}}(k,j)$ from the root to the node where the two branches meet. Thus, $r_{k,j}$ can be computed as the resistance between the two leaves of the tree shown in Figure 5.1(b). Let this tree be called $\mathcal{G}^{\text{tree}}_{n,m}$, and since the only two parameters that define $\mathcal{G}^{\text{tree}}_{n,m}$ are n and m, we can write $r_{k,j}$ as a function of n and m only. That is,

$$r_{k,j} =: r(n,m).$$

In order to compute $r(n,m)$, we will begin by considering the case where $m=1$. Substituting $m=1$ into (5.43) gives

$$r(n,1) = 2(n-1) + 2^{2-n},$$

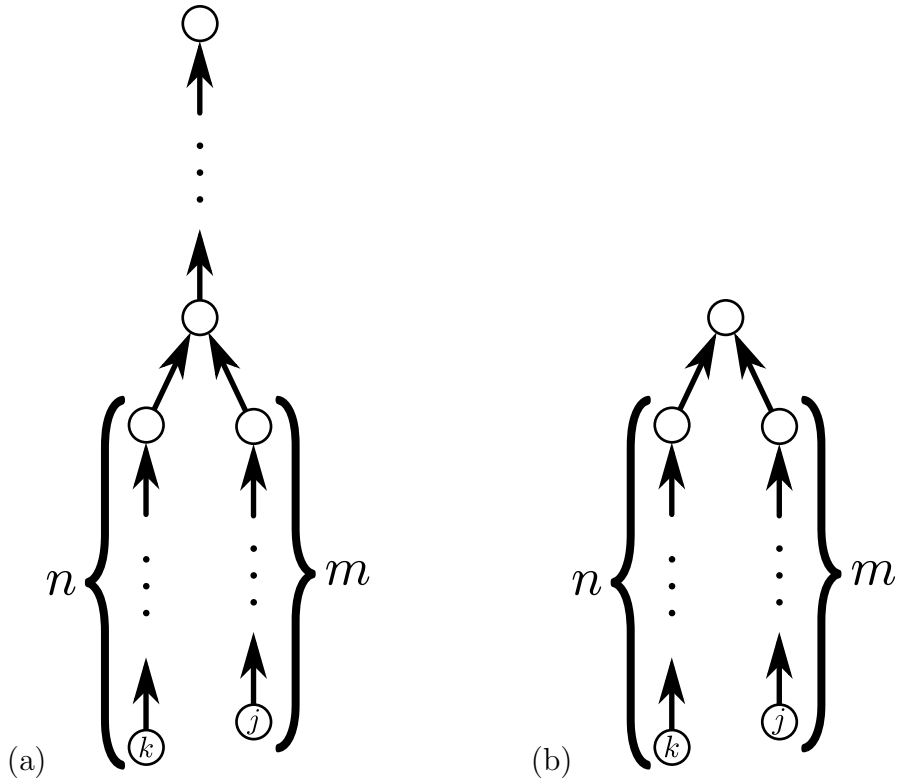

Figure 5.1: (a) The generic form of $\mathcal{C}_{\mathcal{G}}(k,j)$ when it is a directed tree with more than one leaf and unit weights on every edge. (b) A tree, $\mathcal{G}^{\text{tree}}_{n,m}$, in which $r_{k,j}$ is equal to its value in $\mathcal{C}_{\mathcal{G}}(k,j)$ when $\mathcal{C}_{\mathcal{G}}(k,j)$ is a directed tree as shown in part (a).

which follows from Lemma 5.9.

Now, suppose that (5.43) holds for all $n > 0$ and all $m \in \{1, 2, \ldots, \ell\}$ (for some $\ell > 0$). Then $r_{k,j}$ for $m = \ell + 1$ can be computed using Lemma 5.10. In particular, all resistances in the right-hand side of (5.42) are given by (5.43). Using this fact, we find that $r(n, \ell+1)$ matches the expression $s(n,\ell)$ given in Lemma B.8. Therefore, $r(n,\ell+1)$ can be expressed in the form given in (B.81).

Next, suppose that ℓ is odd. That is, $\ell = 2p+1$ for some integer $p \geq 0$. Then (B.81) gives us

$$r(n, 2p+2) = 2\,(n-2p-2) + 2^{1-n-2p} \sum_{i=1}^{p+1} i \binom{n+2p+4}{n+2i+1} + \frac{g(n,p)}{n+\ell+1},$$

where $g(n,p)$ is given by (B.53) in Lemma B.6. But by Lemma B.6, $g(n,p) = 0$ for

any integers $n \geq 0$ and $p \geq 0$. Thus (5.43) holds for $m = \ell + 1$.

Finally, suppose that ℓ is even. That is, $\ell = 2p$ for some integer $p > 0$. Then (B.81) gives us

$$r(n, 2p+1) = 2(n - 2p - 1) + 2^{2-n-2p} \sum_{i=1}^{p} i \binom{n+2p+3}{n+2i+1} + \frac{4p^2 + 2np + 2n + 6p + 2}{n + 2p + 1} 2^{1-n-2p} + \frac{h(n,p)}{n+\ell+1},$$

where $h(n,p)$ is given by (B.67) in Lemma B.7. But by Lemma B.7, $h(n,p) = 0$ for any integers $n \geq 0$ and $p \geq 0$. Thus,

$$r(n, 2p+1) = 2(n - 2p - 1) + 2^{2-n-2p} \sum_{i=1}^{p+1} i \binom{n+2p+3}{n+2i+1},$$

and so (5.43) holds for $m = \ell + 1$.

Therefore, by induction we have that (5.43) also holds for all $n > 0$ and $m > 0$. □

Equation (5.43) is a highly non-intuitive result, not least because on initial inspection it does not appear to be symmetric in n and m (although we know that it must be, by Theorem 4.3). Therefore, it becomes easier to interpret (5.43) if we reformulate it in terms of the shorter path length and the difference between the path lengths. Thus, if we let n be the length of the longer path, that is, $n = m + d$ for some $d \geq 0$, (5.43) becomes

$$\begin{aligned} r_{k,j} &= 2d + 2^{3-2m-d} \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} i \binom{2m+d+2}{m+d+2i+1} \\ &=: 2d + e(m,d). \end{aligned} \tag{5.44}$$

Then, using (2.6), we can write

$$e(m, d+1) = 2^{3-2m-d} \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} i \frac{2m+d+3}{2m+2d+4i+4} \binom{2m+d+2}{m+d+2i+1}$$
$$< \frac{2m+d+3}{2m+2d+4} e(m, d), \tag{5.45}$$

and hence conclude that $\lim_{d\to\infty} e(m, d) = 0$. Thus, when the connection subgraph between two nodes is a directed tree, the resistance between them is twice the difference between the lengths of the paths connecting each node to their closest mutually reachable node, plus some "excess" that disappears as this difference becomes large. Conversely, the excess is significant when the path length difference is small, leading to a resistance that is greater than twice the difference.

One common approach to the analysis of resistive circuits is to replace a section of the network that connects to the rest through a single pair of nodes by a single resistor with an equivalent resistance. The simplest example of this is the replacement of a path with a single edge. If this principle were to extend to the calculation of effective resistance in directed graphs, then $r_{2,3}$ in $\mathcal{G}_3^{\text{star}}$ (as shown in Figure 5.2) would match the formula from Theorem 5.3. However, a simple calculation shows that in $\mathcal{G}_3^{\text{star}}$,

$$r_{2,3} = 2\,(n+m) - \frac{2nm}{n+m},$$

which only matches (5.43) for $n = m = 1$. Thus in more general cases of connection subgraphs like $\mathcal{G}_{n,m}^{\text{tree}}$ but with arbitrary weights on every edge, the resistance between the leaves does *not* depend only on the equivalent resistance of each path.

Theorems 5.1, 5.2 and 5.3 by no means characterise all the possible connection subgraphs in a directed graph. Other connection subgraphs include multiple paths from k to j (some of which could coincide over part of their length), multiple paths from k to j and multiple paths from j to k (again, some of which could partially

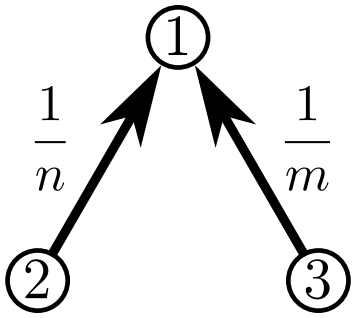

Figure 5.2: A simple 3-node directed graph, $\mathcal{G}_3^{\text{star}}$, with resistances of $2n$ and $2m$ on each edge.

coincide), multiple indirect connections of the type analysed in Theorem 5.3 (which could partially coincide) and a combination of indirect and direct (i.e. path) connections. Further analysis is needed to completely describe how to compute effective resistances in these situations.

5.4 Conclusions

The results of Lemma 5.6 and Theorems 5.1 and 5.2 demonstrate that in some situations our definition of effective resistance for directed graphs behaves as an intuitive extension of effective resistance in undirected graphs. In contrast, Theorem 5.3 demonstrates a fundamental difference between effective resistances in directed and undirected graphs that arises from the indirect connections that are only possible in directed graphs. Nevertheless, the results presented above show that our notion of effective resistance for directed graphs provides an approach that can relate the local structure of a directed graph to its global properties. The familiar properties of effective resistance allow for a firm analysis of directed graphs that behave similarly to undirected graphs, while the unfamiliar properties can provide insight for the design of directed networks which exhibit essential differences to undirected networks.

Chapter 6

Rearranging Trees for Optimal Robustness

In this chapter, we study the robustness of a particular family of graphs, namely trees, according to their $\mathcal{H}_2$ norms. We develop a partial ordering among trees that allows us to find a tree with minimal $\mathcal{H}_2$ norm, given certain constraints. For undirected trees, most of this partial ordering has already been developed in the literature on Wiener indices [30, 31, 32, 104]. Despite this, our methods of proof are new. In particular, we rely only on local changes in which one or more leaf nodes are moved from a single location in the tree to a new location. This approach allows for the development of a decentralised algorithm on trees to improve their robustness. Additionally, we use the results of Chapters 4 and 5 to derive a similar ordering for directed trees that could not be done using the Wiener index literature.

There are a number of reasons to focus our investigation on trees. First, trees contain the minimum number of edges required for connectivity. Thus, if communication is very costly, a tree graph requires the minimum amount of communication possible to maintain group behaviour. Second, every connected graph contains a spanning tree as a subgraph [1, 69]. In undirected graphs, adding an edge can only decrease

effective resistance, and so the Kirchhoff index (and hence $\mathcal{H}_2$ norm) of any spanning tree will provide an upper bound on the Kirchhoff index (respectively, $\mathcal{H}_2$ norm) of the entire graph [45]. Although we cannot make the same claim for directed graphs, any directed tree can be labelled in such a way as to make its Laplacian matrix lower triangular. Thus the edge weights of a directed tree (along with 0) form the eigenvalues of the graph Laplacian and so a normalised directed tree graph will have a convergence speed of 1. This means that (for large N), directed trees have close to the maximum possible convergence speed for any directed graph (see Lemma 3.5).

This chapter is organised as follows. In §6.1 we discuss the relationship between the $\mathcal{H}_2$ norm and other graph indices. In §6.2 we introduce a system of terminology to describe tree graphs. In §6.3 we derive our partial ordering for undirected trees and develop a decentralised algorithm for rearranging them in §6.4. In §6.5 we derive a similar partial ordering for directed trees. Finally, in §6.6 we discuss a decentralised algorithm for rearranging directed trees. Some of the material presented in §6.1, §6.2, §6.3 and §6.4 has been published in [115].

6.1 The $\mathcal{H}_2$ Norm and Other Graph Indices

From Proposition 4.1, we know that the $\mathcal{H}_2$ norm of any connected graph, whether directed or undirected, can be directly related to the Kirchhoff index of the graph. In particular, any ordering imposed by the Kirchhoff index on graphs with the same number of nodes will be the same ordering imposed by the $\mathcal{H}_2$ norm.

Although computing effective resistances can be difficult for most graphs, it is very straightforward for undirected trees. In a tree with unit weights on every edge, the effective resistance between two nodes is given by the length of the path between

them [56]. Hence, the effective resistance of a tree with unit edge weights is

$$K_f = \sum_{i<j} r_{i,j} = \sum_{i<j} d_{i,j}, \tag{6.1}$$

where $d_{i,j}$ is the shortest-path distance between nodes i and j, that is, the smallest sum of the edge weights along any path joining nodes i and j.

In addition to the Kirchhoff index, many other "topological" indices of graphs have arisen out of the mathematical chemistry literature [87]. One of the earliest to arise was the Wiener index, ω [87]. The Wiener index for any (undirected) graph is defined as

$$\omega := \sum_{i<j} d_{i,j}. \tag{6.2}$$

Thus, for undirected trees with unit edge weights, the Kirchhoff and Wiener indices are identical. However, the two indices differ for any other graph. Hence, while the results in §6.3 apply equally to Wiener indices, we choose to interpret them only in terms of the Kirchhoff index and $\mathcal{H}_2$ norm.

Much work has already been done on comparing undirected trees based on their Wiener indices. It is already well-known that the Wiener index of an undirected tree will fall between that of the undirected star and that of the undirected path [31, 39, 45]. For undirected trees with a fixed number of nodes, the 15 trees with smallest Wiener index and the 17 trees with largest Wiener index have been identified [30, 32]. Further, for undirected trees with a fixed number of nodes and a fixed diameter, the tree with smallest Wiener index has been found [104]. Therefore, most of the main results in §6.3 have already been derived. Our contribution in this section includes new methods of proof that rely on local changes of topology and provide constructive means to order undirected trees and derive decentralised strategies for improving robustness. We follow this in §6.5 by deriving a similar partial ordering of directed trees that, to our knowledge, has not been reported previously.

A different graph index, developed in the mathematical literature, is the maximum eigenvalue of the adjacency matrix A [91]. Simić and Zhou developed a partial ordering of trees with fixed diameter according to this index in [91]. Their work, in particular the families of trees they considered and the order in which they proved their results, has motivated the approach taken in this chapter.

6.2 A System of Terminology for Trees

We first introduce a system of terminology relating to trees. Much of our terminology for undirected trees corresponds to that in [91] and earlier papers. We have tried to be as consistent as possible in using similar terminology for directed trees. Let $\mathcal{T}^{\text{un}}_{N,\hat{d}}$ be the set of all undirected trees containing N nodes and with diameter $\hat{d}$. For $N \geq 3$, an undirected tree must have a diameter of at least 2, and $\mathcal{T}^{\text{un}}_{N,2}$ contains only one tree. This tree is the undirected star, and is denoted $\mathcal{S}^{\text{un}}_N$. For all positive N, the maximum diameter of a tree is $N-1$, and $\mathcal{T}^{\text{un}}_{N,N-1}$ contains only one tree. This tree is the undirected path, and is denoted $\mathcal{P}^{\text{un}}_N$.

Let $\mathcal{T}^{\text{dir}}_{N,\hat{d}}$ be the set of all directed trees containing N nodes and with depth $\hat{d}$. For $N \geq 2$, a directed tree must have a depth of at least 1, and $\mathcal{T}^{\text{dir}}_{N,1}$ contains only one tree. This tree is the directed star, and is denoted $\mathcal{S}^{\text{dir}}_N$. For all positive N, the maximum depth of a tree is $N-1$, and $\mathcal{T}^{\text{dir}}_{N,N-1}$ contains only one tree. This tree is the directed path, and is denoted $\mathcal{P}^{\text{dir}}_N$.

Recall that a *leaf* (or pendant) in an undirected tree is a node with degree 1, and that a leaf in a directed tree is a node with zero in-degree. A *bouquet* is a non-empty set of leaf nodes, all adjacent to the same node. A node which is not a leaf is called an *internal* node. Note that the root of a directed tree is also an internal node. In a directed tree, any node with multiple children is referred to as a *branch node*.

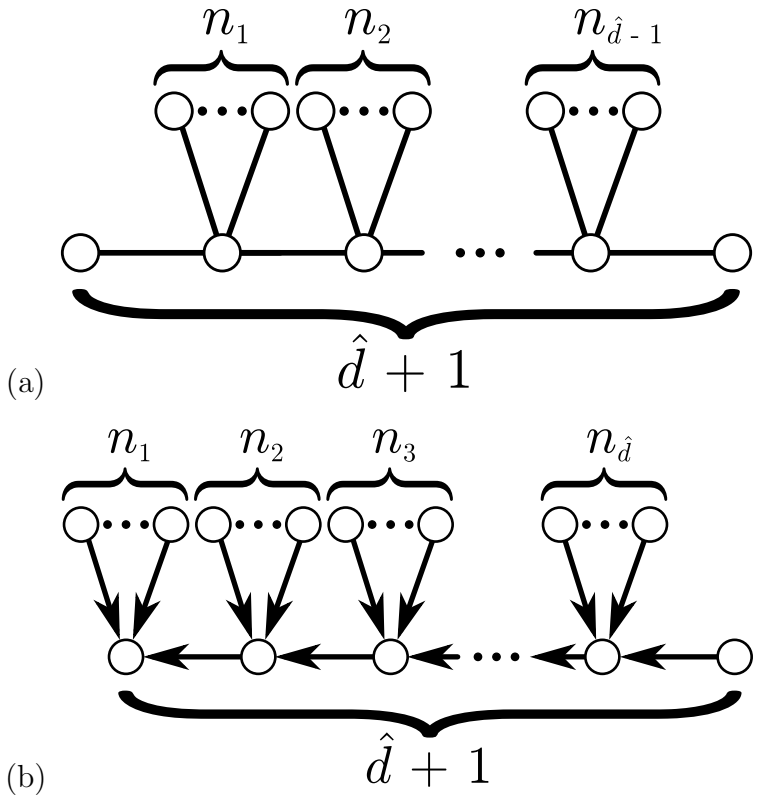

Figure 6.1: General form of a caterpillar in (a) $\mathcal{C}^{\text{un}}_{N,\hat{d}}$ and (b) $\mathcal{C}^{\text{dir}}_{N,\hat{d}}$. Each caterpillar has $n_j \geq 0$ additional leaf nodes attached to each internal node j in the path of length $\hat{d}$.

A *caterpillar* is a tree for which the removal of all leaf nodes would leave a path. The set of all undirected caterpillars with N nodes and diameter $\hat{d}$ is denoted by $\mathcal{C}^{\text{un}}_{N,\hat{d}}$ (see Figure 6.1(a)). The set of all directed caterpillars with N nodes and depth $\hat{d}$ is denoted by $\mathcal{C}^{\text{dir}}_{N,\hat{d}}$ (see Figure 6.1(b)). Any caterpillar in $\mathcal{C}^{\text{un}}_{N,\hat{d}}$ (respectively, $\mathcal{C}^{\text{dir}}_{N,\hat{d}}$) contains an undirected (respectively, directed) path of length $\hat{d}$, with all other nodes adjacent to internal nodes of this path. In particular, we refer to the caterpillar that contains a single bouquet attached to the i^{th} internal node along this path as $\mathcal{P}^{\text{un}}_{N,\hat{d},i}$ in the undirected case and $\mathcal{P}^{\text{dir}}_{N,\hat{d},i}$ in the directed case (see Figure 6.2). To avoid ambiguity, in $\mathcal{P}^{\text{un}}_{N,\hat{d},i}$ we require $1 \leq i \leq \lfloor \frac{\hat{d}}{2} \rfloor$. In contrast, in $\mathcal{P}^{\text{dir}}_{N,\hat{d},i}$, we can have $1 \leq i \leq \hat{d}$ (we exclude $i = \hat{d}+1$ since this would increase the depth of the tree).

The undirected tree formed from $\mathcal{P}^{\text{un}}_{N-1,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$ by attaching an additional node to one of the leaves in the central bouquet is denoted by $\mathcal{N}^{\text{un}}_{N,\hat{d}}$ (see Figure 6.3).

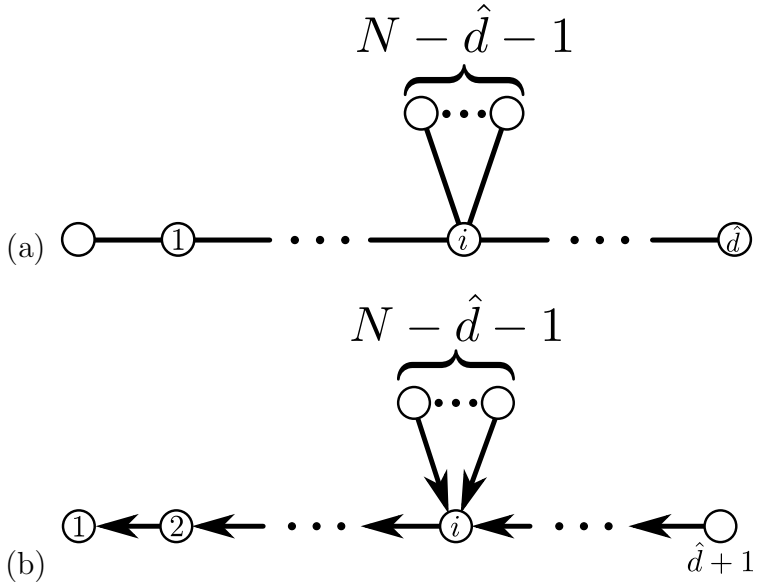

Figure 6.2: Caterpillar graphs (a) $\mathcal{P}^{\text{un}}_{N,\hat{d},i}$ and (b) $\mathcal{P}^{\text{dir}}_{N,\hat{d},i}$. Each contains a path of length $\hat{d}$ with a bouquet containing $N - \hat{d} - 1$ leaf nodes attached to the i^{th} internal node on the path.

A *double palm tree* (also referred to as a dumbbell in [31]) is a caterpillar with two bouquets, one at each end of the path (see Figure 6.4). We use $\mathcal{D}^{\text{un}}_{N,p,q}$ to denote the undirected double palm tree on N nodes, with bouquets of sizes p and q. Similarly, we use $\mathcal{D}^{\text{dir}}_{N,p,q}$ to denote the directed double palm tree on N nodes, with bouquets of sizes p and q. Note that the two ends of a directed path are non-identical (one end is the root while the other is the leaf). In $\mathcal{D}^{\text{dir}}_{N,p,q}$, the bouquet of size p will be attached to the root.

If we take an undirected tree $\mathcal{T}$ and attach two separate paths containing ℓ and k nodes to node r in $\mathcal{T}$, we call the resulting tree a *vine* and denote it by $\mathcal{T}^r_{\ell,k}$ (see Figure 6.5). Note that by a judicious choice of $\mathcal{T}$, *any* undirected tree can be considered to be a vine.

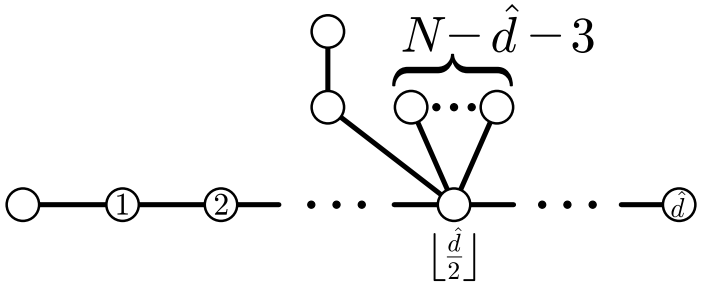

Figure 6.3: The tree $\mathcal{N}^{\text{un}}_{N,\hat{d}}$, formed from $\mathcal{P}^{\text{un}}_{N-1,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$ by attaching an additional node to one of the leaves in the central bouquet. Note that $N - \hat{d} - 3$ must be greater than or equal to 0.

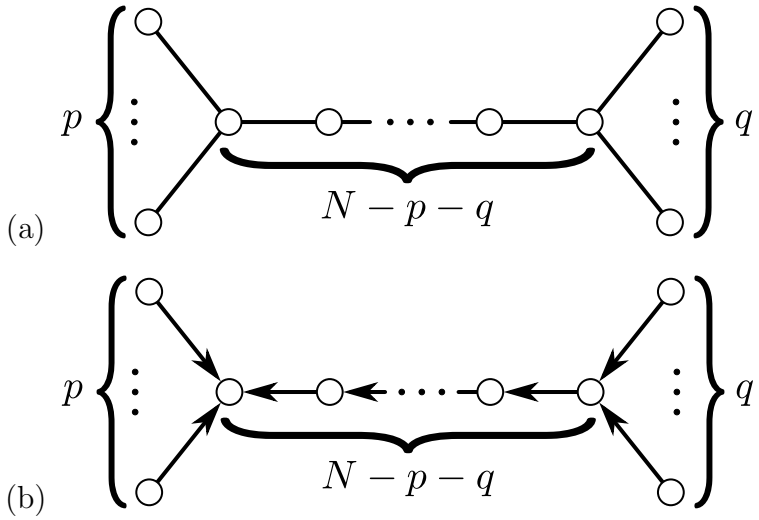

Figure 6.4: Double palm tree graphs (a) $\mathcal{D}^{\text{un}}_{N,p,q}$ and (b) $\mathcal{D}^{\text{dir}}_{N,p,q}$. Each double palm tree contains bouquets of sizes p and q at each end of a path.

6.3 Manipulations to Reduce the Effective Resistance of Undirected Trees

We can now start to describe a partial ordering on undirected trees based on their $\mathcal{H}_2$ norms. Every tree is assumed to have a unit weight on every edge. First, we determine the effect of moving a leaf from one end of a double palm tree to the other, and use this to derive a complete ordering of all trees in $\mathcal{T}^{\text{un}}_{N,3}$ (Theorem 6.1). Second, we consider moving a leaf from one end of a vine to the other, and use this to prove that the path has the largest $\mathcal{H}_2$ norm of any undirected tree with N nodes (Theorem 6.2),

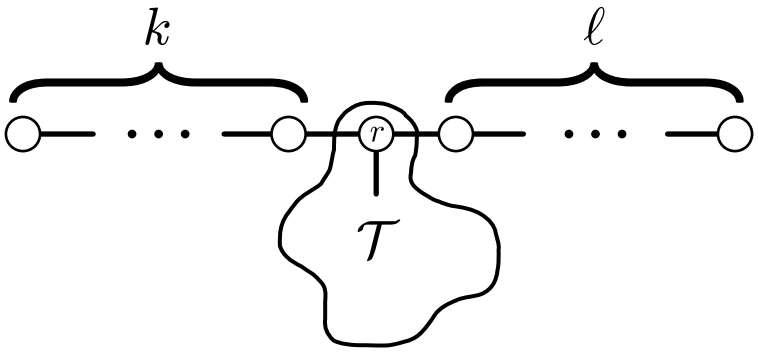

Figure 6.5: The vine $\mathcal{T}^r_{\ell,k}$, formed from any undirected tree $\mathcal{T}$ by separately connecting paths containing ℓ and k nodes to node r.

and to derive a complete ordering of $\mathcal{T}^{\text{un}}_{N,N-2}$ (Theorem 6.3). Finally, by moving all (or almost all) nodes in a bouquet to an adjacent node, we show that $\mathcal{P}^{\text{un}}_{N,\hat{d},\lfloor\frac{\hat{d}}{2}\rfloor}$ has the smallest $\mathcal{H}_2$ norm of any undirected tree with diameter $\hat{d}$ (Theorem 6.4) and that for any tree that is not a star, we can find a tree of smaller diameter with a smaller $\mathcal{H}_2$ norm (Theorem 6.5). From Theorem 6.5 we also conclude that the star has the smallest $\mathcal{H}_2$ norm of any undirected tree with N nodes.

The same ordering will apply to the set of trees with a given constant edge weight, as all effective resistances will be proportional to those in the corresponding tree with unit weights.

6.3.1 Double Palm Trees

We begin our partial ordering by showing that the $\mathcal{H}_2$ norm of an undirected double palm tree is reduced when we move a single node from the smaller bouquet to the larger one.

Lemma 6.1. *Let* $1 < p \leq q$ *and* $p + q \leq N - 2$. *Then* $\eta\left(\mathcal{D}^{un}_{N,p,q}\right) > \eta\left(\mathcal{D}^{un}_{N,p-1,q+1}\right)$.

Proof. In $\mathcal{D}^{\text{un}}_{N,p,q}$, let one of the nodes in the bouquet of size p be node 1. The remaining nodes are labelled 2 through N. To form $\mathcal{D}^{\text{un}}_{N,p-1,q+1}$, we take node 1 and move it to the other bouquet. Since all other nodes remain unchanged, we can use (6.1) to write

$$K_f\left(\mathcal{D}^{\text{un}}_{N,p,q}\right) - K_f\left(\mathcal{D}^{\text{un}}_{N,p-1,q+1}\right) = \left(\sum_{j=2}^{N} d_{1,j}\right)_{\mathcal{D}^{\text{un}}_{N,p,q}} - \left(\sum_{j=2}^{N} d_{1,j}\right)_{\mathcal{D}^{\text{un}}_{N,p-1,q+1}}.$$

Now, in $\mathcal{D}^{\text{un}}_{N,p,q}$, the path length between node 1 and any of the remaining $p-1$ nodes in the bouquet of size p is 2. Similarly, the path length between node 1 and any node in the bouquet of size q is $N-p-q+1$. Finally, the path lengths between node 1 and the internal nodes take on each integer value from 1 to $N-p-q$.

In $\mathcal{D}^{\text{un}}_{N,p-1,q+1}$, the path length between node 1 and any of the nodes in the bouquet of size $p-1$ is now $N-p-q+1$. The path length between node 1 and any of the remaining q nodes in the bouquet of size $q+1$ is now 2. Again, the path lengths between node 1 and the internal nodes take on all integer values from 1 to $N-p-q$.

Thus, $\mathcal{D}^{\text{un}}_{N,p-1,q+1}$ (compared to $\mathcal{D}^{\text{un}}_{N,p,q}$) has more nodes at a distance 2 from node 1 and fewer nodes at a distance $N-p-q+1$, while the sum of distances to all internal nodes remains the same. Therefore $K_f\left(\mathcal{D}^{\text{un}}_{N,p,q}\right) > K_f\left(\mathcal{D}^{\text{un}}_{N,p-1,q+1}\right)$. Hence, by (4.11), the result holds. □

Although Lemma 6.1 applies to double palm trees with any diameter, we can apply it to trees with $\hat{d}=3$ in order to prove our first main result of the chapter.

Theorem 6.1. *For $N \geq 4$, we have a complete ordering of $\mathcal{T}^{un}_{N,3}$, namely*

$$\eta\left(\mathcal{D}^{un}_{N,1,N-3}\right) < \eta\left(\mathcal{D}^{un}_{N,2,N-4}\right) < \ldots < \eta\left(\mathcal{D}^{un}_{N,\lfloor\frac{N-2}{2}\rfloor,\lceil\frac{N-2}{2}\rceil}\right).$$

Proof. Any undirected tree with diameter $\hat{d}=3$ must have a longest path of length 3. Any additional nodes in the tree must be connected through some path to one of the two internal nodes on this longest path. In addition, any node adjacent to one of the internal nodes of the longest path forms a path of length 3 with the node at the far end of the path. Hence all such nodes must be leaves and so every undirected tree with $\hat{d}=3$ is a double palm tree. The ordering follows from Lemma 6.1. □

6.3.2 Vines

Our next task is to find an ordering of undirected trees with the largest possible diameter. Lemma 6.2 applies to trees of any diameter, but again we can specialise it to give the results we need.

Lemma 6.2. *Let $\mathcal{T}$ be an undirected tree containing more than one node, including node r, and let ℓ, k be any positive integers such that $1 \leq \ell \leq k$. Then $\eta\left(\mathcal{T}^r_{\ell,k}\right) < \eta\left(\mathcal{T}^r_{\ell-1,k+1}\right)$.*

Proof. Let the total number of nodes in $\mathcal{T}^r_{\ell,k}$ be N (so $N-k-\ell > 1$), and let the leaf at the end of the path containing ℓ nodes be node 1. Let the remaining nodes in the two paths be nodes 2 through $l+k$, and let node r in $\mathcal{T}$ be node $\ell+k+1$ in $\mathcal{T}^r_{\ell,k}$. The remaining nodes from $\mathcal{T}$ are labelled $\ell+k+2$ through N. To form $\mathcal{T}^r_{\ell-1,k+1}$, we take node 1 and move it to the end of the other path. Since all other nodes remain unchanged, we can use (6.1) to write

$$K_f\left(\mathcal{T}^r_{\ell-1,k+1}\right) - K_f\left(\mathcal{T}^r_{\ell,k}\right) = \left(\sum_{j=2}^{N} d_{1,j}\right)_{\mathcal{T}^r_{\ell-1,k+1}} - \left(\sum_{j=2}^{N} d_{1,j}\right)_{\mathcal{T}^r_{\ell,k}}.$$

Now, in both $\mathcal{T}^r_{\ell,k}$ and $\mathcal{T}^r_{\ell-1,k+1}$, the path lengths between node 1 and all nodes along the paths (including the root of $\mathcal{T}$) take on each integer value between 1 and $\ell+k$. Hence the sum of these path lengths does not change between the two trees. Furthermore, since node r of $\mathcal{T}$ lies on every path between node 1 and any other node in $\mathcal{T}$, we can write

$$d_{1,j} = d_{1,\ell+k+1} + d_{\ell+k+1,j},\ j \geq \ell+k+2.$$

Therefore, for $\mathcal{T}^r_{\ell,k}$, the sum of the distances from node 1 to all the nodes in $\mathcal{T}$ is $(N-\ell-k+1)\ell$ plus the sum of the distances from node r to each node in $\mathcal{T}$. However, in $\mathcal{T}^r_{\ell-1,k+1}$, the sum of the distances from node 1 to all the nodes in $\mathcal{T}$ is

$(N-\ell-k+1)(k+1)$ plus the sum of the distances from node r to each node in $\mathcal{T}$. Thus $K_f\left(\mathcal{T}^r_{\ell-1,k+1}\right) > K_f\left(\mathcal{T}^r_{\ell,k}\right)$ and so by (4.11), the result holds $\square$

The first consequence of Lemma 6.2 is that the undirected tree with largest diameter (i.e. $\hat{d} = N-1$) also has the largest $\mathcal{H}_2$ norm.

Theorem 6.2. *The path $\mathcal{P}^{un}_N$ has the largest $\mathcal{H}_2$ norm of any undirected tree with N nodes.*

Proof. Any undirected tree $\mathcal{T}_1$ which is not a path will contain a node with degree greater than 2. We can locate one such node that has two paths (each with fewer than N nodes; one of length ℓ and one of length k) attached. Let $\mathcal{T}$ be the tree formed by removing these two paths from $\mathcal{T}_1$, and let this node be node r in $\mathcal{T}$. Then $\mathcal{T}_1 = \mathcal{T}^r_{\ell,k}$, and by Lemma 6.2 we can find a tree with larger $\mathcal{H}_2$ norm. $\square$

We can also use Lemma 6.2 to derive an ordering of the undirected trees with diameter one less than its maximum value (i.e. $\hat{d} = N-2$). In this case, a tree will consist of a path of length $N-2$ with one leaf attached to an internal node. Thus these trees are both vines and caterpillars.

Theorem 6.3. *For $N \geq 4$, we have a complete ordering of $\mathcal{T}^{un}_{N,N-2}$, namely*

$$\eta\left(\mathcal{P}^{un}_{N,N-2,\lfloor\frac{N-2}{2}\rfloor}\right) < \eta\left(\mathcal{P}^{un}_{N,N-2,\lfloor\frac{N-2}{2}\rfloor-1}\right) < \ldots < \eta\left(\mathcal{P}^{un}_{N,N-2,1}\right).$$

Proof. Every tree in $\mathcal{T}^{\text{un}}_{N,N-2}$ must contain a path of length $N-2$ (which contains $N-1$ nodes), and one additional node. This node must be adjacent to an internal node of the path, since otherwise we would have a path of length $N-1$. Thus every tree in $\mathcal{T}^{\text{un}}_{N,N-2}$ is of the form $\mathcal{P}^{\text{un}}_{N,N-2,i}$, for some $1 \leq i \leq \lfloor\frac{N-2}{2}\rfloor$.

Now, if we let $\mathcal{T}$ be an undirected path containing 2 nodes (labelled 1 and 2), we observe that $\mathcal{P}^{\text{un}}_{N,N-2,i} = \mathcal{T}^1_{i,N-i-2}$. Suppose that $i < \lfloor\frac{N-2}{2}\rfloor$. Then $i < N-i-2$, and so by Lemma 6.2, $\eta\left(\mathcal{P}^{\text{un}}_{N,N-2,i}\right) = \eta\left(\mathcal{T}^1_{i,N-2-i}\right) > \eta\left(\mathcal{T}^1_{i+1,N-3-i}\right) = \eta\left(\mathcal{P}^{\text{un}}_{N,N-2,i+1}\right)$. $\square$

The only difference between trees in $\mathcal{T}^{\text{un}}_{N,N-2}$ is which internal node has the additional leaf attached. Theorem 6.3 ensures that the $\mathcal{H}_2$ norm is smallest when this internal node is at the centre of the path.

6.3.3 Caterpillars

We now have complete orderings for $\mathcal{T}^{\text{un}}_{N,2}$ (trivial, since $\mathcal{T}^{\text{un}}_{N,2}$ contains only the undirected star), $\mathcal{T}^{\text{un}}_{N,3}$ (by Theorem 6.1), $\mathcal{T}^{\text{un}}_{N,N-2}$ (by Theorem 6.3) and $\mathcal{T}^{\text{un}}_{N,N-1}$ (trivial, since $\mathcal{T}^{\text{un}}_{N,N-1}$ contains only the undirected path). We next consider the remaining families of trees with diameters in the range $4 \leq \hat{d} \leq N-3$ (and hence, $N \geq 7$).

Rather than deriving complete orderings, the main goal of the next two lemmas is to find the tree in $\mathcal{T}^{\text{un}}_{N,\hat{d}}$ with lowest $\mathcal{H}_2$ norm. However, we use two steps to attain our result since this provides greater insight into the ordering amongst the remaining trees. Lemma 6.5 then allows us to combine the results to prove (in Theorem 6.4) that among trees of diameter $\hat{d}$, the one with lowest $\mathcal{H}_2$ norm is $\mathcal{P}_{N,\hat{d},\lfloor\frac{\hat{d}}{2}\rfloor}$. Theorem 6.5 then provides a comparison of trees with different diameters.

Lemma 6.3. *Suppose $N \geq 7$ and $4 \leq \hat{d} \leq N-3$. If $\mathcal{T} \in \mathcal{C}^{un}_{N,\hat{d}}$, then $\eta(\mathcal{T}) \geq \eta\left(\mathcal{P}^{un}_{N,\hat{d},\lfloor\frac{\hat{d}}{2}\rfloor}\right)$, with equality if and only if $\mathcal{T} = \mathcal{P}^{un}_{N,\hat{d},\lfloor\frac{\hat{d}}{2}\rfloor}$.*

Proof. Since $\hat{d} \leq N-3$ and $\mathcal{T} \in \mathcal{C}_{N,\hat{d}}$, a longest path in $\mathcal{T}$ contains $N-\hat{d}-1 \geq 2$ leaves attached to internal nodes (other than the two leaves in the longest path). Suppose that $\mathcal{P}_{\mathcal{T}}$ is a longest undirected path subgraph of $\mathcal{T}$. For the rest of this proof, when we refer to leaf nodes and bouquets, we mean leaves not part of $\mathcal{P}_{\mathcal{T}}$, and bouquets made up of these leaves.

Suppose $\mathcal{T}$ contains a single bouquet. Thus $\mathcal{T} = \mathcal{P}^{\text{un}}_{N,d,i}$ for some $1 \leq i \leq \lfloor\frac{d}{2}\rfloor$. If $i \neq \lfloor\frac{d}{2}\rfloor$, then by Lemma 6.2, $\eta\left(\mathcal{P}^{\text{un}}_{N,d,i}\right) > \eta\left(\mathcal{P}^{\text{un}}_{N,d,i+1}\right)$.

Suppose $\mathcal{T}$ contains multiple bouquets. Locate a bouquet furthest from the centre of $\mathcal{P}_{\mathcal{T}}$, and move every leaf in this bouquet one node further from the closest end of

$\mathcal{P}_{\mathcal{T}}$. Call this new tree $\mathcal{T}'$, and label the nodes that were moved 1 through n. Then between $\mathcal{T}$ and $\mathcal{T}'$, the path lengths between each of these leaves and any other leaf decrease by 1. The path lengths between each of these leaves and $\leq \lfloor \frac{\hat{d}+1}{2} \rfloor$ nodes on $\mathcal{P}_{\mathcal{T}}$ increase by 1, and the path lengths between each of these leaves and $\geq \lfloor \frac{\hat{d}+1}{2} \rfloor$ nodes on $\mathcal{P}_{\mathcal{T}}$ decrease by 1. Thus the sum of the path lengths in $\mathcal{T}'$ is less than the sum in $\mathcal{T}$, and so by (6.1) and (4.11), $\eta(\mathcal{T}') < \eta(\mathcal{T})$.

Thus, if $\mathcal{T}$ is not $\mathcal{P}^{\text{un}}_{N,\hat{d},\lfloor \frac{d}{2} \rfloor}$, there is a tree in $\mathcal{C}^{\text{un}}_{N,d}$ with strictly smaller $\mathcal{H}_2$ norm. $\square$

Lemma 6.4. *Suppose that $N \geq 7$ and $4 \leq \hat{d} \leq N-3$. Let $\mathcal{T}$ be a tree in $\mathcal{T}^{un}_{N,\hat{d}} \setminus \mathcal{C}^{un}_{N,\hat{d}}$. Then $\eta(\mathcal{T}) \geq \eta\left(\mathcal{N}^{un}_{N,\hat{d}}\right)$, with equality if and only if $\mathcal{T} = \mathcal{N}^{un}_{N,\hat{d}}$.*

Proof. Let $\mathcal{P}_{\mathcal{T}}$ be a longest undirected path subgraph of $\mathcal{T}$ (of length $\hat{d}$), and let m be the number of nodes with distances to $\mathcal{P}_{\mathcal{T}}$ greater than 1 (the distance between a node and $\mathcal{P}_{\mathcal{T}}$ is given by the shortest distance between that node and any node on the path).

If $m > 1$, locate a bouquet with the greatest distance from $\mathcal{P}_{\mathcal{T}}$, label the leaves in this bouquet 1 through n, and label the adjacent node $n+1$. Suppose that either the distance between this bouquet and $\mathcal{P}_{\mathcal{T}}$ is greater than 2, or the distance is 2 and another bouquet exists at the same distance from $\mathcal{P}_{\mathcal{T}}$. Let $\mathcal{T}'$ be the tree formed by moving all leaves in this bouquet one node closer to $\mathcal{P}_{\mathcal{T}}$. By our assumptions, $\mathcal{T}' \in \mathcal{T}^{\text{un}}_{N,\hat{d}} \setminus \mathcal{C}^{\text{un}}_{N,\hat{d}}$. Then $d_{i,n+1}$ increases by 1 for $i = 1, \ldots, n$. Conversely, $d_{i,j}$ decreases by 1 for $i = 1, \ldots, n$ and $j > n+1$. Since there must be at least $\hat{d}+2 \geq 6$ of these other nodes (with labels above $n+1$), the sum of all distances in $\mathcal{T}'$ is smaller than the sum of all distances in $\mathcal{T}$. Thus $\eta(\mathcal{T}') < \eta(\mathcal{T})$.

If the bouquet we found has a distance of 2 to $\mathcal{P}_{\mathcal{T}}$, and is the only such bouquet, form $\mathcal{T}'$ by moving leaves 1 through $n-1$ one node closer to $\mathcal{P}_{\mathcal{T}}$. Then $\mathcal{T}' \in \mathcal{T}^{\text{un}}_{N,\hat{d}} \setminus \mathcal{C}^{\text{un}}_{N,\hat{d}}$. Now, $d_{i,n}$ and $d_{i,n+1}$ both increase by 1 for $i = 1, \ldots, n-1$. However, $d_{i,j}$ decreases by 1 for $i = 1, \ldots, n-1$ and j one of the remaining $\geq \hat{d}+1 \geq 5$ nodes. Thus $\eta(\mathcal{T}') < \eta(\mathcal{T})$.

If $m = 1$, then $\mathcal{T}$ contains a single node at a distance 2 from $\mathcal{P}_\mathcal{T}$, and all other nodes in $\mathcal{T}$ are either on $\mathcal{P}_\mathcal{T}$ or adjacent to nodes on $\mathcal{P}_\mathcal{T}$. Locate a node on $\mathcal{P}_\mathcal{T}$ with additional nodes attached that is furthest from the centre of the path. Label all nodes attached to this node 1 through n (including the node at distance 2 from $\mathcal{P}_\mathcal{T}$ if it is connected to $\mathcal{P}_\mathcal{T}$ through this node), and label this node $n+1$. If $n+1$ is not the $\lfloor \frac{\hat{d}}{2} \rfloor^{\text{th}}$ internal node on the path (i.e. if $\mathcal{T} \neq \mathcal{N}^{\text{un}}_{N,\hat{d}}$), then form $\mathcal{T}'$ by moving all nodes not on $\mathcal{P}_\mathcal{T}$ that are adjacent to $n+1$ (carrying along the node at distance 2, if present) one node further from the closest end of $\mathcal{P}_\mathcal{T}$. Then $\mathcal{T}' \in \mathcal{T}^{\text{un}}_{N,\hat{d}} \setminus \mathcal{C}^{\text{un}}_{N,\hat{d}}$. Furthermore, for $i = 1, \ldots, n$, $d_{i,j}$ decreases by 1 for any j not on $\mathcal{P}_\mathcal{T}$, increases by 1 for $\leq \lfloor \frac{\hat{d}}{2} \rfloor$ nodes on $\mathcal{P}_\mathcal{T}$ and decreases by 1 for $\geq \lceil \frac{\hat{d}}{2} \rceil + 1$ nodes on $\mathcal{P}_\mathcal{T}$. Thus the sum of the distances in $\mathcal{T}'$ is less than the sum in $\mathcal{T}$, and so $\eta(\mathcal{T}') < \eta(\mathcal{T})$.

Hence for every tree in $\mathcal{T}^{\text{un}}_{N,\hat{d}} \setminus \mathcal{C}^{\text{un}}_{N,\hat{d}}$ other than $\mathcal{N}^{\text{un}}_{N,\hat{d}}$, there exists another tree in $\mathcal{T}^{\text{un}}_{N,\hat{d}} \setminus \mathcal{C}^{\text{un}}_{N,\hat{d}}$ with smaller $\mathcal{H}_2$ norm. □

Lemma 6.5. *Suppose that $N \geq 7$ and $4 \leq \hat{d} \leq N-3$. Then $\eta\left(\mathcal{P}^{un}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}\right) < \eta\left(\mathcal{N}^{un}_{N,\hat{d}}\right)$.*

Proof. Label the node in $\mathcal{N}^{\text{un}}_{N,\hat{d}}$ that is a distance 2 from the longest path as node 1, and label the node it is adjacent to as node 2. Then we can form $\mathcal{P}^{\text{un}}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$ from $\mathcal{N}^{\text{un}}_{N,\hat{d}}$ by moving node 1 one node closer to the longest path. Then $d_{1,j}$ decreases by 1 for $j = 3, \ldots, N$, and $d_{1,2}$ increases by 1. Since $N \geq 7$, the sum of all path lengths in $\mathcal{P}^{\text{un}}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$ is less than in $\mathcal{N}^{\text{un}}_{N,\hat{d}}$. Thus, by (6.1) and (4.11), $\eta\left(\mathcal{P}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}\right) < \eta\left(\mathcal{N}_{N,\hat{d}}\right)$. □

Now, we have enough to determine the tree in $\mathcal{T}^{\text{un}}_{N,\hat{d}}$ with smallest $\mathcal{H}_2$ norm.

Theorem 6.4. *Let $N \geq 4$ and $2 \leq \hat{d} \leq N-2$. The tree in $\mathcal{T}^{un}_{N,\hat{d}}$ with smallest $\mathcal{H}_2$ norm is $\mathcal{P}^{un}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$.*

Proof. For $\hat{d} = 2$, $\mathcal{T}^{\text{un}}_{N,\hat{d}}$ only contains $\mathcal{S}^{\text{un}}_N$, which is the same as $\mathcal{P}^{\text{un}}_{N,2,1}$. For $\hat{d} = 3$, the result follows from Theorem 6.1 since $\mathcal{D}^{\text{un}}_{N,1,N-3} = \mathcal{P}^{\text{un}}_{N,3,1}$. For $4 \leq \hat{d} \leq N-3$, this is a simple consequence of Lemmas 6.3, 6.4 and 6.5. For $\hat{d} = N-2$, the result follows from Theorem 6.3. □

Finally, we can combine several of our earlier results to obtain a basic comparison between undirected trees of different diameters.

Theorem 6.5. *Let $3 \leq \hat{d} \leq N-1$. For any tree in $\mathcal{T}^{un}_{N,\hat{d}}$, there is a tree in $\mathcal{T}^{un}_{N,\hat{d}-1}$ with a smaller $\mathcal{H}_2$ norm. Hence, the star $\mathcal{S}^{un}_N$ has the smallest $\mathcal{H}_2$ norm of any tree with N nodes.*

Proof. By Lemma 6.2, $\eta\left(\mathcal{P}^{\text{un}}_{N,N-2,i}\right) < \eta\left(\mathcal{P}^{\text{un}}_N\right)$ (for any $1 \leq i \leq \lfloor \frac{N-2}{2} \rfloor$).

Let $4 \leq \hat{d} \leq N-2$. Suppose $\mathcal{T} \in \mathcal{T}^{\text{un}}_{N,\hat{d}}$. Then by Theorem 6.4, $\eta(\mathcal{T}) \geq \eta\left(\mathcal{P}^{\text{un}}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}\right)$. But by Lemma 6.2, $\eta\left(\mathcal{P}^{\text{un}}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}\right) > \eta\left(\mathcal{P}^{\text{un}}_{N,\hat{d}-1,\lfloor \frac{\hat{d}-1}{2} \rfloor}\right)$. Thus

$$\eta\left(\mathcal{P}^{\text{un}}_{N,\hat{d}-1,\lfloor \frac{\hat{d}-1}{2} \rfloor}\right) < \eta(\mathcal{T}).$$

Let $\mathcal{T} \in \mathcal{T}^{\text{un}}_{N,3}$. Then by Theorem 6.1, $\eta(\mathcal{T}) \geq \eta\left(\mathcal{D}^{\text{un}}_{N,1,N-3}\right)$. But by Lemma 6.2, $\eta\left(\mathcal{S}^{\text{un}}_N\right) < \eta\left(\mathcal{D}^{\text{un}}_{N,1,N-3}\right)$. Thus $\eta\left(\mathcal{S}^{\text{un}}_N\right) < \eta(\mathcal{T})$. □

6.4 A Decentralised Algorithm For Rearranging Undirected Trees

We were mostly able to derive the results in §6.3 by determining the effect of moving leaves on effective resistances. In particular, we showed that for a non-star tree, the $\mathcal{H}_2$ norm can always be reduced either by moving a single node to somewhere else in the tree, or by moving a bouquet of nodes to an adjacent node. These manipulations are "local" in the sense that nodes are moved only from a single location in the tree at a time, and the rest of the nodes in the tree are not required to take any additional action.

In order for agents interacting over a network to make changes to their neighbours that will be guaranteed to improve the $\mathcal{H}_2$ norm of the graph, each agent must have *some* knowledge about the network. However, requiring *total* knowledge

of the network structure is inefficient (due to the time required to learn such information) and does not scale well - knowing an entire undirected network is equivalent to knowing the $\frac{N(N-1)}{2}$ elements below the diagonal in the graph adjacent matrix, while knowing an entire undirected tree requires the knowledge of $N-1$ undirected edges. Therefore, an efficient algorithm for rearranging a network should ideally only rely on information about an agent's neighbours.

Based the approach taken in §6.3, we are able to propose a decentralised algorithm for rearranging undirected trees that only requires three pieces of information to be transmitted between neighbours - node degrees, node locations (note that location information might already be transmitted for the purposes of collective behaviour) and number of new neighbours. Furthermore, our algorithm can proceed asynchronously, with each agent proceeding through the steps as its own rate. As above, we assume that all edge weights in the communication graph are 1.

Algorithm 6.1. *1. Each agent transmits its degree and location to each of its neighbours and then proceeds to step 2.*

2. *If an agent has $n > 1$ neighbours, and has precisely one neighbour with degree greater then 1 (neighbour m) then it informs neighbour m to expect $n-1$ new neighbours and instructs its neighbours with degree equal to 1 to connect to neighbour m. Otherwise, the agent should wait for new information from its neighbours and then proceed to step 3.*

3. *If an agent has been instructed to make a new connection, it breaks its current connection, changes its location (if necessary) and forms a connection with its new neighbour. Otherwise, proceed to step 4.*

4. *If an agent is expecting new neighbours, it maintains its current location until all expected connections have formed. Any agent not expecting new connections can return to step 1.*

Before we analyse the effects of Algorithm 6.1, we must first verify that it cannot give rise to conflicting instructions or lead to the graph becoming disconnected (except temporarily, as connections are rearranged). The only step in the algorithm that results in changes to the graph is step 3, which (by step 2) will only be undertaken by agents that are currently leaf nodes. Since leaf nodes only have one neighbour, they can only receive one instruction to form a new connection at a time. Furthermore, since the only edges in the graph that become broken are those between a leaf and an internal node, any disconnections that occur will break the graph into a tree and one or more isolated nodes. Then, by the instructions sent in step 2, the isolated nodes will know where to find their target node to reconnect to the graph, and the target node will not change its location until all isolated nodes seeking it have found it.

In addition, since one connection is formed for every one broken, the graph will always remain an undirected tree.

Finally, we observe that any agent that satisfies the conditions of step 2 will subsequently become a leaf node. In this case the remaining neighbour will wait for all of its new connections in step 4 before it can issue any instructions to this new leaf node.

A potential flaw with implementing Algorithm 6.1 in a physical system (e.g. a group of mobile robots) is that a leaf node could experience a failure during step 3 and be unable to reconnect to the network. In this case, the algorithm as written would cause its target node to wait indefinitely for an additional node. One solution to this issue could be to include a time limit in step 4, and assume that any nodes which do not connect in that time have been permanently lost.

We are now in a position to determine the effect of Algorithm 6.1 on the $\mathcal{H}_2$ norm of the tree.

Proposition 6.1. *Suppose a network of agents that are connected by an undirected tree graph $\mathcal{T}$ carries out Algorithm 6.1. If this causes the graph to change to the undirected tree $\mathcal{T}'$, then*

$$\eta(\mathcal{T}') < \eta(\mathcal{T}).$$

Proof. Due to the asynchronous nature of Algorithm 6.1 and the discussion above, we can assume that the only difference between $\mathcal{T}$ and $\mathcal{T}'$ is the result of a single node in $\mathcal{T}$ issuing instructions according to step 2 in the algorithm. Suppose that in $\mathcal{T}$ this node is labelled as node 1, its leaf neighbours are nodes $2, \ldots, n$ and its non-leaf neighbour is node $n+1$. Note that since node $n+1$ is not a leaf, it must have at least one additional neighbour and so $N > n+1$.

By step 2 of Algorithm 6.1, $\mathcal{T}'$ is formed from $\mathcal{T}$ by nodes $2, \ldots, n$ changing their neighbour from node 1 to node $n+1$. Thus for $i = 2, \ldots, n$, $d_{i,j}$ decreases by 1 for any $j > n$, increases by 1 for $j = 1$ and remains the same for $j = 2, \ldots, n$. All other distances in the graph are unchanged. Since there are at least 2 nodes j with $j > n$, the sum of all distances in $\mathcal{T}'$ is less than the sum of all distances in $\mathcal{T}$. The result follows by (6.1) and (4.11). □

Corollary 6.1. *If Algorithm 6.1 is implemented on an undirected tree of finite size, that tree will become an undirected star after a finite number of steps.*

Proof. By Proposition 6.1, any changes that occur as a result of Algorithm 6.1 will strictly decrease the $\mathcal{H}_2$ norm of the graph, and hence produce a tree that has not arisen before. Since the number of trees with N nodes is finite, Algorithm 6.1 must converge in a finite number of steps.

The only time when Algorithm 6.1 will not lead to changes in the graph is when there are no nodes in the tree that satisfy the conditions of step 2. That is, any node that is not a leaf in a limiting undirected tree of Algorithm 6.1 must have either zero or more than one non-leaf neighbours. But if all the leaves of a tree are removed,

the resulting graph (containing the non-leaf nodes of the original tree) is still a tree and so it must either contain leaves (that is, non-leaf nodes in the original tree that only have one non-leaf neighbour) or it must be a single isolated node. Since the only undirected tree containing one non-leaf node with all other nodes being leaves is the undirected star, Algorithm 6.1 must converge to $\mathcal{S}_N^{\text{un}}$. □

From Proposition 6.1 and Corollary 6.1, we see that Algorithm 6.1 is a decentralised algorithm that will monotonically improve the $\mathcal{H}_2$ norm of an undirected tree until it reaches its lowest possible value. However, situations can be envisioned where agents would not have complete freedom over which other agents to connect with. For example, a spanning tree of minimal $\mathcal{H}_2$ norm might be sought for a fixed network, some agents may have a maximum number of neighbours they can connect with at any one time, or certain connections in the network may need to be maintained for other reasons. In these cases, it should be possible to modify Algorithm 6.1 to satisfy such constraints while still monotonically improving the $\mathcal{H}_2$ norm. For example, a node satisfying the conditions in step 2 only needs to send $n-2$ of its leaf neighbours to its non-leaf neighbour to guarantee improvement in the $\mathcal{H}_2$ norm. Alternatively, with more knowledge about the network, it could be possible to make even fewer changes at once.

6.5 Manipulations to Reduce the Effective Resistance of Directed Trees

We now return our attention to directed trees with unit edge weights and seek a partial ordering similar to that derived in §6.3. However, effective resistances in directed trees are not necessarily equivalent to path lengths between nodes, but instead are given by (5.43). Throughout this section we will use $r(n, m)$ to denote the effective resistance between two nodes in a directed tree with unit edge weights and connected to their

closest mutually reachable node by paths of length n and m (as in §5.3). Then $r(n,m)$ can be computed using (5.43). This formula is difficult to evaluate in general, but any two nodes in a directed caterpillar will either have a direct connection between them, or one node will be joined to their closest mutually reachable node by a path of length 1. Therefore, (5.43) can be evaluated explicitly for all effective resistances in a directed caterpillar graph. This allows us to derive results for directed trees that are similar to Lemma 6.1 and Lemma 6.3.

Lemma 6.6. *Let $p \geq 0$, $q > 1$ and $p + q \leq N - 2$. Then*

$$\eta\left(\mathcal{D}^{dir}_{N,p,q}\right) - \eta\left(\mathcal{D}^{dir}_{N,p+1,q-1}\right) \begin{cases} = 0 & \text{if } q = p+3 \\ > 0 & \text{if } q < p+3 \\ < 0 & \text{if } q > p+3. \end{cases} \tag{6.3}$$

Proof. In $\mathcal{D}^{\text{dir}}_{N,p,q}$, let one of the nodes in the bouquet of size q be node 1. The remaining nodes are labelled 2 through N. To form $\mathcal{D}^{\text{dir}}_{N,p+1,q-1}$, we take node 1 and move it to the other bouquet. Since all other nodes and edges remain unchanged, we can use (4.10) to write

$$K_f\left(\mathcal{D}^{\text{dir}}_{N,p,q}\right) - K_f\left(\mathcal{D}^{\text{dir}}_{N,p+1,q-1}\right) = \left(\sum_{j=2}^{N} r_{1,j}\right)_{\mathcal{D}^{\text{dir}}_{N,p,q}} - \left(\sum_{j=2}^{N} r_{1,j}\right)_{\mathcal{D}^{\text{dir}}_{N,p+1,q-1}}.$$

Now, in $\mathcal{D}^{\text{dir}}_{N,p,q}$, the effective resistance between node 1 and any of the remaining $q-1$ nodes in the bouquet of size q is, by (5.43),

$$r(1,1) = 2.$$

In addition, the effective resistance between node 1 and any node in the bouquet of

size p is

$$r(N-p-q,1) = 2\,(N-p-q-1) + 2^{2+p+q-N}.$$

Finally, by Theorem 5.1, the effective resistances between node 1 and the internal nodes take on each even integer value from 2 to $2\,(N-p-q)$. Thus, using (B.26), we find that

$$\left(\sum_{j=2}^{N} r_{1,j}\right)_{\mathcal{D}^{\mathrm{dir}}_{N,p,q}} = 2\,(q-1) + p\left[2\,(N-p-q-1) + 2^{2+p+q-N}\right] + (N-p-q)\,(N-p-q+1)\,. \quad (6.4)$$

Conversely, in $\mathcal{D}^{\mathrm{dir}}_{N,p+1,q-1}$, the effective resistance between node 1 and any of the nodes in the bouquet of size $q-1$ is $2\,(N-p-q-1)+2^{2+p+q-N}$, while the effective resistance between node 1 and any of the remaining p nodes in the bouquet of size $p+1$ is 2. This time, the path lengths between node 1 and the internal nodes take on the values

$$r(n,1) = 2(n-1) + 2^{2-n}$$

for all integer values of n from 0 to $N-p-q-1$. Therefore, by using (B.26) and (B.28), we obtain

$$\left(\sum_{j=2}^{N} r_{1,j}\right)_{\mathcal{D}^{\mathrm{dir}}_{N,p+1,q-1}} = 2p + (q-1)\left[2\,(N-p-q-1) + 2^{2+p+q-N}\right] + (N-p-q)\,(N-p-q-3) + 8 - 2^{3+p+q-N}. \quad (6.5)$$

Taking the difference between (6.4) and (6.5) gives us

$$K_f\left(\mathcal{D}^{\mathrm{dir}}_{N,p,q}\right) - K_f\left(\mathcal{D}^{\mathrm{dir}}_{N,p+1,q-1}\right) = (p-q+3)\left[2\,(N-p-q) - 4 + 2^{2+p+q-N}\right]. \quad (6.6)$$

Now, we know that $N-p-q \geq 2$ by assumption, and so $2(N-p-q)-4+2^{2+p+q-N}$ must be strictly positive. Hence, by (4.11), the result holds. □

Lemma 6.6 implies that moving a node from the "lower" bouquet (i.e. the one not attached to the root) of a directed double palm tree to the "upper" bouquet will improve the $\mathcal{H}_2$ norm if $q < p+3$. Conversely, it also implies that moving a node from the upper bouquet to the lower one will improve the $\mathcal{H}_2$ norm if $q > p+1$. Thus if $q = p+2$, the $\mathcal{H}_2$ norm can be improved by moving a node in either direction. It is interesting to note that the robustness of an undirected double palm tree could always be improved by moving a node from the smaller bouquet to the larger one. This principle still holds for directed double palm trees if we treat the bouquet attached to the root as if it always contains two additional nodes.

We can now derive a result for directed trees that is similar to Lemma 6.3.

Lemma 6.7. *Suppose $N \geq 4$ and $2 \leq \hat{d} \leq N-2$. If $\mathcal{T} \in \mathcal{C}^{dir}_{N,\hat{d}}$, then $\eta(\mathcal{T}) \geq \eta\left(\mathcal{P}^{dir}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}\right)$, with equality if and only if $\mathcal{T} = \mathcal{P}^{dir}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}$.*

Proof. Since $\hat{d} \leq N-2$ and $\mathcal{T} \in \mathcal{C}_{N,\hat{d}}$, a longest path in $\mathcal{T}$ contains $N-\hat{d}-1 \geq 1$ leaves attached to internal nodes (other than the leaf in the longest path). Suppose that $\mathcal{P}_{\mathcal{T}}$ is a longest directed path subgraph of $\mathcal{T}$, and suppose that the nodes in $\mathcal{P}_{\mathcal{T}}$ are labelled from 1 to $\hat{d}+1$ in the order they appear along the path, starting with the root. For the rest of this proof, when we refer to leaf nodes and bouquets, we mean leaves not part of $\mathcal{P}_{\mathcal{T}}$, and bouquets made up of these leaves.

Let i be the closest node in $\mathcal{P}_{\mathcal{T}}$ to the root that has a bouquet attached in $\mathcal{T}$, and suppose that

$$i < \left\lfloor \frac{\hat{d}+1}{2} \right\rfloor. \tag{6.7}$$

Now, let $\mathcal{T}'$ be the tree formed by taking the bouquet attached to node i and moving it to node $i+1$. Then by Theorem 5.1 the effective resistances between a node in this bouquet and nodes $1,\ldots,i$ will all increase by 2. However, by (5.43), the

effective resistance between a node in this bouquet and node $j \in \left\{i+2, \ldots, \hat{d}+1\right\}$ will decrease by $2\left(1-2^{1+i-j}\right)$, while the effective resistance between a node in the bouquet and node $i+1$ will be unchanged. Thus the sum of the effective resistances between a node in the bouquet and all nodes in $\mathcal{P}_{\mathcal{T}}$ will change by

$$\begin{aligned} 2i - 2\sum_{j=i+2}^{\hat{d}+1}\left(1-2^{1+i-j}\right) &= 2i - 2\left(\hat{d}-i\right) + 2\sum_{k=1}^{\hat{d}-i} 2^{-k} \\ &= 2\left(2i - \hat{d} + 1 - 2^{i-\hat{d}}\right) \text{ (by (B.28))}, \end{aligned}$$

which is negative since (6.7) implies $2i \leq \hat{d}-1$. In addition, the effective resistance between nodes in this bouquet and any other leaf nodes will decrease since these leaves are all attached to $\mathcal{P}_{\mathcal{T}}$ at a node further from the root than node i. Thus

$$K_f\left(\mathcal{T}'\right) < K_f\left(\mathcal{T}\right).$$

Alternatively, let i be the furthest node in $\mathcal{P}_{\mathcal{T}}$ from the root that has a bouquet attached in $\mathcal{T}$, and suppose that

$$\hat{d}+1 > i > \left\lfloor \frac{\hat{d}+1}{2} \right\rfloor. \tag{6.8}$$

Now, let $\mathcal{T}'$ be the tree formed by taking the bouquet attached to node i and moving it to node $i-1$. Then by Theorem 5.1 the effective resistances between a node in this bouquet and nodes $1, \ldots, i-1$ will all decrease by 2. However, by (5.43), the effective resistance between a node in this bouquet and node $j \in \left\{i+1, \ldots, \hat{d}+1\right\}$ will increase by $2\left(1-2^{i-j}\right)$, while the effective resistance between a node in the bouquet and node i will be unchanged. Thus the sum of the effective resistances

between a node in the bouquet and all nodes in $\mathcal{P}_{\mathcal{T}}$ will change by

$$-2(i-1)+2\sum_{j=i+1}^{\hat{d}+1}\left(1-2^{i-j}\right)=-2i+2+2\left(\hat{d}-i+1\right)-2\sum_{k=1}^{\hat{d}-i+1}2^{-k}$$
$$=2\left(\hat{d}-2i+1+2^{i-\hat{d}-1}\right)\ \text{(by (B.28))},$$

which is negative since (6.8) implies $2i \geq \hat{d}+2$ and $2^{i-\hat{d}-1} < 1$. In addition, the effective resistance between nodes in this bouquet and any other leaf nodes will decrease since these leaves are all attached to $\mathcal{P}_{\mathcal{T}}$ at a node closer to the root than node i. Thus

$$K_f(\mathcal{T}') < K_f(\mathcal{T}).$$

The only case where neither of the two above assumptions can hold is when the only node on $\mathcal{P}_{\mathcal{T}}$ with a bouquet attached is node $i = \left\lfloor \frac{\hat{d}+1}{2} \right\rfloor$, that is, when $\mathcal{T} = \mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$. Thus, by (4.11), the result holds. □

In order to analyse non-caterpillar directed trees, we must gauge the effects of changing path lengths by 1 in arbitrary connections in directed trees. Based on (5.43), we make the following conjecture.

Conjecture 6.1. *Let $r(n,m)$ denote the effective resistance between two nodes in a directed tree with unit edge weights that are connected to their closest mutually reachable node by paths of length $n \geq 0$ and $m \geq 0$, as given by Theorem 5.3. Then*

$$r(n+1,n+1) = r(n,n+1), \tag{6.9}$$

$$0 < r(n+1,m) - r(n,m) \leq 2 \text{ if } m < n+1, \tag{6.10}$$

$$-2 < r(n+1,m) - r(n,m) < 0 \text{ if } m > n+1, \text{ and} \tag{6.11}$$

$$r(n+1,m+1) - r(n,m) > 0. \tag{6.12}$$

Justification. The upper bound in (6.10) and the lower bound in (6.11) follow from the observation made in (5.45) about the "excess" term $e(m, d)$, as defined in (5.44). By the same argument, the lower bound in (6.10) will hold for $n \gg m$ and the upper bound in (6.11) will hold for $m \gg n$. (6.9) occupies a "middle ground" between (6.10) and (6.11). We can verify these inequalities numerically in other cases.

(6.12) is equivalent to the statement that $e(m + 1, d) > e(m, d)$. We can verify this numerically.

Note that by Theorem 5.3, for $m = 0$ (6.10) and (6.12) reduce to

$$0 < 2 \leq 2, \text{ and}$$

$$2^{1-n} > 0,$$

which clearly hold. Similarly, for $m = 1$ (6.10) and (6.12) (with $n > 0$) reduce to

$$0 < 2\left(1 - 2^{-n}\right) \leq 2, \text{ and}$$

$$n2^{-n} > 0,$$

which are also both clearly true. □

(6.9), (6.10) and (6.11) together imply that increasing the length of the longer path in an indirect connection will increase the resistance between the two leaves by somewhere between 0 and 2. If the two paths in an indirect connection have equal lengths, then decreasing the length of either path by 1 will not change the resistance between the two leaves. Finally, (6.12) implies that simultaneously moving two nodes closer to their mutually reachable node will decrease the resistance between them. Assuming that Conjecture 6.1 holds, we can continue our ordering of directed trees.

Lemma 6.8 (Dependent on Conjecture 6.1)**.** *Suppose that $N \geq 5$ and $2 \leq \hat{d} \leq N-3$. Let $\mathcal{T}$ be a tree in $\mathcal{T}_{N,\hat{d}}^{dir} \setminus \mathcal{C}_{N,\hat{d}}^{dir}$. Then there is another tree in $\mathcal{T}_{N,\hat{d}}^{dir}$ with smaller $\mathcal{H}_2$ norm than $\mathcal{T}$.*

Proof. Let $\mathcal{P}_{\mathcal{T}}$ be a longest directed path subgraph of $\mathcal{T}$ (of length $\hat{d}$), and let the nodes in $\mathcal{P}_{\mathcal{T}}$ be labelled from 1 to $\hat{d}+1$ in the order they appear along the path, starting with the root.

Let i be the closest node in $\mathcal{P}_{\mathcal{T}}$ to the root that has multiple children attached in $\mathcal{T}$, and suppose that

$$i < \left\lfloor \frac{\hat{d}+1}{2} \right\rfloor.$$

If node i only has a bouquet attached, let $\mathcal{T}'$ be the tree formed by taking the bouquet attached to node i and moving it to node $i+1$. Then by the proof of Lemma 6.7, $K_f(\mathcal{T}') < K_f(\mathcal{T})$. Alternatively, node i must be a branch node with a child (say node j) that is not a leaf. Then let $\mathcal{T}'$ be the tree formed by taking all children of node j in $\mathcal{T}$ and attaching them to node $i+1$. Then the effective resistance is unchanged between any node in the branch of $\mathcal{T}$ below j and any node along $\mathcal{P}_{\mathcal{T}}$ up to node i, or any leaf node attached to node i. By (5.43), the effective resistance between any node in the branch of $\mathcal{T}$ below j and node j will increase by the same amount that the effective resistance will decrease between that node and node $i+1$. However, by (6.12), the effective resistance will decrease between any node in this branch and any node that is connected to $\mathcal{P}_{\mathcal{T}}$ somewhere below node i. Thus $K_f(\mathcal{T}') < K_f(\mathcal{T})$.

Alternatively, suppose that $\ell_{\max} > \left\lfloor \frac{\hat{d}+1}{2} \right\rfloor$ is the length of a longest path in $\mathcal{T}$ other than $\mathcal{P}_{\mathcal{T}}$ and let node i be the parent of the leaf of this path. Note that all the children of node i must be leaves, since otherwise there would be a longer path in $\mathcal{T}$. Let $\mathcal{T}'$ be the tree formed by taking the bouquet attached to node i and moving it to the parent of node i. Then, since every node not part of $\mathcal{P}_{\mathcal{T}}$ must be no further from the root than the children of node i, (6.10) and (6.9) imply that the effective

resistance will not increase between any node in this bouquet and any node not in $\mathcal{P}_{\mathcal{T}}$. By the same argument, the effective resistance will not increase between any node in this bouquet and any node in $\mathcal{P}_{\mathcal{T}}$ from 1 to $\ell_{\max}$. In addition, the effective resistances between any node in this bouquet and the $\ell_{\max}-1$ nodes between node i and node 1 will all decrease by 2. Conversely, by (6.11) the effective resistances between any node in this bouquet and the $\hat{d}-\ell_{\max}$ nodes at the end of $\mathcal{P}_{\mathcal{T}}$ will all increase by somewhere between 0 and 2. Thus by the assumption on $\ell_{\max}$, we know that $K_f(\mathcal{T}') < K_f(\mathcal{T})$.

If neither of the above assumptions hold, then $\mathcal{T}$ contains no nodes attached to $\mathcal{P}_{\mathcal{T}}$ above node $\left\lfloor \frac{\hat{d}+1}{2} \right\rfloor$, and no nodes not on $\mathcal{P}_{\mathcal{T}}$ at a greater depth than this node. The only tree which satisfies these assumptions is $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor \frac{\hat{d}}{2} \rfloor}$, which is a caterpillar. Then the result follows by (4.11). □

In a similar way to §6.3, Lemmas 6.7 and 6.8 allow us to identify the tree in $\mathcal{T}^{\text{dir}}_{N,\hat{d}}$ with smallest $\mathcal{H}_2$ norm.

Proposition 6.2 (Dependent on Conjecture 6.1)**.** *Let $N \geq 3$ and $1 \leq \hat{d} \leq N-2$. The tree in $\mathcal{T}^{dir}_{N,\hat{d}}$ with smallest $\mathcal{H}_2$ norm is $\mathcal{P}^{dir}_{N,\hat{d},\lfloor \frac{\hat{d}+1}{2} \rfloor}$.*

Proof. For $\hat{d}=1$, $\mathcal{T}^{\text{dir}}_{N,\hat{d}}$ only contains $\mathcal{S}^{\text{dir}}_N$, which is the same as $\mathcal{P}^{\text{dir}}_{N,1,1}$. For $\hat{d}=N-2$, every tree in $\mathcal{T}^{\text{dir}}_{N,\hat{d}}$ is of the form $\mathcal{P}^{\text{dir}}_{N,\hat{d},i}$, and so the result follows from Lemma 6.7. For $2 \leq \hat{d} \leq N-3$, this is a simple consequence of Lemmas 6.7 and 6.8. □

Finally, we can obtain an equivalent result to Theorem 6.5.

Proposition 6.3 (Dependent on Conjecture 6.1)**.** *Let $2 \leq \hat{d} \leq N-1$. For any tree in $\mathcal{T}^{dir}_{N,\hat{d}}$, there is a tree in $\mathcal{T}^{dir}_{N,\hat{d}-1}$ with a smaller $\mathcal{H}_2$ norm. Hence, the star $\mathcal{S}^{dir}_N$ has the smallest $\mathcal{H}_2$ norm of any tree with N nodes.*

Proof. Let $\mathcal{T}$ be a tree in $\mathcal{T}^{\text{dir}}_{N,\hat{d}}$. Then by Proposition 6.2, $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor \frac{\hat{d}+1}{2} \rfloor}$ has a $\mathcal{H}_2$ norm that is less than $\eta(\mathcal{T})$ (or equal to $\eta(\mathcal{T})$ if $\mathcal{T} = \mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor \frac{\hat{d}+1}{2} \rfloor}$).

Let $\mathcal{T}'$ be the tree formed by taking the leaf node of the longest path in $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}$ and moving it to the parent of its parent in $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}$. Then the effective resistance between this node and every node in $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}$ other than its original parent will decrease, by (5.43). Conversely, the resistance between this node and its original parent will be unchanged. Since the leaves in the bouquet attached to the middle of $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}$ have depth $\lfloor\frac{\hat{d}+1}{2}\rfloor < \hat{d}$, $\mathcal{T}'$ will have a depth of $\hat{d}-1$ and also (by (4.11)) a smaller $\mathcal{H}_2$ norm than both $\mathcal{P}^{\text{dir}}_{N,\hat{d},\lfloor\frac{\hat{d}+1}{2}\rfloor}$ and $\mathcal{T}$.

Since the star is the only directed tree with a depth of 1, it must have the smallest $\mathcal{H}_2$ norm of any tree on N nodes. □

6.6 A Decentralised Algorithm For Rearranging Directed Trees

In much the same way as §6.3, we were able to derive the results in §6.5 by determining the effect of moving individual nodes on effective resistances. In particular, we showed that for a non-star tree, the $\mathcal{H}_2$ norm can always be reduced either by moving a single node to somewhere else in the tree, or by moving a group of sibling nodes to a nearby node. These manipulations are still "local", as was the case for undirected trees.

Designing a decentralised algorithm to rearrange directed trees poses something of a problem, since we are forced to (at least temporarily) break either the directedness of the communication or the tree structure of the graph. If we were to completely respect the direction of all edges, then each node would only be able to observe its parent and could only transmit information to its children. Since the root node would know it had no parent, every node could transmit an estimate of their depth in the tree and update their value when they learned their parent's depth. Thus each node in the tree could learn their depth, but no other structural information would be available. Due to this, the design of a decentralised rearrangement algorithm for

directed trees must consider not only how much information to share, but how to do so in a way that respects the directed nature of the network as much as possible. In order to proceed, therefore, we will assume that each directed link in the graph permits a limited amount of communication in both directions for a small fraction of the time.

An alternative to our approach could be to consider the directed tree to be a spanning tree of a strongly connected digraph. In this case, it is possible to transmit information throughout the network, while maintaining the directed tree for consensus behaviour.

If we assume that limited bidirectional communication is possible, then it is plausible that every node could learn the number of its children (i.e., its in-degree) and (eventually) the maximum depth of any node in the tree. Although this last piece of information is a piece of global knowledge and would require some time to learn, we saw in §6.5 that knowing the relative depth of nodes in a directed tree is critical to ensuring that robustness is improved by any rearrangement. In addition, we assume that (due to the original directed tree graph), every node can learn its own depth and its parent's location. As in §6.4, we assume that all edge weights in the graph are 1 and that the algorithm proceeds asynchronously.

Algorithm 6.2. *1. Each agent learns its in-degree and the maximum depth of any node in the tree, then proceeds to step 2.*

2. If the depth of a non-root agent is one less than the depth of the tree, and the in-degree of the agent is $n > 0$, then it contacts its parent (when possible) and informs it to expect n new children. Following this, it broadcasts to its children an instruction to connect to its parent at a given location. Otherwise, the agent should wait for new information from its parent or children and then proceed to step 3.

3. *If an agent has been instructed to make a new connection, it breaks its current connection, changes its location (if necessary) and forms a connection with its new parent by observing its parent and, when possible, informing the parent of the new connection. Otherwise, proceed to step 4.*

4. *If an agent is expecting new children, it maintains its current location until all expected connections have formed. Any agent not expecting new connections can return to step 1.*

Once again, we must now verify that Algorithm 6.2 cannot give rise to conflicting instructions or lead to the graph becoming disconnected (except temporarily, as connections are rearranged). As before, the only step in the algorithm that results in changes to the graph is step 3, which (by step 2) will only be undertaken upon instruction by a parent. Since each node has no more than one parent, it can receive at most one instruction to form a new connection at a time. Furthermore, since the only nodes that receive instructions to move are at a depth equal to the depth of the tree, they must all be leaf nodes. Therefore, the only edges in the graph that become broken are those between a leaf and an internal node, and so any disconnections that occur will break the graph into a tree and one or more isolated nodes. Then, by the instructions sent in step 2, the isolated nodes will know where to find their target node to reconnect to the graph, and the target node will not change its location until all isolated nodes seeking it have found it.

In addition, since one edge is formed for every one broken, the graph will always remain a directed tree.

Finally, we observe that any agent that satisfies the conditions of step 2 will subsequently become a leaf node. In this case the parent of this node will wait for all of its new connections in step 4 as well as the current value of the depth of the tree in step 1 before it can issue any instructions to this new leaf node.

As above, a potential flaw in implementing Algorithm 6.2 in a physical network is that a leaf node could experience a failure during step 3 and be unable to reconnect to the network. In this case, its target node would wait indefinitely for an additional child. Again, a solution to this issue could be to include a time limit in step 4, and assume that any nodes which do not connect in that time have been permanently lost.

We are now in a position to determine the effect of Algorithm 6.2 on the $\mathcal{H}_2$ norm of the tree.

Proposition 6.4 (Dependent on Conjecture 6.1)**.** *Suppose a network of agents that are connected by a directed tree graph $\mathcal{T}$ carries out Algorithm 6.2. If this causes the graph to change to the directed tree $\mathcal{T}'$, then*

$$\eta\left(\mathcal{T}'\right) < \eta\left(\mathcal{T}\right).$$

Proof. Due to the asynchronous nature of Algorithm 6.2 and the discussion above, we can assume that the only difference between $\mathcal{T}$ and $\mathcal{T}'$ is the result of a single node in $\mathcal{T}$ issuing instructions according to step 2 in the algorithm. Suppose that in $\mathcal{T}$ this node is labelled as node 1, its children are nodes $2, \ldots, n+1$ and its parent is node $n+2$. Note that this implies that $N > n+1$.

By step 2 of Algorithm 6.2, $\mathcal{T}'$ is formed from $\mathcal{T}$ by nodes $2, \ldots, n+1$ changing their parent from node 1 to node $n+2$. Suppose that the depth of $\mathcal{T}$ is $\hat{d}$ (and so this is also the depth of nodes $2, \ldots, n+1$ in $\mathcal{T}$). Since every node in $\mathcal{T}$ has a depth no greater than $\hat{d}$, we can say by (6.10) and (6.9) that for $i = 2, \ldots, n+1$, $r_{i,j}$ decreases for any node $j \neq 1$ at a depth lower than $\hat{d}$ and remains the same for any node j at a depth of $\hat{d}$. By (5.43), the effective resistance between node 1 and any of the nodes $2, \ldots, n+1$ remains unchanged. All other effective resistances in the graph are also unchanged. Since there is at least 1 node with a depth less than $\hat{d}-1$ (i.e., the

parent of node 1), the sum of all effective resistances in $\mathcal{T}'$ is less than the sum of all effective resistances in $\mathcal{T}$. The result follows by (4.11). □

Corollary 6.2 (Dependent on Conjecture 6.1)**.** *If Algorithm 6.2 is implemented on a directed tree of finite size, that tree will become a directed star after a finite number of steps.*

Proof. By Proposition 6.4, any changes that occur as a result of Algorithm 6.2 will strictly decrease the $\mathcal{H}_2$ norm of the graph, and hence produce a tree that has not arisen before. Since the number of trees with N nodes is finite, Algorithm 6.2 must converge in a finite number of steps.

The only time when Algorithm 6.2 will not lead to changes in the graph is when there are no nodes in the tree that satisfy the conditions of step 2. That is, if $\hat{d}$ is the depth of the tree, then any node at depth $\hat{d}-1$ must be the root, and so $\hat{d}=1$. Since the only directed tree with a depth of 1 is the directed star, Algorithm 6.2 must converge to $\mathcal{S}_N^{\text{dir}}$. □

From Proposition 6.4 and Corollary 6.2, we see that Algorithm 6.2 is a decentralised algorithm that will monotonically improve the $\mathcal{H}_2$ norm of a directed tree until it reaches its lowest possible value. However, convergence will happen very slowly since each iteration of the algorithm cannot proceed until the entire network has learnt the depth of the tree. It would be interesting to investigate whether it is possible to monotonically improve the robustness of a directed tree without the need for such global knowledge.

Chapter 7

Robustness of Starling Flocks

In this chapter, we investigate the connection between the number of neighbours used by each bird within a flock of starlings (*Sturnus vulgaris*) for social information and the robustness of the flock as a whole. We evaluate robustness for starling flocks using three-dimensional positions of birds studied in [4, 11] and the $\mathcal{H}_2$ norm of the interaction graph, as described in Chapter 3.

Our approach makes it possible to evaluate robustness to uncertainty over a parameterised family of hypothesised individual sensing strategies, given observations of the group. For the starling flocks we evaluate the set of strategies corresponding to each individual sensing and responding to a fixed number of closest neighbours. Since the interaction structure of each starling flock network is determined by the measured spatial distribution of the birds and the strategy that each bird uses to determine which neighbours it senses, we can apply our measure to the starling flock data to distinguish which strategy (i.e., which number of neighbours), among a parameterised family of strategies (i.e., the family parameterised by number of neighbours), minimises the influence of uncertainty on how close the birds come to consensus.

Assuming that every bird in a flock responds to a fixed number of neighbours (m) and that each interaction poses some cost in effort to the bird, we compute the

per-neighbour contribution to robustness as a function of m. The interaction cost, accounted for by the per-neighbour calculation, is associated with the understanding that achieving consensus is not the only behaviour undertaken by the birds: in addition to remaining with the flock, each bird must watch for and avoid predators, seek food or a roosting site, etc. Thus, the flock must be responsive to external signals in addition to remaining cohesive, and this requires that each individual use as little effort as possible for maintaining cohesion. We show that across all flocks in the data set, interaction networks with six or seven neighbours maximise the per-neighbour contribution to robustness. These networks match the observed interactions among the same starling flocks [4, 11].

By analysing variations between different flocks, we show further that for the range of flocks observed the optimal number of neighbours (m^*) does not depend on the size of a flock (N). Instead, both the optimal number of neighbours and the peak value of robustness per neighbour depend on the shape (in particular the thickness) of the flock. The results presented in this chapter have been published in [116].

7.1 Robustness Analysis of Starling Flocks

Most models of flocking are based on consensus behaviour [26, 27, 28, 51, 59, 99]. Accordingly, we are able to apply our robustness measure to a flock carrying out linear consensus dynamics [54, 70, 75] on some quantity of interest, such as a direction of travel.

Recent work has shown that (3.1) is the minimal model consistent with experimental correlations in natural flocks of birds [11]. Even though directions are inherently nonlinear quantities, linear consensus is a good model for consensus on direction of travel when the relative differences in directions are small [70].

Given an interaction network, encoded by a graph Laplacian L, the reduced Laplacian $\overline{L}$ can be computed from (3.6) and the steady-state disagreement covariance metric Σ determined from (3.12). The steady-state dispersion is then computed from (3.19) in the case that every agent's response is corrupted by noise with intensity 1. If the noise has some intensity other than 1, the resulting $\mathcal{H}_2$ norm is simply scaled by the intensity of the noise.

The $\mathcal{H}_2$ norm depends on N since it is a distance in N-dimensional space. This dependence is removed by dividing by the square root of N, to obtain the expected disagreement due to each individual. By inverting this quantity, we obtain the "nodal $\mathcal{H}_2$ robustness" which is small (large) when individuals contribute a large (small) amount of disagreement (see Definition 3.5). The robustness is zero precisely when the graph is not connected, and the individuals are unable to reach consensus even in the absence of noise (see §3.2.1).

Our robustness measure is most suitable to our purposes: since the $\mathcal{H}_2$ norm only depends on the sensing graph, we can evaluate robustness for different sensing strategies (e.g., choice of m neighbours), provided we can construct the resulting graphs.

Previous analysis of the observed positions of starlings within large flocks (containing 440 to 2600 birds) has shown that the birds interact with approximately seven nearest neighbours, irrespective of flock density [4, 11]. Since the starling data were collected during flocking events with no apparent direct targets or threats to the birds [4], we assume that a primary goal of each bird was to remain with the flock, i.e., to maintain consensus on a direction of flight. In more complicated scenarios, such as goal-oriented behaviour, different metrics may be used to evaluate individual performance, such as an individual's average speed in the direction of the goal [23]. In addition, other robustness measures, such as the $\mathcal{H}_\infty$ norm, may be more relevant if the disturbances in the system are non-random. However, in our scenario, it is

most natural to use the nodal $\mathcal{H}_2$ robustness to obtain a measure of how well the starling flock networks managed uncertainty: first we re-construct the sensing graph by applying to the three-dimensional positions of birds the strategy in which each bird uses information from its seven closest neighbours, and then we compute the robustness metric for that graph. We can likewise compute and compare the sensing graph and robustness metric corresponding to any interaction strategy by applying it to the same position data; here we focused on the strategies in which each bird uses information from its m closest neighbours, and we examined the set with m ranging from 1 to 11.

The cost for an individual starling to sense the behaviour of each neighbour comes from sensory and neurological requirements as well as time lost for the purpose(s) of watching for predators or searching for a roosting site, etc. It is known that birds have a limited and thus costly capability for tracking multiple objects [38]. To account for these costs, which increase with increasing m, we evaluated the "nodal robustness per neighbour" for each value of m, which is computed as the nodal $\mathcal{H}_2$ robustness divided by the number of neighbours m. This allows us to identify ranges of m of increasing (decreasing) return, where the robustness per neighbour increases (decreases) with m. We define the optimum m for robustness, m^*, as the value that maximises the robustness per neighbour.

It is possible that the birds weigh information from different neighbours differently, for example, depending on their distance or how well they are sensed. To be conservative and consistent, our basic assumption is that each individual uses an unweighted average of the information from its m nearest neighbours, so $a_{i,j}$ equals $\frac{1}{m}$ when an edge is present. However, a plausible alternative is that greater weight is given to closer neighbours. To test the effects of alternative edge weighting schemes, we computed the average nodal $\mathcal{H}_2$ robustness per neighbour for two different starling flocks under three different weighting schemes (as shown in Figure 7.1). In each case,

the edge weights in the sensing graph were normalised so that the sum of all weights used by any individual bird was 1 (i.e. the out-degree of every node in the graph was 1). The first scheme was our equal weight assumption - that is,

$$a_{i,j} = \frac{1}{m}$$

when an edge was present. The second scheme took weights to be inversely proportional to the distance between birds - that is,

$$a_{i,j} \propto \frac{1}{d_{i,j}}$$

when an edge was present, where $d_{i,j}$ was the physical distance between birds i and j. The third scheme took weights to be decreasing linearly according to the ordering of the neighbours from closest to furthest, such that the $(m+1)^{\text{st}}$ neighbour had a weight of 0. Both additional weighting schemes decreased the importance of neighbours that were further away, although the order-based scheme was more "radical" since neighbours tended to be spaced closer than in a geometric progression. In every case shown in Figure 7.1, decreasing the weight given to further neighbours tended to decrease the overall robustness, but the location of the peak remained unchanged except for the order-based scheme in flock 17-06. These results suggest that a substantial variation in edge weights is required to move the peak of the robustness per neighbour curve, and furthermore that overall robustness is decreased by doing so. For these reasons, the equal-weight assumption was used in the rest of the analysis presented in this chapter.

We realise that our calculations evaluate robustness at steady-state for fixed sensing graphs, while in flocks of starlings the graph is dynamic and no steady state is reached. However, as we saw in Proposition 3.1, the steady-state assumption in our computation is only required to remove transient dependence on initial conditions.

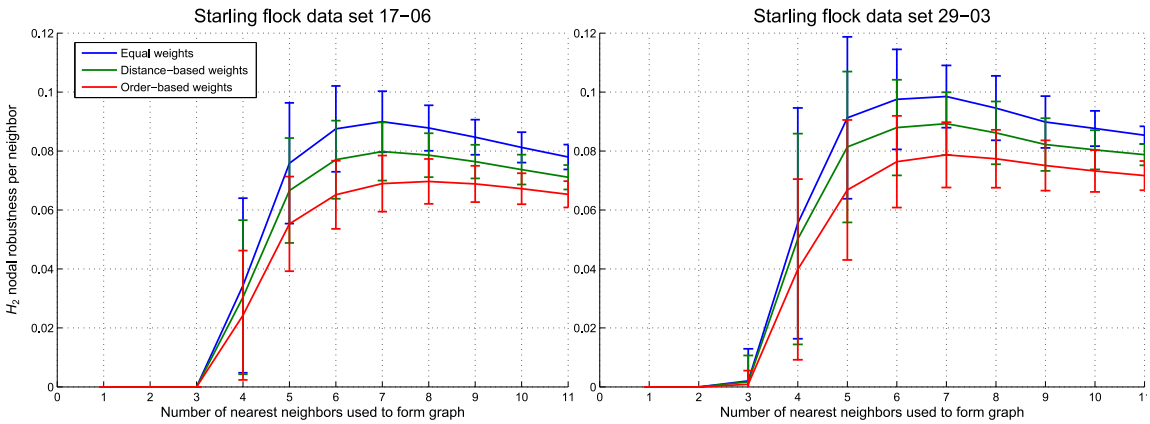

Figure 7.1: Average nodal $\mathcal{H}_2$ robustness per neighbour as a function of the number of nearest neighbours (m) used to form the graph for two different flocks, with the edge weights computed in three different ways. Although equal edge weights were used throughout this chapter, a plausible alternative is that greater weight is given to closer neighbours. The blue curves show results with equal edge weights. The green curves show results with edge weights inversely proportional to the distance between birds. The red curves show results with edge weights decreasing linearly according to the ordering of the neighbours from closest to furthest.

Hence for a group already close to consensus but with a time-varying graph, our steady-state calculation reflects the instantaneous performance of the flock.

We computed robustness per neighbour (nodal $\mathcal{H}_2$ robustness divided by m, number of neighbours sensed by each bird) for data sets from twelve starling flocks: all ten flocks that were studied in [4] and two additional flocks that were studied in [11]. From each flock there were between 16 and 80 snapshots over time, for a total of 394 snapshots. The number of birds in these flocks ranged from 440 to 2600.

7.2 Results

For small values of m, robustness will be zero since the sensing graph will not be connected [113]. Robustness per neighbour will increase with increasing m to non-zero values when m is sufficiently large for the graph to be connected. Further, robustness is bounded above by the value for the complete graph (by Proposition 3.6), and so robustness per neighbour can be expected to be a decreasing function of m for large

values of m. Thus, *a priori* we can expect a peak in robustness per neighbour as a function of m.

For all ten flocks studied in [4] and two additional flocks studied in [11], we computed the robustness per neighbour for each snapshot for $m = 1, \ldots, 11$. In every case, the graphs remained disconnected for m equal to 1 and 2, but almost all graphs were connected when m was equal to 5. Each flock attained its peak robustness per neighbour value for m between 5 and 9 (Figure 7.2), i.e., at higher values of m than were required for connectivity; this demonstrates that our robustness measure is not simply recording the onset of connectivity. The average robustness per neighbour across all flocks reached its peak value at either $m = 7$ or $m = 6$ (Figure 7.2). Therefore, the observed behaviour of the starlings ($m = 7$) from [4, 11] places them at a point that maximises the robustness per neighbour.

We further investigated observed variation in the value of m^* for different flocks. When the average robustness per neighbour was computed by averaging every snapshot from every flock, rather than by averaging the flock averages, we obtained almost identical results, as seen in Figure 7.3. Since the results of the two averages match so closely while the number of snapshots taken of each flock varied between 16 and 80, this suggested that we could treat all 394 snapshots as independent data points and strengthened the generality of the result compared to the case in which we treated the flocks as the independent observations. Furthermore, treating the snapshots as the independent observations greatly reduced the standard error of our average robustness per neighbour curve, as seen in Figure 7.4. Thus, despite the variation seen between different snapshots and different flocks, we can be confidant that on average, robustness per neighbour peaks for starling flocks at $m = 6$ or $m = 7$.

In a fully random group, the number of neighbours required for connectivity, and hence m^*, grows weakly with the size of the group (on the order of $\log N$ [113]). However, even when connectivity is attained, noise has a crucial role in determining

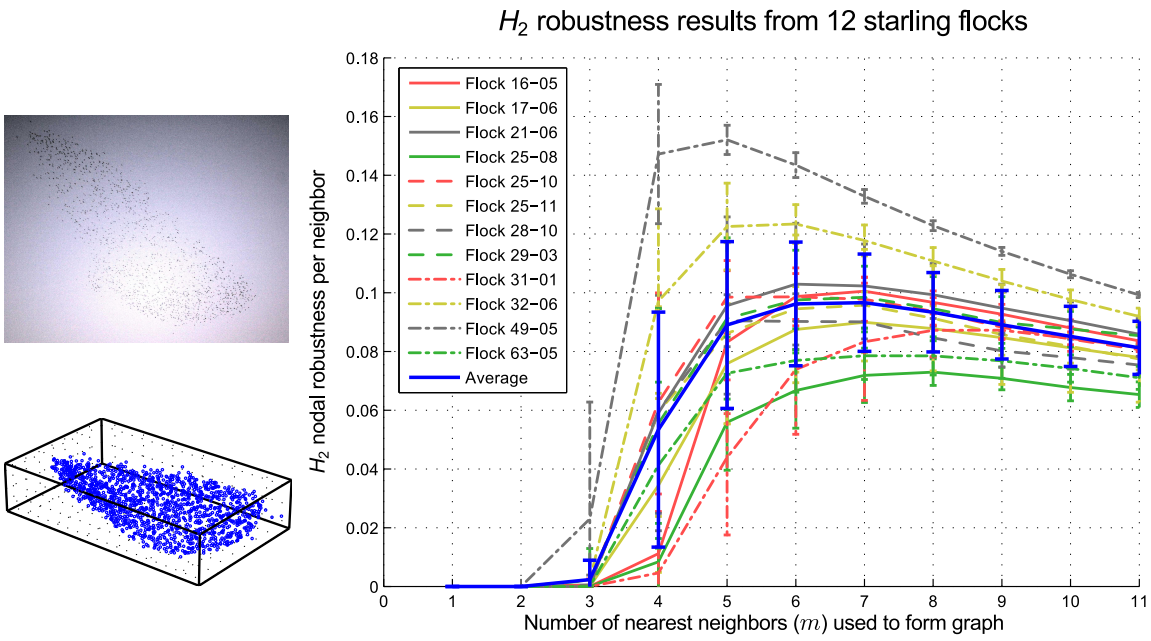

Figure 7.2: $\mathcal{H}_2$ nodal robustness per neighbour as a function of the number of nearest neighbours (m) used to form the graph, for twelve separate flocks as well as an overall average. For each flock the curve shown is the average of all snapshots taken of that flock, with error bars showing the standard deviation. The overall average, shown as the blue curve, is an average of the twelve flocks, with error bars showing the standard deviation. On the left is shown a snapshot of starling flock 25-08 in flight and the corresponding tracked positions, rotated to fit inside a rectangular bounding box.

whether or not global order can be reached. First, above a certain noise threshold (critical temperature), global order is lost, whether or not the network is connected. Second, even in the low noise phase and on a connected static network, depending on the physical dimension of space and on the topology of the network [18, 68] there are cases where order can be reached only if the number of neighbours scales with N. Given that our method is static in nature (it does not take into account birds' motion) and that the topology of flocks' network is nontrivial, the dependence of m^* on N may be a concern. However, the variation observed here in m^* was not a result of varying flock size since neither the value of m^* nor the peak robustness per neighbour showed a significant dependence on the number of birds in the flock, as shown in Figures 7.5 and 7.6. In both cases, the best linear fit to the data has

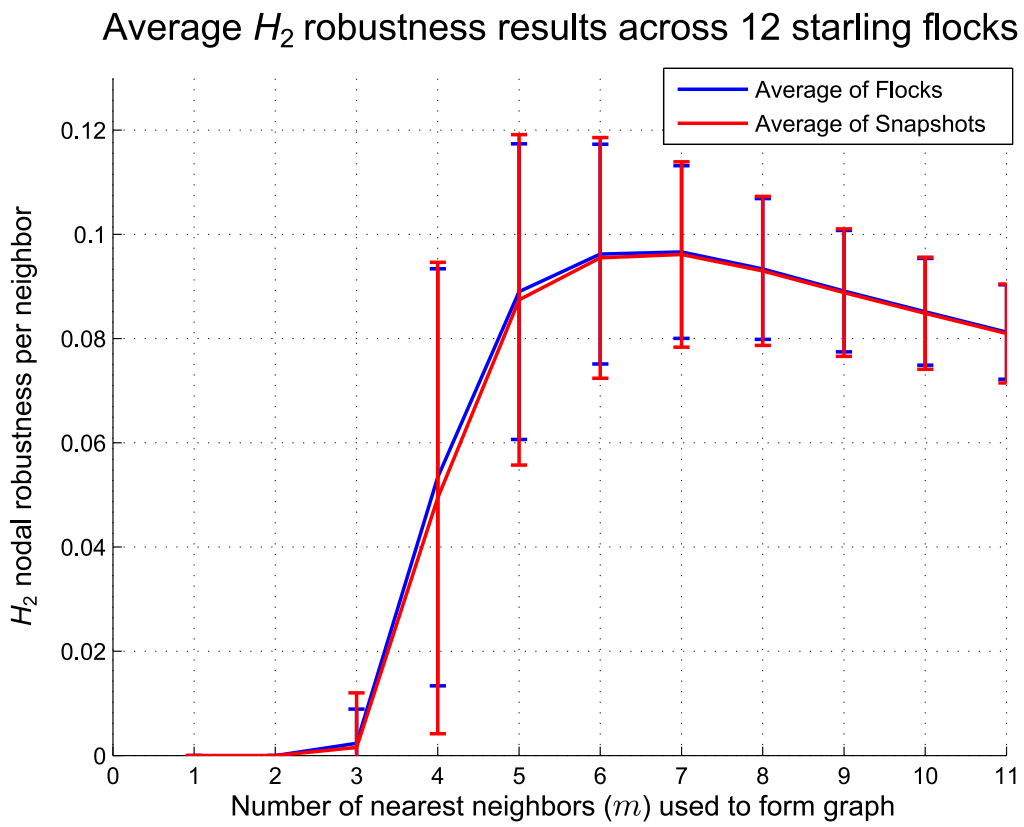

Figure 7.3: Average nodal $\mathcal{H}_2$ robustness per neighbour as a function of the number of nearest neighbours (m) used to form the graph, with the average taken in two different ways. The blue curve shows the average of the twelve flock averages (as in Figure 7.2), while the red curve shows the average of the 394 snapshots taken across all flocks. In each case, the error bars show standard deviation.

negligible slope with an R^2 value[‡] of 0.0178 in the case of m^* and 0.0230 in the case of peak robustness per neighbour.

Instead we observed a strong dependence of both m^* and peak robustness per neighbour on flock thickness, as seen in Figures 7.7 and 7.8. We measured flock thickness as the ratio of smallest to largest dimension of an ellipsoid having the same principal moments of inertia as the flock. Thus a two-dimensional flock has a thickness of 0 while a flock with an equal distribution of birds in all directions has a thickness of 1. We found that the starling flocks had thicknesses between 0.13 and 0.44, with most

[‡]In our case, R^2 measures how well a linear model fits the data, and can be thought of as the fraction of the variation in the data that is captured by the linear fit.

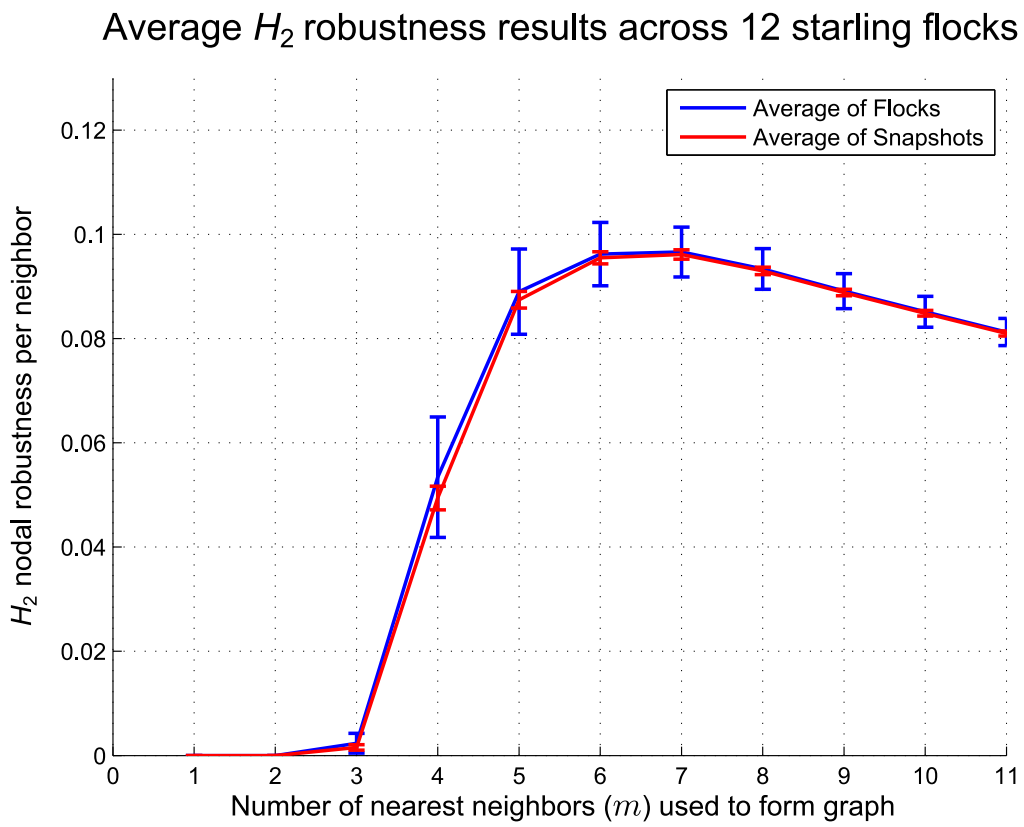

Figure 7.4: Average nodal $\mathcal{H}_2$ robustness per neighbour with standard error as a function of the number of nearest neighbours (m) used to form the graph, with the average taken in two different ways. The blue curve shows the average of the twelve flock averages (as in Figure 7.2), while the red curve shows the average of the 394 snapshots taken across all flocks. In each case, the error bars show standard error.

between 0.13 and 0.27. Across this range, both the variation in m^* and the average value of m^* decreased significantly with thickness (Figure 7.7). The best linear fit to the data displays a negative slope with an R^2 value of 0.1816, which is relatively low due to the changing variance. Furthermore, the peak robustness per neighbour increased significantly with thickness (Figure 7.8). The best linear fit to the data has a positive slope and an R^2 value of 0.6435.

To further understand the dependence on thickness, we generated random flocks of varying thicknesses (each containing 1200 individuals) within a square prism as described below. For uniformly distributed flocks, m^* initially decreased with thickness before levelling out, while peak robustness per neighbour showed an increase with

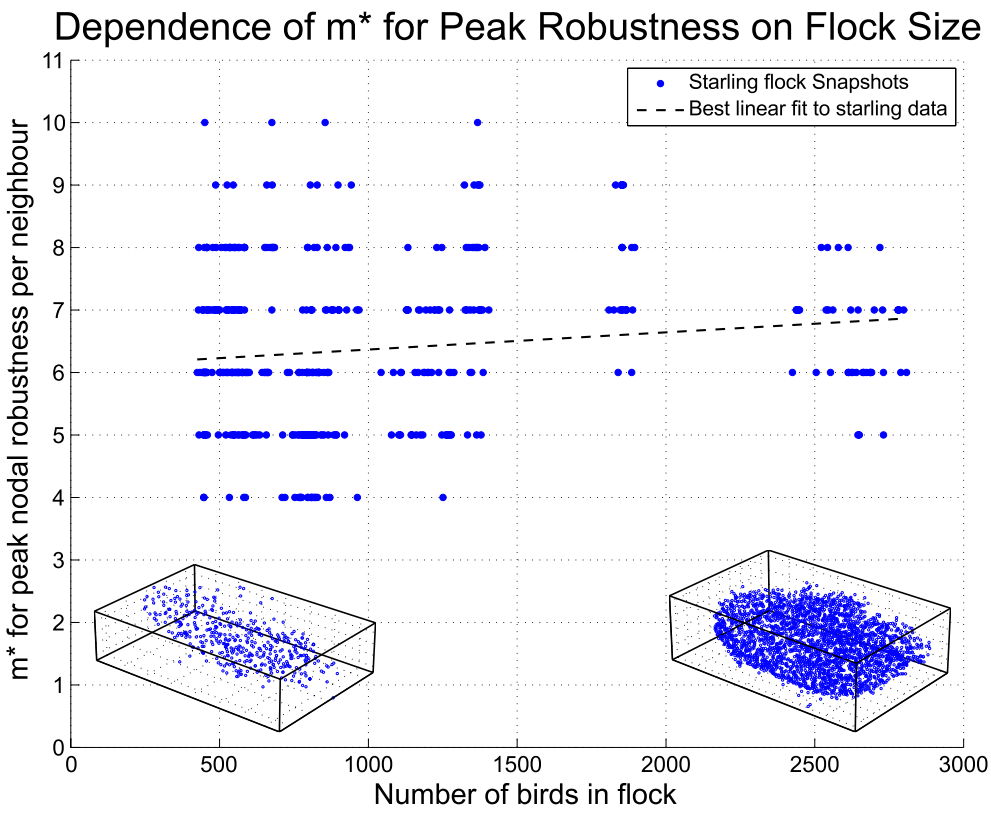

Figure 7.5: Dependence of the optimum number of neighbours (m^*) on the number of birds in the flock (N). Different snapshots from the same flock have different numbers of birds due to occlusions. Results for each snapshot are shown rather than averaged across flocks since we can take each snapshot to be an independent observation (see Figure 7.3). Under the plot are the bird positions (rotated to fit inside a rectangular bounding box) for two snapshots corresponding to the smallest and largest flocks studied.

thickness according to a sigmoidal shape, as seen in Figures 7.7 and 7.8. Starling flocks, however, cannot be uniformly distributed since (at least) two birds cannot be positioned arbitrarily close together. We therefore also generated random flocks with non-uniform, more ordered distributions. For a "medium" amount of ordering, we drew points from three-dimensional Halton sequences [49] with random starting points, while for a "large" amount of ordering we placed points in a three-dimensional grid and perturbed them by Gaussian noise. As the random flocks become more ordered, the m^* values tended to decrease and there was less of a dependence on thickness, with the most ordered flocks showing no thickness dependence in m^*, as seen in

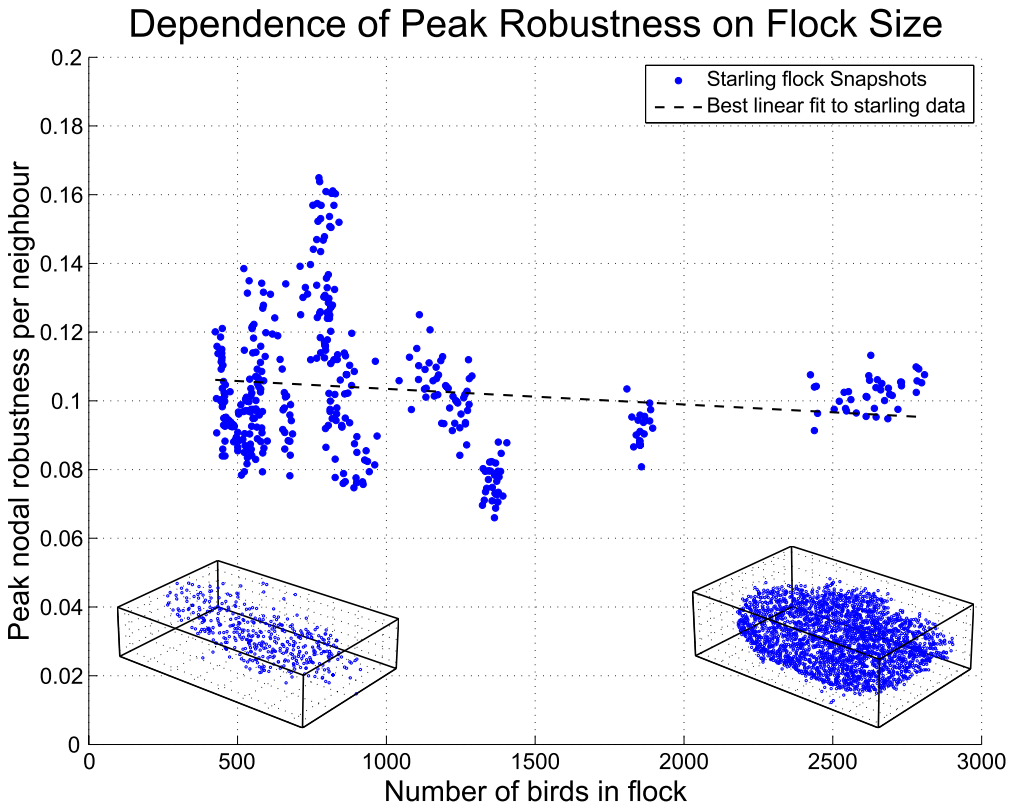

Figure 7.6: Dependence of the peak value of robustness per neighbour on the number of birds in the flock (N). Different snapshots from the same flock have different numbers of birds due to occlusions. Results for each snapshot are shown rather than averaged across flocks since we can take each snapshot to be an independent observation (see Figure 7.3). Under the plot are the bird positions (rotated to fit inside a rectangular bounding box) for two snapshots corresponding to the smallest and largest flocks studied.

Figure 7.9. Additionally, as the random flocks become more ordered, the peak values of robustness increased but the same thickness trends were apparent, i.e., the curves all showed a sigmoidal shape in the increase in peak value with increasing thickness, as seen in Figure 7.10. Compared to these three distributions, the starling flocks appear closest to uniform, with slightly more "order." This is consistent with the fact that starlings have a more regularly separated distribution than uniformly distributed points, although it suggests that a uniform random flock is not too dissimilar to an actual starling flock.

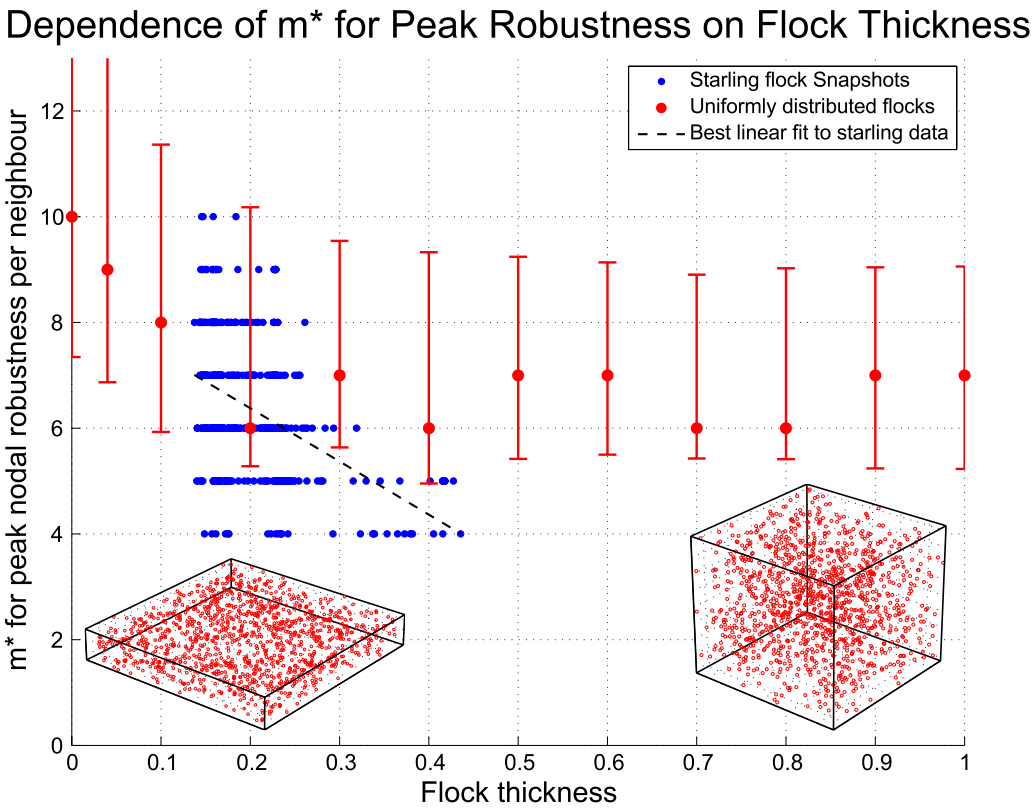

Figure 7.7: Dependence of the optimum number of neighbours (m^*) on the thickness of the flock. Results are shown in blue from each snapshot of starling data and in red from flocks randomly generated from a uniform distribution within a rectangular prism. Each data point shown from the random flocks is the average result from generating 100 separate flocks, each containing 1200 individuals. The error bars show the range of values for which the robustness per neighbour is within 90% of the peak. Under the plot are the positions of two randomly generated flocks, with thicknesses of 0.15 and 0.85.

We compared the optimum number of neighbours and peak robustness per neighbour to the width of the flock, as shown in Figure 7.11 and Figure 7.12. Flock width is defined as the ratio of intermediate to largest dimension of an ellipsoid having the same principal moments of inertia as the flock. In the case of m^*, no significant dependence on width was observed, with the best linear fit having negligible slope and an R^2 value of 0.0064. Similarly, no significant dependence of peak robustness on width was observed, with the best linear fit having a slight positive slope and an R^2 value of 0.0635.

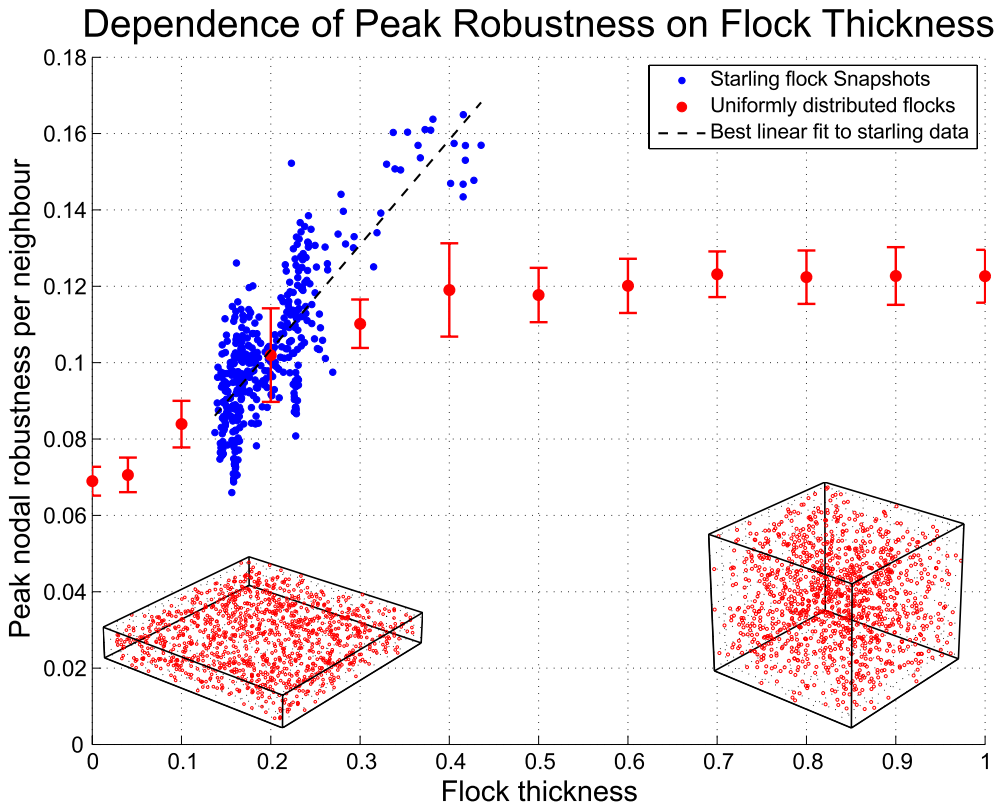

Figure 7.8: Dependence of the peak value of robustness per neighbour on the thickness of the flock. Results are shown in blue from each snapshot of starling data and in red from flocks randomly generated from a uniform distribution within a rectangular prism. Each data point shown from the random flocks is the average result from generating 100 separate flocks, each containing 1200 individuals. The error bars shown are the standard deviation. Under the plot are the positions of two randomly generated flocks, with thicknesses of 0.15 and 0.85.

7.3 Discussion

Our analysis shows that the size (seven) of each starling's neighbourhood [4, 11] optimally trades off gains from robustness with costs associated with sensing and attention; this suggests that robustness to uncertainty may have been a factor in the evolution of flocking. The fact that the same number of neighbours is optimal over a range of flock sizes and densities (as well as, to a certain extent, typical flock thicknesses) suggests that the number of neighbours that a bird interacts with could be an evolved trait. This is consistent with the fact that the ability to follow more neigh-

bours requires additional sensory and cognitive apparatus. A bird that fully utilises whatever capability it has will contribute most to maximising the absolute robustness of the group; however, our results provide an explanation for why the evolved capacity of starlings should be limited to seven neighbours. Further investigation is required to discern whether evolutionary processes could lead to the optimisation of efficient robustness at the level of the group.

The trade-off seen here between robustness and sensing cost was not observed for performance metrics related to responsiveness, such as the speed of convergence to consensus, as seen in Figure 7.13. Recall that by Lemma 3.5, the maximum possible speed for a graph with a normalised Laplacian is approximately 1 for large values of N. On average, speed per neighbour increased with m, with no maximum observed for $m < 20$. In fact, when m is equal to the number of birds in the flock, the interaction graph will be a complete graph and so by (3.41) the speed per neighbour will be $\frac{1}{N-1}$, approximately 7.7×10^{-4} for this flock. This number is significantly larger than the values for small m, implying that the speed per neighbour can be expected to continue increasing for values of m larger than those shown in Figure 7.13. Although responsiveness is an important property of group behaviour, our results correspond with the previous observation [4] that the primary benefit of the observed interaction rule within starling flocks is to improve robustness. Other aspects of behaviour, such as the way in which individuals respond to external signals, may be required for an analysis that seeks to explain the responsiveness of flocks.

Although we observed variability in our computed values of m^* across different flocks, and variability was also observed in the estimated number of interacting neighbours for each flock in [4] (called "topological range" in [4] and denoted by n_c), no correlation can be seen between these two values across flocks, as shown in Figure 7.14. The correlation coefficient between the two values was approximately -0.24, with a p-value of approximately 0.46. This is not surprising since it seems unlikely that indi-

vidual starlings would interact with more or fewer neighbours based on the thickness of the flock (or any other bulk flock parameter). In addition, this suggests that the two analyses are independent and there is no underlying mathematical reason why m^* should be so close to n_c.

Although here we have focussed on the sensing strategy of interacting with m nearest neighbours, our methods can also be applied to networks resulting from any sensing strategy. For example, our methods could be used to evaluate the robustness to noise of zonal sensing strategies like those used in [26, 51, 99] as a function of a parameter such as zone size. Provided that a real or hypothesised sensing network can be constructed, its robustness can be calculated. However, care must be taken when comparing different strategies. For consistency, the weights in the graph should be scaled so that the sum of $a_{i,j}$ over all the neighbours of any individual (when neighbours are present) is equal to 1.

The nonlinear dependence on thickness observed in the random flocks suggests that a transition between "2-d" and "3-d" behaviour takes place as thickness increases, with a flock behaving as fully 3-d when its thickness is above about 0.4. There appear to be aerodynamic reasons why starling flocks should be thin and sheet-like [51], and it is telling that the observed thicknesses lie near the transition point to fully three-dimensional behaviour in terms of robustness. This suggests that groups with different characteristic thicknesses, such as schools of fish, swarms of insects, and herds of animals (with a thickness of zero), should interact with more (fewer) neighbours if they have a smaller (larger) thickness. Testing this hypothesis would provide important insight into the generality of this work for the analysis of animal groups. It should be noted, however, that factors that were not significant for starling flocks, such as flock width (Figures 7.11 and 7.12) or distribution (Figures 7.9 and 7.10) could play a larger role in 2-d groups.

More generally, our work demonstrates the significant role of who is interacting

with whom in the ability of a network to efficiently manage uncertainty when seeking to maintain consensus. This suggests possibilities for understanding and evaluating uncertainty management in other social and technological networks. Our approach to evaluating robustness to uncertainty in consensus can be applied to interaction networks in these other contexts; distinguishing interaction strategies that yield networks that optimise robustness can be useful both for better understanding observed group behaviour and, when control is available, for designing high performing groups.

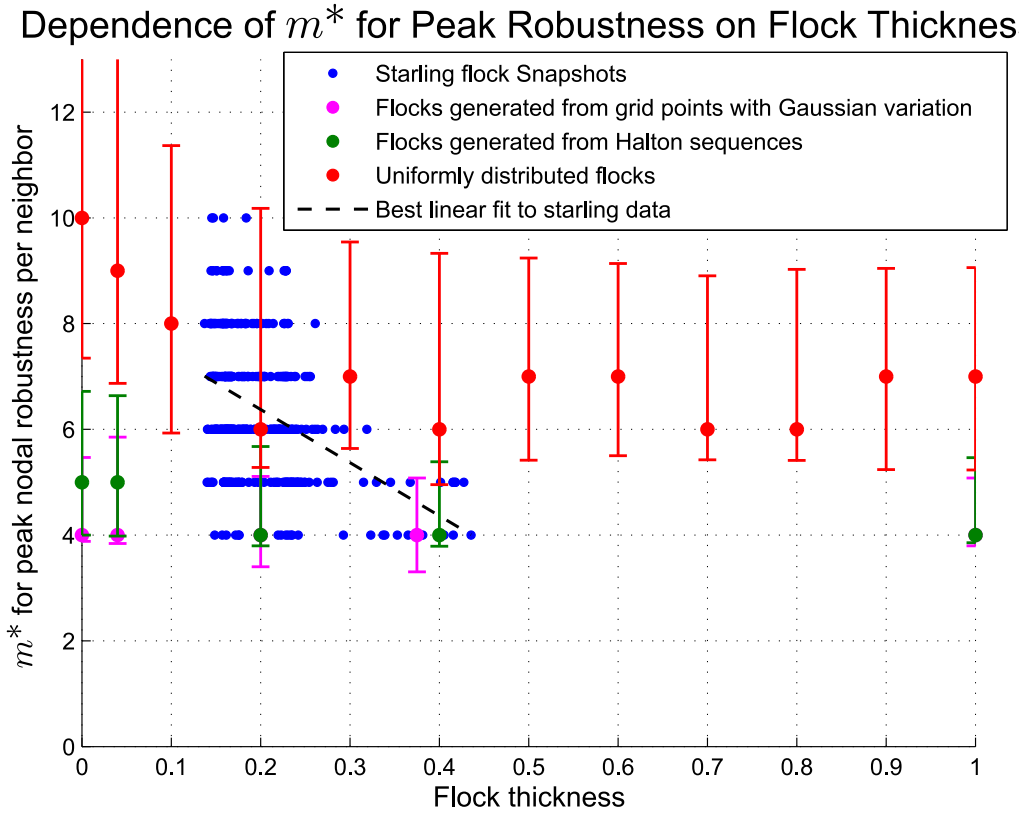

Figure 7.9: Dependence of the optimum number of neighbours (m^*) on the thickness of the flock. In addition to starling data plotted in blue, results are shown from flocks randomly generated from three different distributions within a rectangular prism. These distributions are as follows: points arranged in a grid and then perturbed with Gaussian noise (in magenta), points generated from Halton sequences (in green) and points taken from a uniform distribution (in red). Each data point shown from the random flocks is the average result from generating 100 separate flocks, each containing approximately 1200 individuals. The error bars show the range of values for which the robustness per neighbour is within 90% of the peak.

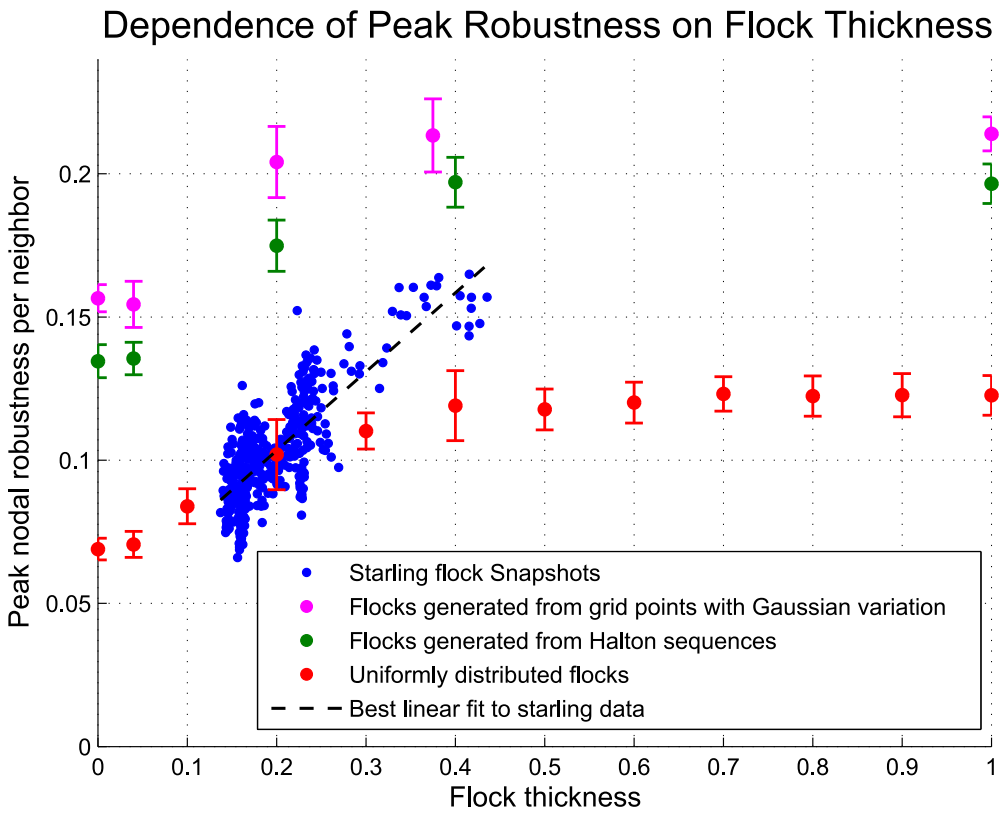

Figure 7.10: Dependence of the peak value of robustness per neighbour on the thickness of the flock. In addition to starling data plotted in blue, results are shown from flocks randomly generated from three different distributions within a rectangular prism. These distributions are as follows: points arranged in a grid and then perturbed with Gaussian noise (in magenta), points generated from Halton sequences (in green) and points taken from a uniform distribution (in red). Each data point shown from the random flocks is the average result from generating 100 separate flocks, each containing approximately 1200 individuals. The error bars show standard deviation.

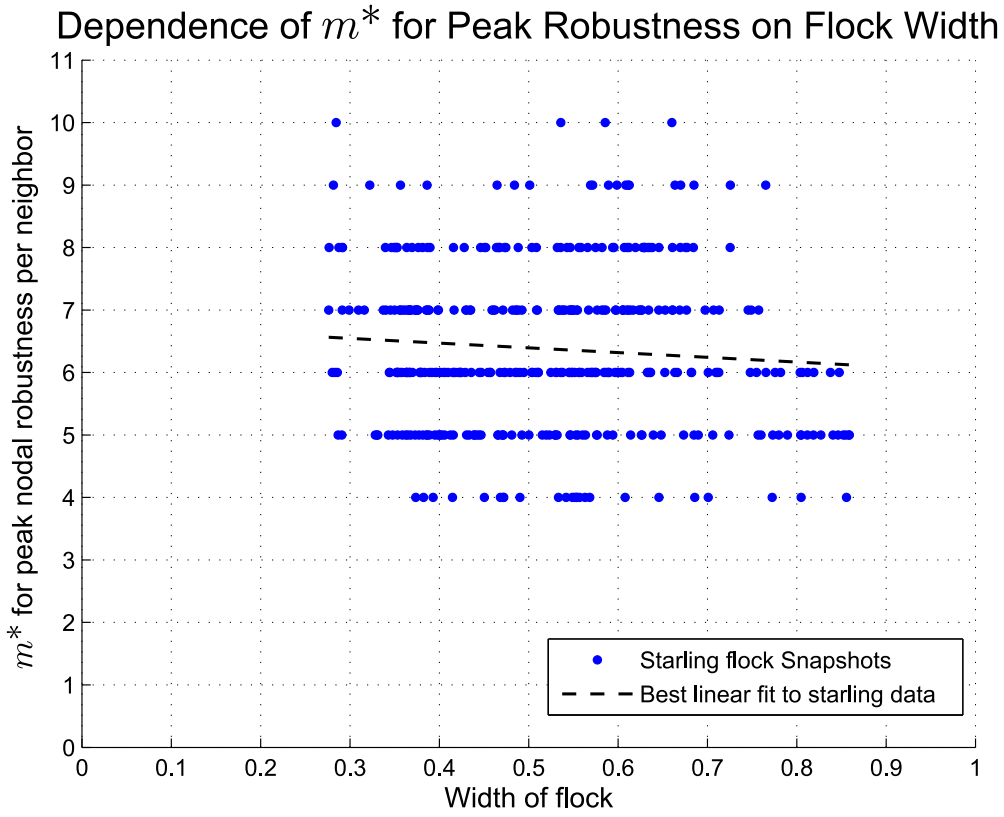

Figure 7.11: Dependence of the optimum number of neighbours (m^*) on the width of the flock for 394 snapshots of starling flocks.

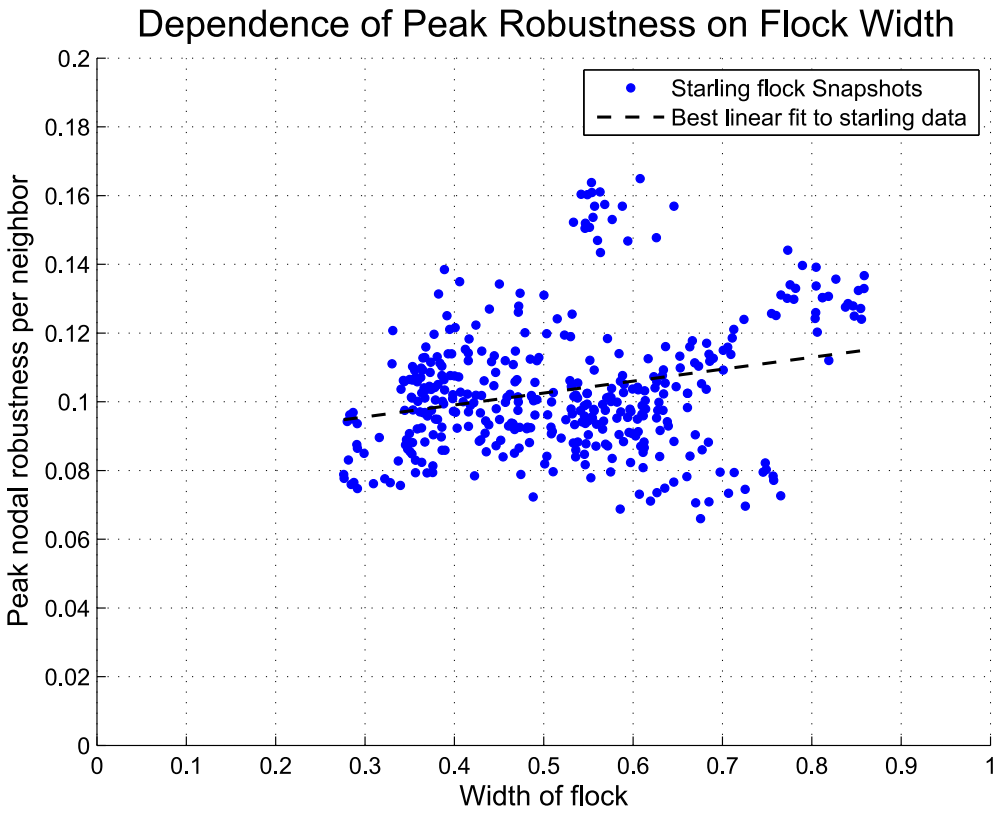

Figure 7.12: Dependence of the peak value of robustness per neighbour on the width of the flock for 394 snapshots of starling flocks.

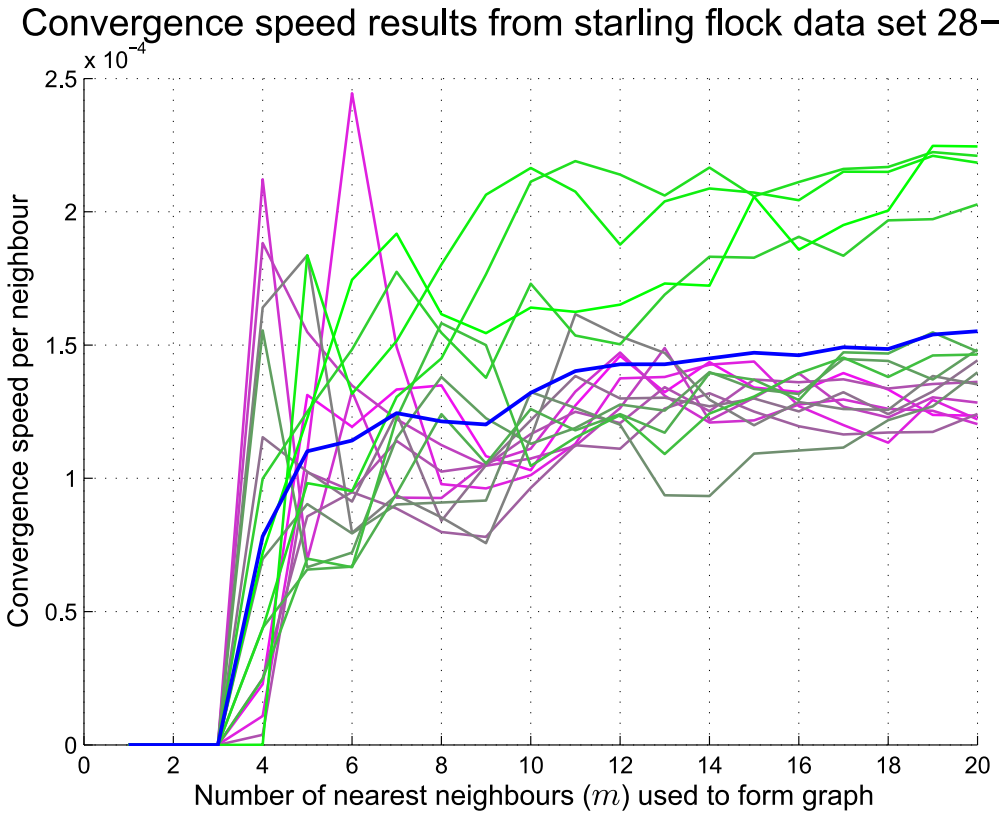

Figure 7.13: Speed of convergence to consensus per neighbour as a function of the number of nearest neighbours (m) used to form the graph, for one starling flock containing approximately 1300 birds. The thin lines show results for each snapshot, while the thick blue line shows the average over all snapshots.

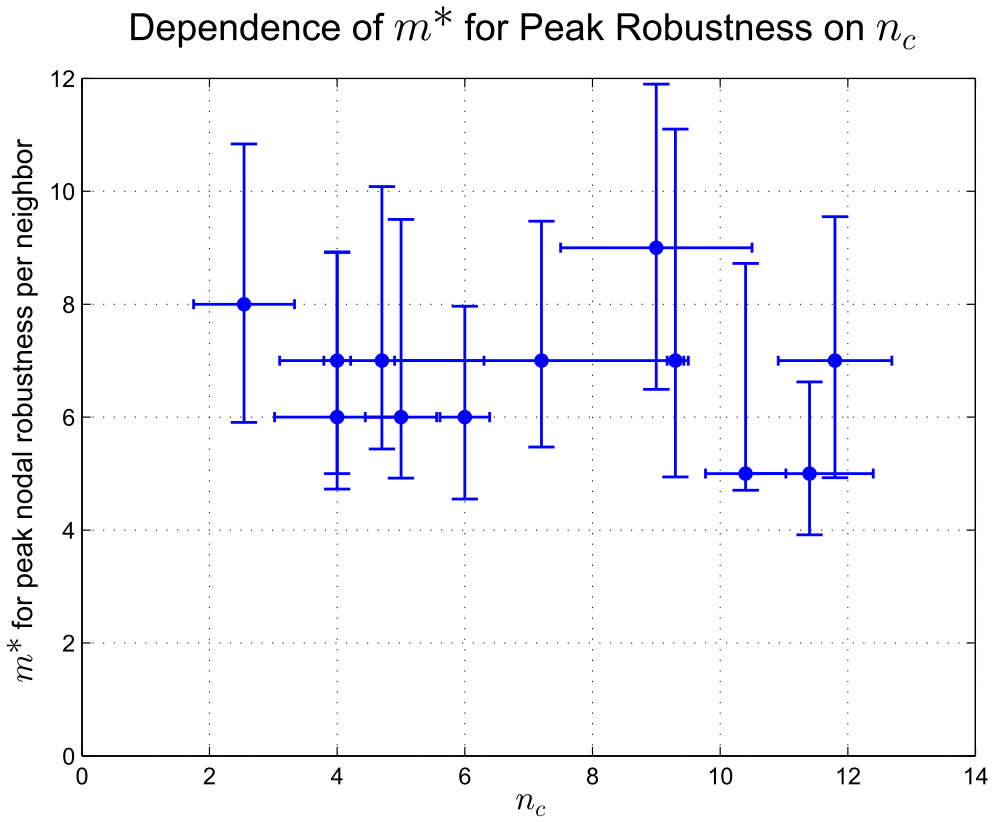

Figure 7.14: A comparison between the optimal number of neighbours (m^*) and the observed topological range (n_c) [4] for each flock studied in this chapter. The vertical error bars show the range of values for which the robustness per neighbour was within 90% of the peak and the horizontal error bars show the error in the estimates of n_c.

Chapter 8

Conclusions and Future Directions

This dissertation presented an analysis of the robustness of consensus to noise on directed networks and demonstrated some cases in which the network can be chosen to optimise its robustness. We investigated how the robustness of consensus depends on the eigenvalues of the graph Laplacian, and presented a generalisation of the notion of effective resistance to directed graphs. We used our new notion of effective resistance to explore some of the fundamental connections that occur in directed graphs. This understanding was then applied to the study of tree graphs, allowing for the development of methods to rearrange trees in ways that improve robustness. Finally, we validated our approach with an investigation of the robustness of starling flocks that demonstrated that the observed interactions between starlings optimise the tradeoff between robust group performance and individual sensing cost.

In Chapter 3, we described how the robustness of consensus to noise is naturally characterised by a $\mathcal{H}_2$ norm that can be computed from the graph Laplacian matrix. We were able to show that for a certain class of graphs, namely those with normal reduced Laplacian matrices, this $\mathcal{H}_2$ norm can be computed directly from the eigenvalues of the Laplacian. The $\mathcal{H}_2$ norm of any graph was then shown to be lower bounded by the same eigenvalue expression. For graphs with positive algebraic

connectivity, we were able to derive an upper bound on the $\mathcal{H}_2$ norm from the eigenvalues of the symmetric part of the reduced Laplacian. Finally, we examined the performance of several graph families and demonstrated that although the complete graph achieves the best possible robustness and speed of convergence, its performance can be approached by directed graphs containing many fewer edges.

Motivated by the connection between the $\mathcal{H}_2$ norm and Kirchhoff index of a graph, in Chapter 4 we presented a generalisation of the notion of effective resistance that applied to directed graphs. Although our definition was an algebraic one, we proceeded to show that it is a well-defined pairwise property of nodes in a graph, that it depends only on the connections between nodes in a graph and that its square root is a metric. Since our new notion of effective resistance has been shown to be applicable in some situations other than measuring the robustness of consensus, we are hopeful that it will prove to be useful in others as well.

In order to gain insight into the ways in which effective resistance depends on the structure of a directed graph, in Chapter 5 we computed the effective resistance of several fundamental directed graphs. We were first able to derive three sets of sufficient conditions for effective resistances in a directed graph to be equal to those in an equivalent undirected graph. The first set of conditions indicated that graphs with normal Laplacian matrices have the same effective resistances as their underlying undirected graphs, suggesting (along with the results of §3.2.2) that digraphs satisfying this normality condition behave in many essential respects as if they were undirected. We proceeded to demonstrate that the familiar rules for combining resistances in series and parallel still apply to the computation of effective resistance in directed path and cycle graphs. However, by examining the indirect connections that are only possible in directed graphs, we derived a formula for effective resistance in directed trees that has no analogue in the study of effective resistance in undirected graphs.

In Chapter 6, we used effective resistance to examine how local changes to the structure of tree graphs affects their $\mathcal{H}_2$ norms. From this, we were able to derive a partial ordering of both undirected and directed trees according to their robustness. Our ordering for directed trees was, in part, dependent on Conjecture 6.1. Although we can prove some of the bounds in this conjecture, and verify the others numerically, a complete proof is still needed. Furthermore, we were able to translate our understanding into decentralised algorithms for rearranging trees that are guaranteed to improve robustness and, without additional constraints, converge to the most robust tree graph.

Finally, in Chapter 7, we examined the robustness of interaction networks derived from the measured positions of individuals within flocks of starlings. By comparing the network performance across a family of sensing strategies (parameterised by the number of nearest neighbours each bird interacted with), we were able to demonstrate that the observed interactions between starlings optimise the tradeoff between maximising robustness and minimising sensing cost. Furthermore, we were able to show that the optimal behaviour for starling-like flocks depends most strongly on the thickness of the flock and not (over the observed conditions) on the size or width of the flock. By examining randomly-generated flocks we observed that as flock thickness increases, a transition takes place between two-dimensional and three-dimensional behaviour. Interestingly, the starling flocks tended to have thicknesses that placed them precisely within this region of transition. These results demonstrated the applicability of our approach to measuring the performance of real-world groups and also suggest that a reason for the observed interactions between starlings is the need to efficiently maintain consensus in a noisy environment.

The work presented in this dissertation was inspired by the desire to design robotic groups capable of performing sophisticated tasks in variable and uncertain environments. Although we have focused on only a small part of this larger problem, we

have made progress in understanding the role of the communication network in the group's performance and in developing strategies that each individual could follow in order to maximise the robustness of the group.

8.1 Future Directions

Although we have attempted throughout this dissertation to answer some of the open questions identified in §1.1, §1.2 and §1.3, there are many questions that remain unanswered. Here we attempt to summarise some of the topics that, in our opinion, would be particularly useful and enlightening to address.

The work presented in this dissertation focussed on static communication graphs. In most real-world multi-agent systems, the interactions will change over time, leading to a dynamic network. Thus an important generalisation of this work will be to investigate robustness of consensus to noise in dynamic networks. In particular, it has been shown that periodic motion can enhance group performance [96], and so a starting point could be the analysis of periodic networks. Furthermore, some of the graphs identified in this dissertation as being particularly robust to noise (e.g. the directed star graph, as shown in §3.3.4) are not robust to other kinds of uncertainty, such as link or node failures. In a periodic system, however, it is possible that switching between graphs that are robust to noise could maintain this robustness while adding other kinds of robustness as well. Mathematically, a periodically time-varying stochastic system is unlikely to converge to a stationary distribution, but rather to a periodic one. Thus in order to proceed in this direction, the basic definition of robustness will need to be adjusted to account for this kind of behaviour. Ultimately, a theory of general time-varying systems is desirable, although the analysis of time-varying consensus networks becomes progressively more challenging as assumptions on the graph are relaxed [70]. An alternative could be to provide a more rigorous

understanding of *instantaneous* performance in dynamic networks.

In several results in this dissertation, the assumption of normal matrices allowed directed graphs to be closely linked to their undirected counterparts. However, normality is an algebraic condition and we have obtained little insight into how the normality of the Laplacian matrix depends on the structure of the graph (other than Lemma 3.3, for which the converse does not hold), or whether a group may be able to ensure that their communication graph has a normal Laplacian. Based on bounds such as Proposition 3.4, insight of this nature could lead to an ability to maximise the performance of directed networks.

Our definition of effective resistance has proved to be useful in our study of robust consensus and in the analysis of networks of stochastic decision-makers. However, the concept of effective resistance is widely used for undirected graphs in many fields, including other areas of control theory. Therefore, it is our hope that more applications can be found for our directed graph version of effective resistance, particularly in areas where the assumption of undirected graphs is not fundamental to the problem. On the theoretical side, there is more work to be done in understanding which connections in directed graphs affect the effective resistance between node pairs and determining how the effective resistances from multiple connections combine together.

In Chapter 6 we demonstrated how effective resistance can be used to derive analytical results for improving the robustness of graphs. However, our results are incomplete. Firstly, we have not fully explored the implications of our expression for effective resistance in directed trees and secondly, we have not yet analysed any more complicated graphs. Both directions are important if we hope to derive more general design rules for robust directed networks.

Finally, the analysis of starling flocks presented in Chapter 7 offered an explanation of the observed behaviour in starlings from an optimisation perspective. However, it is unclear exactly how a robust flock translates to a benefit for each individual

within the flock, and how evolutionary pressures could drive this optimisation. From a biological perspective, this work is necessary to determine whether the robustness-cost tradeoff is the true reason for the starlings' behaviour or if it is the result of other factors. For engineers, such individual-focused work could provide a guide to strategies that could be implemented by individual robots in order to improve the performance of the entire group.

Appendix A

Eigenvalues of Laplacian Matrices

A.1 Cycle Graphs

A.1.1 Proof of Lemma 3.6

Proof. From (3.43), the Laplacian matrix of the directed cycle graph is

$$L_N^{\text{cycle}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

Now, the characteristic equation of L_N^{cycle} may be easily derived by observing that

$$\begin{aligned}
\det\left(\lambda I_N - L_N^{\text{cycle}}\right) &= \underbrace{\begin{vmatrix} \lambda-1 & 0 & \cdots & 0 & 1 \\ 1 & \lambda-1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \lambda-1 & 0 \\ 0 & 0 & \cdots & 1 & \lambda-1 \end{vmatrix}}_{N\times N} \\
&= (\lambda-1)\underbrace{\begin{vmatrix} \lambda-1 & \cdots & 0 & 0 \\ 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \lambda-1 & 0 \\ 0 & \cdots & 1 & \lambda-1 \end{vmatrix}}_{(N-1)\times(N-1)} - (-1)^N \underbrace{\begin{vmatrix} 1 & \lambda-1 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \lambda-1 \\ 0 & 0 & \cdots & 1 \end{vmatrix}}_{(N-1)\times(N-1)} \\
&= (\lambda-1)^N - (-1)^N,
\end{aligned}$$

since the two $(N-1)\times(N-1)$ submatrices on the second line are triangular. Thus the characteristic equation of L_N^{cycle} is

$$(\lambda-1)^N - (-1)^N = 0, \tag{A.1}$$

and so the eigenvalues of L_N^{cycle} are equal to 1 plus the N^{th} roots of -1. We can therefore write the solutions to (A.1) as

$$\lambda_j = 1 + \mathrm{e}^{\mathrm{i}\pi\left(1-\frac{2j}{N}\right)}, \quad j = 0, 1, \ldots, N-1. \tag{A.2}$$

Then, if we use our regular convention of labelling the eigenvalues of L_N^{cycle} in ascending order of real part, we obtain (3.44) □

A.1.2 Proof of Lemma 3.7

Proof. Let L_N^{cycle} be the Laplacian matrix of the directed cycle graph, as given by (3.43), and let $L_N^{\text{u cycle}}$ be the Laplacian matrix of the undirected cycle graph, as given by (3.45). Since L_N^{cycle} is circulant, and hence normal [46, §3.1], by the Spectral Theorem we can find a unitary matrix $U \in \mathbb{C}^{N\times N}$ such that

$$L_N^{\text{cycle}} = U\Lambda U^*,$$

where Λ is the diagonal matrix containing the eigenvalues of L_N^{cycle}. Furthermore, as L_N^{cycle} is real we know that

$$L_N^{\text{cycle}^T} = L_N^{\text{cycle}^*} = U\Lambda^* U^*.$$

Then, using (3.43) and (3.45), we can see that

$$\begin{aligned} L_N^{\text{u cycle}} &= \frac{1}{2}\left(L_N^{\text{cycle}} + L_N^{\text{cycle}^T}\right) \\ &= \frac{1}{2} U\left(\Lambda + \Lambda^*\right) U^*, \end{aligned}$$

and since the eigenvalues of a matrix are unchanged under similarity transforms [52, Corollary 1.3.4], the eigenvalues of $L_N^{\text{u cycle}}$ are the real parts of the eigenvalues of L_N^{cycle}, as given by (3.44). □

A.2 Undirected Path Graphs

In order to prove Lemma 3.8, the following lemma will prove useful.

Lemma A.1. *Let M_n be the tridiagonal matrix*

$$M_n := \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}. \tag{A.3}$$

Then the eigenvalues of M_n are given by

$$\lambda_k\left(M_n\right) = 2\sin^2\left(\frac{k\pi}{2(n+1)}\right), \tag{A.4}$$

for $k = 1, 2, \ldots, n$.

Proof. Suppose $\mathbf{v} \in \mathbb{C}^n$ is an eigenvector of M_n with eigenvalue λ, that is

$$M_n\mathbf{v} = \lambda\mathbf{v}, \tag{A.5}$$

and let

$$v_0 := 0 =: v_{n+1}. \tag{A.6}$$

Then, since M_n is tridiagonal with equal entries along each diagonal, (A.5) (along with (A.6)) gives us

$$\left(1-\lambda\right)v_k - \frac{1}{2}v_{k-1} - \frac{1}{2}v_{k+1} = 0, \;\; 1 \leq k \leq n. \tag{A.7}$$

Now, (A.7) and (A.6) form a second-order, linear, constant-coefficient difference equation boundary value problem. We will hypothesise that a solution exists of the form

$$v_k = \nu_1 \mathrm{e}^{k\gamma} + \nu_2 \mathrm{e}^{k\delta},$$

with ν_1, ν_2, γ and $\delta \in \mathbb{C}$. However, in order to satisfy the condition that $v_0 = 0$, we must have $\nu_1 = -\nu_2 = \nu$ for some $\nu \in \mathbb{C}$. Therefore, our expression for v_k becomes

$$v_k = \nu \left(\mathrm{e}^{k\gamma} - \mathrm{e}^{k\delta} \right).$$

Note that we must have $\gamma \neq \delta$, $\mathrm{e}^{\gamma} \neq \mathrm{e}^{\delta}$ and $\nu \neq 0$ for v_k to not be identically 0. With this hypothesis, (A.7) becomes

$$\begin{aligned}
&(1-\lambda)\,\nu \left(\mathrm{e}^{k\gamma} - \mathrm{e}^{k\delta} \right) - \frac{1}{2}\nu \left(\mathrm{e}^{(k-1)\gamma} - \mathrm{e}^{(k-1)\delta} \right) - \frac{1}{2}\nu \left(\mathrm{e}^{(k+1)\gamma} - \mathrm{e}^{(k+1)\delta} \right) = 0 \\
\Rightarrow\ & \mathrm{e}^{k\gamma} \left(1 - \lambda - \frac{1}{2}\mathrm{e}^{-\gamma} - \frac{1}{2}\mathrm{e}^{\gamma} \right) - \mathrm{e}^{k\delta} \left(1 - \lambda - \frac{1}{2}\mathrm{e}^{-\delta} - \frac{1}{2}\mathrm{e}^{\delta} \right) = 0 \\
\Rightarrow\ & \mathrm{e}^{k(\gamma-\delta)} \left(1 - \lambda - \cosh\gamma \right) = \left(1 - \lambda - \cosh\delta \right). \qquad \text{(A.8)}
\end{aligned}$$

But the left hand side of (A.8) is a function of k (since $\mathrm{e}^{\gamma} \neq \mathrm{e}^{\delta}$) while the right hand side is not, so the terms in parentheses on each side of (A.8) must be both equal to 0. Thus

$$\lambda = 1 - \cosh\gamma = 1 - \cosh\delta, \qquad \text{(A.9)}$$

and so

$$\cosh\gamma = \cosh\delta. \qquad \text{(A.10)}$$

But since $\gamma \neq \delta$ and $\mathrm{e}^{\gamma} \neq \mathrm{e}^{\delta}$, the only way that (A.10) can hold is if

$$\delta = -\gamma + 2\ell\pi\mathrm{i}$$

for some $\ell \in \mathbb{Z}$. However, since v_k only depends on δ through the exponential function, without loss of generality we can take

$$\delta = -\gamma.$$

Thus we can write v_k as

$$v_k = \nu \mathrm{e}^{-k\gamma} \left(\left(\mathrm{e}^{2\gamma} \right)^k - 1 \right). \tag{A.11}$$

The final condition that v_k must satisfy is that $v_{n+1} = 0$. Since $\nu \neq 0$ and $\mathrm{e}^{-(n+1)\gamma}$ cannot equal 0, this implies that

$$\left(\mathrm{e}^{2\gamma} \right)^{n+1} = 1,$$

and so

$$\mathrm{e}^{2\gamma} = \mathrm{e}^{\mathrm{i}\frac{2k\pi}{n+1}}, \;\; 1 \leq k \leq n. \tag{A.12}$$

Note that k cannot equal 0 in (A.12) since that would imply $\gamma = \delta$. From (A.12), we can see that

$$\gamma = \pm \mathrm{i} \frac{k\pi}{n+1}, \;\; 1 \leq k \leq n,$$

and so

$$\cosh \gamma = \cos \left(\frac{k\pi}{n+1} \right), \;\; 1 \leq k \leq n.$$

Finally, since $1 - \cos\left(\frac{k\pi}{n+1}\right) = 2\sin^2\left(\frac{k\pi}{2(n+1)}\right)$, (A.9) simplifies to (A.4). □

A.2.1 Proof of Lemma 3.8

Proof. From (3.48), the Laplacian matrix of the undirected path is

$$L_N^{\text{u path}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Now, let $p_N(\lambda) := \det\left(\lambda I_N - L_N^{\text{u path}}\right)$ be the characteristic polynomial of $L_N^{\text{u path}}$. Also, let M_N be the matrix given by (A.3) with $n = N$ and let

$q_N(\lambda) := \det(\lambda I_N - M_N)$ be the characteristic polynomial of M_N. Note that M_N is closely related to $L_N^{\text{u path}}$ by the fact that

$$L_N^{\text{u path}} = M_N - \frac{1}{2}\mathbf{e}_N^{(1)}\mathbf{e}_N^{(1)T} - \frac{1}{2}\mathbf{e}_N^{(N)}\mathbf{e}_N^{(N)T}.$$

Then we can see that for $N \geq 3$,

$$\begin{aligned}
q_N(\lambda) &= \underbrace{\begin{vmatrix} \lambda-1 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \lambda-1 & \frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} & \lambda-1 & \frac{1}{2} \\ 0 & \cdots & 0 & \frac{1}{2} & \lambda-1 \end{vmatrix}}_{N\times N} \\
&= (\lambda-1)\underbrace{\begin{vmatrix} \lambda-1 & \frac{1}{2} & \cdots & 0 \\ \frac{1}{2} & \ddots & \ddots & \vdots \\ \cdots & \ddots & \lambda-1 & \frac{1}{2} \\ \cdots & 0 & \frac{1}{2} & \lambda-1 \end{vmatrix}}_{(N-1)\times(N-1)} - \frac{1}{2}\underbrace{\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \lambda-1 & \frac{1}{2} & \cdots & 0 \\ \vdots & \frac{1}{2} & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \lambda-1 & \frac{1}{2} \\ 0 & \cdots & 0 & \frac{1}{2} & \lambda-1 \end{vmatrix}}_{(N-1)\times(N-1)} \\
&= (\lambda-1)\,q_{N-1}(\lambda) - \frac{1}{4}\underbrace{\begin{vmatrix} \lambda-1 & \frac{1}{2} & \cdots & 0 \\ \frac{1}{2} & \ddots & \ddots & \vdots \\ \cdots & \ddots & \lambda-1 & \frac{1}{2} \\ \cdots & 0 & \frac{1}{2} & \lambda-1 \end{vmatrix}}_{(N-2)\times(N-2)} \\
&= (\lambda-1)\,q_{N-1}(\lambda) - \frac{1}{4}q_{N-2}(\lambda). \qquad \text{(A.13)}
\end{aligned}$$

Furthermore, we can say that for $N \geq 5$,

$$\begin{aligned}
\lambda q_{N-1}(\lambda) &= \left(\lambda^2 - \lambda\right) q_{N-2}(\lambda) - \frac{1}{4}\lambda q_{N-3}(\lambda) \\
&= \left(\lambda - \frac{1}{2}\right)^2 q_{N-2}(\lambda) - \frac{1}{4} q_{N-2}(\lambda) - \frac{1}{4}\lambda q_{N-3}(\lambda) \\
&= \left(\lambda - \frac{1}{2}\right)^2 q_{N-2}(\lambda) - \frac{1}{2}\left(\lambda - \frac{1}{2}\right) q_{N-3}(\lambda) + \frac{1}{16} q_{N-4}(\lambda) \text{ (by (A.13))}.
\end{aligned} \tag{A.14}$$

Next, we can compute $p_N(\lambda)$ for $N \geq 5$ as

$$\begin{aligned}
p_N(\lambda) &= \underbrace{\begin{vmatrix}
\lambda - \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \lambda - 1 & \frac{1}{2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{1}{2} & \lambda - 1 & \frac{1}{2} \\
0 & \cdots & 0 & \frac{1}{2} & \lambda - \frac{1}{2}
\end{vmatrix}}_{N \times N} \\
&= \left(\lambda - \frac{1}{2}\right) \underbrace{\begin{vmatrix}
\lambda - 1 & \frac{1}{2} & \cdots & 0 \\
\frac{1}{2} & \ddots & \ddots & \vdots \\
\cdots & \ddots & \lambda - 1 & \frac{1}{2} \\
\cdots & 0 & \frac{1}{2} & \lambda - \frac{1}{2}
\end{vmatrix}}_{(N-1)\times(N-1)} - \frac{1}{2} \underbrace{\begin{vmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
0 & \lambda - 1 & \frac{1}{2} & \cdots & 0 \\
\vdots & \frac{1}{2} & \ddots & \ddots & \vdots \\
0 & \cdots & \ddots & \lambda - 1 & \frac{1}{2} \\
0 & \cdots & 0 & \frac{1}{2} & \lambda - \frac{1}{2}
\end{vmatrix}}_{(N-1)\times(N-1)} \\
&= \left(\lambda - \frac{1}{2}\right)^2 q_{N-2}(\lambda) - \frac{1}{2}\left(\lambda - \frac{1}{2}\right) \underbrace{\begin{vmatrix}
\lambda - 1 & \frac{1}{2} & \cdots & 0 \\
\frac{1}{2} & \ddots & \ddots & \vdots \\
\cdots & \ddots & \lambda - 1 & \frac{1}{2} \\
\cdots & 0 & \frac{1}{2} & \frac{1}{2}
\end{vmatrix}}_{(N-2)\times(N-2)}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{4}\underbrace{\begin{vmatrix} \lambda-1 & \frac{1}{2} & \cdots & 0 \\ \frac{1}{2} & \ddots & \ddots & \vdots \\ \cdots & \ddots & \lambda-1 & \frac{1}{2} \\ \cdots & 0 & \frac{1}{2} & \lambda-\frac{1}{2} \end{vmatrix}}_{(N-2)\times(N-2)} \\
&= \left(\lambda-\frac{1}{2}\right)^2 q_{N-2}(\lambda) - \frac{1}{4}\left(\lambda-\frac{1}{2}\right) q_{N-3}(\lambda) - \frac{1}{4}\left(\lambda-\frac{1}{2}\right) q_{N-3}(\lambda) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{8}\underbrace{\begin{vmatrix} \lambda-1 & \frac{1}{2} & \cdots & 0 \\ \frac{1}{2} & \ddots & \ddots & \vdots \\ \cdots & \ddots & \lambda-1 & 0 \\ \cdots & 0 & 0 & \frac{1}{2} \end{vmatrix}}_{(N-3)\times(N-3)} \\
&= \left(\lambda-\frac{1}{2}\right)^2 q_{N-2}(\lambda) - \frac{1}{2}\left(\lambda-\frac{1}{2}\right) q_{N-3}(\lambda) + \frac{1}{16} q_{N-4}(\lambda). \qquad \text{(A.15)}
\end{aligned}
$$

Comparing (A.15) and (A.14) reveals that

$$p_N(\lambda) = \lambda q_{N-1}(\lambda)$$

for $N \geq 5$. Furthermore, by direct computation we can see that

$$
\begin{aligned}
p_2(\lambda) &= \lambda^2 - \lambda = \lambda q_1(\lambda), \\
p_3(\lambda) &= \lambda^3 - 2\lambda^2 + \frac{3}{4}\lambda = \lambda q_2(\lambda), \text{ and} \\
p_4(\lambda) &= \lambda^4 - 3\lambda^3 + \frac{5}{2}\lambda^2 - \frac{1}{2}\lambda = \lambda q_3(\lambda).
\end{aligned}
$$

Thus $p_N(\lambda) = \lambda q_{N-1}(\lambda)$ for all $N \geq 2$, and so the eigenvalues of the undirected path are 0 and the eigenvalues of M_{N-1}. Then (A.4) from Lemma A.1 gives us (3.49). $\square$

A.3 Undirected Star Graphs

A.3.1 Proof of Lemma 3.9

Proof. From (3.48), the Laplacian matrix of the undirected star is

$$L_N^{\text{u star}} = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & 0 & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & \frac{N-1}{2} \end{bmatrix}.$$

Now, suppose $\mathbf{v} \in \mathbb{C}^N$ is an eigenvector of $L_N^{\text{u star}}$ with eigenvalue λ, that is

$$L_N^{\text{u star}}\mathbf{v} = \lambda\mathbf{v}. \tag{A.16}$$

Then multiplying out (A.16) gives us

$$\frac{1}{2}(1-2\lambda)\,v_i = \frac{1}{2}v_N, \quad 1 \le i \le N-1 \text{ and} \tag{A.17}$$

$$\left(\frac{N-1}{2} - \lambda\right) v_N = \frac{1}{2}\sum_{i=1}^{N-1} v_i. \tag{A.18}$$

If $\lambda = \frac{1}{2}$, (A.17) becomes

$$v_N = 0,$$

and so (A.18) reduces to

$$\sum_{i=1}^{N-1} v_i = 0. \tag{A.19}$$

Now (A.19) has $N-2$ free parameters, and so we can find $N-2$ linearly independent vectors that satisfy (A.16) for $\lambda = \frac{1}{2}$. Thus $L_N^{\text{u star}}$ has an eigenvalue of $\frac{1}{2}$ with algebraic multiplicity $N-2$.

To find the remaining 2 eigenvalues of $L_N^{\text{u star}}$, we assume that $\lambda \neq \frac{1}{2}$. Then (A.17) gives us

$$v_i = \frac{1}{1-2\lambda} v_N, \;\; 1 \leq i \leq N-1, \tag{A.20}$$

which, when substituted into (A.18), produces

$$\begin{aligned} \left(\frac{N-1}{2} - \lambda\right) v_N &= \frac{N-1}{2} \frac{1}{1-2\lambda} v_N \\ \Rightarrow \left(\frac{N-1}{2} - \lambda - \frac{N-1}{2-4\lambda}\right) v_N &= 0. \end{aligned} \tag{A.21}$$

Now, if $v_N = 0$, (A.20) implies that $v_i = 0 \;\forall\; i$, which is the trivial solution to (A.16). Thus, we must have

$$\begin{aligned} \frac{N-1}{2} - \lambda - \frac{N-1}{2-4\lambda} &= 0 \\ \Rightarrow \frac{4\lambda^2 - 2N\lambda}{2-4\lambda} &= 0 \\ \Rightarrow \lambda\,(2\lambda - N) &= 0, \end{aligned}$$

which holds for $\lambda = 0$ and $\lambda = \dfrac{N}{2}$. Hence (3.53) defines the complete set of eigenvalues of $L_N^{\text{u star}}$. □

Appendix B

Technical Results for Computing Effective Resistances

B.1 Proofs for §5.1

B.1.1 Proof of Lemma 5.1

Proof. (i) (5.5) follows from the fact that the rows (or columns) of P form an orthonormal set. See, e.g., Theorem 2.1.4 in [52].

(ii) Since P contains precisely one 1 in each row and column, $P\mathbf{1}_n = \mathbf{1}_n$ and $\mathbf{1}_n^T = \mathbf{1}_n^T P$. Thus, using (2.2), we find that

$$\begin{aligned} P\Pi &= P - \frac{1}{n} P\mathbf{1}_n\mathbf{1}_n^T \\ &= P - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T P = \Pi P. \end{aligned}$$

(iii) The first part of (5.7) follows directly from (5.6). Then by (2.2), and again

using the fact that $P\mathbf{1}_n = \mathbf{1}_n$, we have

$$\begin{aligned}(P - I_n)\Pi &= P - I_n - \frac{1}{n}P\mathbf{1}_n\mathbf{1}_n^T + \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T \\ &= P - I_n.\end{aligned}$$

□

B.1.2 Proof of Lemma 5.2

Proof. (i) Let $D := AP$, which is diagonal by assumption. Then, by Lemma 5.1, we can see that $A = DP^T$. Thus $PA = PDP^T$, which implies that PA is formed by permuting the rows and columns of a diagonal matrix, and is therefore diagonal.

(ii) Since AP and PA are diagonal, they are both symmetric. Thus A^TP^T ($= (PA)^T = PA$) is symmetric too. Since $\left(-A - A^T\right)$ is also symmetric, the result follows.

(iii) Recall (2.5), that $\overline{M} = QMQ^T$ for any matrix $M \in \mathbb{R}^{n\times n}$ and some $Q \in \mathbb{R}^{(n-1)\times n}$ that satisfies (2.3). Now, we recognise that since AP is diagonal, it is symmetric and commutes with its transpose (i.e. itself). Thus

$$\begin{aligned}P^TA^TAP &= APP^TA^T \\ &= AA^T \text{ (by (5.5)).}\end{aligned}$$

Similarly, by part (i), PA is also diagonal and so it too is symmetric and commutes with its transpose. Hence

$$PAA^TP^T = A^TP^TPA$$

$$= A^T A \text{ (by (5.5))}.$$

Using these facts, we can observe that

$$\left(P^T - I_n\right) A^T A \left(P - I_n\right) = \left(P - I_n\right) A A^T \left(P^T - I_n\right).$$

Now, adding $\left(P - I_n\right) A^2 \left(P - I_n\right)$ to both sides gives us

$$\left[\left(P - I_n\right) A + \left(P^T - I_n\right) A^T\right] A \left(P - I_n\right) = \left(P - I_n\right) A \left[A \left(P - I_n\right) + A^T \left(P^T - I_n\right)\right].$$

But we can use (5.8) to write this as

$$A \left(P - I_n\right) A \left(P - I_n\right) + A^T \left(P^T - I_n\right) A \left(P - I_n\right) = \left(P - I_n\right) A \left(P - I_n\right) A + \left(P - I_n\right) A \left(P^T - I_n\right) A^T. \quad \text{(B.1)}$$

Now, by (5.7), we can pre- or post-multiply any factor of $\left(P - I_n\right)$ or $\left(P^T - I_n\right)$ by Π without changing the matrix. Therefore, we can subtract $\left(P - I_n\right) A \Pi A \left(P - I_n\right)$ from both sides of (B.1), obtain a common factor of $\Pi A \left(P - I_n\right)$ on the left hand side and $\left(P - I_n\right) A \Pi$ on the right hand side, then use (5.8) to obtain

$$\left(P^T - I_n\right) A^T \Pi A \left(P - I_n\right) = \left(P - I_n\right) A \Pi A^T \left(P^T - I_n\right),$$

which is equivalent to (using (5.7) again)

$$\left(P^T - I_n\right) \Pi A^T \Pi A \Pi \left(P - I_n\right) = \left(P - I_n\right) \Pi A \Pi A^T \Pi \left(P^T - I_n\right).$$

Finally, pre-multiplying by Q, post-multiplying by Q^T and using (2.3) gives us our desired result.

□

B.1.3 Proof of Lemma 5.3

Proof. (i) We know that $J = [\hat{\jmath}_{i,j}]$ where $\hat{\jmath}_{i,j} = \delta_{n+1-i,j}$ and δ is the Kronecker delta. If $J^T = [\tilde{\jmath}_{i,j}]$, then $\tilde{\jmath}_{i,j} = \hat{\jmath}_{j,i} = \delta_{n+1-j,i} = \delta_{n+1-i,j} = \hat{\jmath}_{i,j}$. Thus $J^T = J$.

(ii) Let $J^2 = K = [k_{i,j}]$. Then $k_{i,j} = \sum_{\ell=1}^{n} \hat{\jmath}_{i,\ell}\, \hat{\jmath}_{\ell,j} = \sum_{\ell=1}^{n} \delta_{n+1-i,\ell}\, \delta_{n+1-\ell,j} = \delta_{i,j}$. Thus $J^2 = I_n$ and so $J^{-1} = J$.

(iii) Now, J contains precisely one 1 in each row and column, so $J\mathbf{1}_n = \mathbf{1}_n$ and $\mathbf{1}_n^T = \mathbf{1}_n^T J$. Thus

$$\begin{aligned} J\Pi &= J - \frac{1}{n} J\mathbf{1}_n\mathbf{1}_n^T \\ &= J - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T J \\ &= \Pi J. \end{aligned}$$

(iv) By (2.5), $\overline{J} = QJQ^T$ for some matrix $Q \in \mathbb{R}^{(n-1)\times n}$ that satisfies (2.3). Thus $\overline{J}^T = QJ^TQ^T$. But by (5.11), $J^T = J$. Thus $\overline{J}^T = \overline{J}$.

(v) Using (2.5) again, $\overline{J}^2 = QJQ^TQJQ^T = QJ\Pi JQ^T$. By (5.13), we can exchange the order of either J and the Π. Furthermore, by (5.12), $J^2 = I_n$. Thus

$$\overline{J}^2 = QQ^TQQ^T = I_{n-1},$$

and so $\overline{J}^{-1} = \overline{J}$.

□

B.1.4 Proof of Lemma 5.4

Proof. By assumption, we have

$$JL^T\Pi L = L^T\Pi LJ.$$

Since L is a Laplacian matrix, we know that $L\Pi = L$ (and $\Pi L^T = L^T$). Thus, we can say

$$J\Pi L^T\Pi L = L^T\Pi L\Pi J.$$

Pre-multiplying by Q and post-multiplying by Q^T gives us

$$QJQ^TQL^TQ^TQLQ^T = QL^TQ^TQLQ^TQJQ^T,$$

that is

$$\overline{J}\,\overline{L}^T\,\overline{L} = \overline{L}^T\,\overline{L}\,\overline{J}.$$

By pre- and post-multiplying by $\overline{L}^{-1}$ (which exists since the graph is connected), we find

$$\overline{L}^{-T}\,\overline{J}\,\overline{L}^T = \overline{L}\,\overline{J}\,\overline{L}^{-1}.$$

Hence

$$I_{N-1} + \overline{L}^{-T}\,\overline{J}\,\overline{L}^T\,\overline{J} = I_{N-1} + \overline{L}\,\overline{J}\,\overline{L}^{-1}\,\overline{J}.$$

□

B.1.5 Proof of Lemma 5.5

Proof. By (2.5), $\overline{L}_u = QL_uQ^T$. Thus

$$\overline{L}_u = \frac{1}{2}\left(QLQ^T + QJLJQ^T\right)$$

$$= \frac{1}{2}\left(\overline{L} + QQ^TQJLQ^TQJQ^T\right)$$

since $QQ^T = I$ and $L = L\Pi = LQ^TQ$. But by (5.13), $Q^TQJ = JQ^TQ$. Thus

$$\begin{aligned}\overline{L}_u &= \frac{1}{2}\left(\overline{L} + QJQ^TQLQ^TQJQ^T\right)\\ &= \frac{1}{2}\left(\overline{L} + \overline{J}\,\overline{L}\,\overline{J}\right).\end{aligned}$$

□

B.2 Proof of Lemma 5.10

Proof. As stated in the lemma, we will assume that n and ℓ are positive integers throughout this proof. Let $N_{n,\ell}$ be the number of nodes in $\mathcal{G}^{\text{tree}}_{n,\ell}$. The branch of length n contains n nodes (excluding the root), while the other branch contains ℓ nodes (excluding the root). Therefore, we have $N_{n,\ell} = n + \ell + 1$. Let us label the nodes in $\mathcal{G}^{\text{tree}}_{n,\ell}$ from 1 to $n+1$ along the branch of length n, in reverse order of the edge directions and starting with the root (thus the leaf of this branch is node $n+1$). Then let us label the nodes in the branch of length ℓ from $n+2$ to $N_{n,\ell} = n + \ell + 1$ in reverse order of the edge directions. Thus the second leaf is node $N_{n,\ell}$.

In the following, we will denote the adjacency matrix of $\mathcal{G}^{\text{tree}}_{n,\ell}$ by $A_{n,\ell}$, its matrix of node out-degrees by $D_{n,\ell}$ and its Laplacian matrix by $L_{n,\ell}$. Furthermore, we will let $Q_{n,\ell}$ be a $(N_{n,\ell} - 1) \times N_{n,\ell}$ matrix that satisfies (2.3) and $\overline{L}_{n,\ell}$ and $\Sigma_{n,\ell}$ be the corresponding matrices from (3.6) and (3.12) using $L_{n,\ell}$ and $Q_{n,\ell}$. Finally, $X_{n,\ell}$ will be the matrix from (4.7), computed using $\Sigma_{n,\ell}$ and $Q_{n,\ell}$. Then, by Lemma 5.7, the entries of $X_{n,\ell}$ are related to the resistances in $\mathcal{G}^{\text{tree}}_{n,\ell}$ by (5.20).

As in the proof of Lemma 5.9, let $A^{\text{path}}_{N_p}$, $D^{\text{path}}_{N_p}$ and $L^{\text{path}}_{N_p}$ denote the adjacency matrix, matrix of out-degrees and Laplacian matrix of a directed path containing N_p

nodes and unit weights on every edge. Let the nodes in this path be labelled from 1 to N_p in the order in which they appear, starting with the root. Then we can write $A_{n,\ell}$, $D_{n,\ell}$ and $L_{n,\ell}$ in terms of $A^{\text{path}}_{N_p}$, $D^{\text{path}}_{N_p}$ and $L^{\text{path}}_{N_p}$ as follows:

$$A_{n,\ell} = \begin{bmatrix} A^{\text{path}}_{n+1} & \emptyset \\ \mathbf{e}^{(1)}_{\ell}\mathbf{e}^{(1)T}_{n+1} & A^{\text{path}}_{\ell} \end{bmatrix},$$

$$D_{n,\ell} = \begin{bmatrix} D^{\text{path}}_{n+1} & \emptyset \\ \emptyset & D^{\text{path}}_{\ell} + \mathbf{e}^{(1)}_{\ell}\mathbf{e}^{(1)T}_{\ell} \end{bmatrix} \text{ and}$$

$$L_{n,\ell} = \begin{bmatrix} L^{\text{path}}_{n+1} & \emptyset \\ -\mathbf{e}^{(1)}_{\ell}\mathbf{e}^{(1)T}_{n+1} & L^{\text{path}}_{\ell} + \mathbf{e}^{(1)}_{\ell}\mathbf{e}^{(1)T}_{\ell} \end{bmatrix}.$$

Using these expressions as well as (5.29) and (5.30), we can observe that

$$\mathbf{1}^T_{N_{n,\ell}} L_{n,\ell} = -2\mathbf{e}^{(1)T}_{N_{n,\ell}} + \mathbf{e}^{(n+1)T}_{N_{n,\ell}} + \mathbf{e}^{(N_{n,\ell})T}_{N_{n,\ell}}, \text{ and} \tag{B.2}$$

$$\mathbf{e}^{(i)T}_{N_{n,\ell}} L_{n,\ell} = \begin{cases} \mathbf{e}^{(i)T}_{N_{n,\ell}} - \mathbf{e}^{(i-1)T}_{N_{n,\ell}} & \text{if } 1 < i \leq N_{n,\ell},\ i \neq n+2, \\ \mathbf{e}^{(n+2)T}_{N_{n,\ell}} - \mathbf{e}^{(1)T}_{N_{n,\ell}} & \text{if } i = n+2, \\ \mathbf{0}^T & \text{if } i = 1. \end{cases} \tag{B.3}$$

Let us now consider $\mathcal{G}^{\text{tree}}_{n,\ell+1}$. By our labelling convention, the resistance between the two leaves of $\mathcal{G}^{\text{tree}}_{n,\ell+1}$ is given by

$$r(n, \ell+1) = r_{n+1,n+\ell+2}.$$

Now, we can write the adjacency matrix of $\mathcal{G}^{\text{tree}}_{n,\ell+1}$ in terms of $A_{n,\ell}$ as

$$A_{n,\ell+1} = \begin{bmatrix} A_{n,\ell} & \mathbf{0} \\ \mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})T} & 0 \end{bmatrix}.$$

In a similar fashion, we can write the matrix of node out-degrees for $\mathcal{G}^{\text{tree}}_{n,\ell+1}$ as

$$D_{n,\ell+1} = \begin{bmatrix} D_{n,\ell} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

and the Laplacian matrix as

$$L_{n,\ell+1} = \begin{bmatrix} L_{n,\ell} & \mathbf{0} \\ -\mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})T} & 1 \end{bmatrix}.$$

Now, let

$$Q_{n,\ell+1} = \begin{bmatrix} Q_{n,\ell} & \mathbf{0} \\ \alpha \mathbf{1}_{N_{n,\ell}}^T & -\beta \end{bmatrix},$$

where

$$\alpha = \frac{1}{\sqrt{N_{n,\ell}(N_{n,\ell}+1)}} = \frac{1}{\sqrt{(n+\ell+1)(n+\ell+2)}} \text{ and}$$

$$\beta = \sqrt{\frac{N_{n,\ell}}{N_{n,\ell}+1}} = \sqrt{\frac{n+\ell+1}{n+\ell+2}}.$$

Then $Q_{n,\ell+1}$ satisfies (2.3). We can therefore use (3.6), (B.2) and the facts that $L_{n,\ell}\mathbf{1}_{N_{n,\ell}} = \mathbf{0}_{N_{n,\ell}}$ and $\beta\,(\alpha+\beta) = 1$ to compute $\overline{L}_{n,\ell}$ as

$$\overline{L}_{n,\ell} = \begin{bmatrix} \overline{L}_{n,\ell} & \mathbf{0} \\ -2\alpha \mathbf{e}_{N_{n,\ell}}^{(1)T} Q_{n,\ell}^T + \alpha \mathbf{e}_{N_{n,\ell}}^{(n+1)T} Q_{n,\ell}^T + (\alpha+\beta)\, \mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})T} Q_{n,\ell}^T & 1 \end{bmatrix}.$$

In order to compute resistances in $\mathcal{G}^{\text{tree}}_{n,\ell+1}$, we must find the matrix $\Sigma_{n,\ell+1}$ which

solves (3.12). Since we have partitioned $\overline{L}_{n,\ell+1}$ into a 2×2 block matrix, we will do the same for $\Sigma_{n,\ell+1}$. Let

$$\Sigma_{n,\ell+1} = \begin{bmatrix} S & \mathbf{t} \\ \mathbf{t}^T & u \end{bmatrix},$$

where $S \in \mathbb{R}^{(N_{n,\ell}-1)\times(N_{n,\ell}-1)}$ is a symmetric matrix, $\mathbf{t} \in \mathbb{R}^{N_{n,\ell}-1}$ and $u \in \mathbb{R}$. Then multiplying out the matrices in (3.12) and equating blocks in this matrix equation gives us

$$\overline{L}_{n,\ell} S + S\overline{L}_{n,\ell}^T = I_{N_{n,\ell}-1}, \tag{B.4}$$

$$\overline{L}_{n,\ell}\mathbf{t} + \mathbf{t} - 2\alpha S Q_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(1)} + \alpha S Q_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(n+1)} + (\alpha+\beta)\, S Q_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})} = \mathbf{0}, \text{ and} \tag{B.5}$$

$$2u - 4\alpha \mathbf{e}_{N_{n,\ell}}^{(1)T} Q_{n,\ell}^T \mathbf{t} + 2\alpha \mathbf{e}_{N_{n,\ell}}^{(n+1)T} Q_{n,\ell}^T \mathbf{t} + 2\,(\alpha+\beta)\, \mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})T} Q_{n,\ell}^T \mathbf{t} = 1. \tag{B.6}$$

From (B.4), it is clear that $S = \Sigma_{n,\ell}$. In addition, we can rewrite (B.6) as

$$u = \frac{1}{2} + 2\alpha \mathbf{e}_{N_{n,\ell}}^{(1)T} Q_{n,\ell}^T \mathbf{t} - \alpha \mathbf{e}_{N_{n,\ell}}^{(n+1)T} Q_{n,\ell}^T \mathbf{t} - (\alpha+\beta)\, \mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})T} Q_{n,\ell}^T \mathbf{t}. \tag{B.7}$$

Thus in order to find a complete solution for $\Sigma_{n,\ell+1}$, we must solve (B.5) for $\mathbf{t}$. However, resistances are computed from the entries of $X_{n,\ell+1}$, which, if we let $\mathbf{v} := Q_{n,\ell}^T \mathbf{t} = [v_i]$ and use (4.7), can be written as

$$X_{n,\ell+1} = \begin{bmatrix} X_{n,\ell} + 2\alpha \mathbf{v}\mathbf{1}_{N_{n,\ell}}^T + 2\alpha \mathbf{1}_{N_{n,\ell}}\mathbf{v}^T + 2\alpha^2 u \mathbf{1}_{N_{n,\ell}}\mathbf{1}_{N_{n,\ell}}^T & -2\beta\mathbf{v} - 2\alpha\beta u \mathbf{1}_{N_{n,\ell}} \\ -2\beta\mathbf{v}^T - 2\alpha\beta u \mathbf{1}_{N_{n,\ell}}^T & 2\beta^2 u \end{bmatrix}.$$

Hence, in order to compute resistances in $\mathcal{G}_{n,\ell+1}^{\text{tree}}$, we need only compute $\mathbf{v}$, not $\mathbf{t}$. We should also note that as $X_{n,\ell+1}$ does not depend on our choice of $Q_{n,\ell+1}$ (by Lemma 4.1), neither does $\mathbf{v}$. In fact, we can write (B.7) as

$$u = \frac{1}{2} + 2\alpha v_1 - \alpha v_{n+1} - (\alpha+\beta)\, v_{N_{n,\ell}},$$

and the resistance we seek as

$$r(n,\ell+1) = x_{n,\ell\, n+1,n+1} + (\alpha+\beta)^2 + 4\alpha\,(\alpha+\beta)^2\, v_1 + 2\,(\alpha+\beta)\,[2-\alpha\,(\alpha+\beta)]\, v_{n+1} - 2(\alpha+\beta)^3 v_{N_{n,\ell}}. \quad \text{(B.8)}$$

Thus we only need to find v_1, v_{n+1} and $v_{N_{n,\ell}}$ in order to compute $r(n,\ell+1)$.

Now, $v_i = \mathbf{e}_{N_{n,\ell}}^{(i)T}\mathbf{v} = \mathbf{e}_{N_{n,\ell}}^{(i)T} Q_{n,\ell}^T \mathbf{t}$. We will therefore proceed by left-multiplying (B.5) by $\mathbf{e}_{N_{n,\ell}}^{(i)T} Q_{n,\ell}^T$. Using the fact that $S = \Sigma_{n,\ell}$, we obtain

$$\mathbf{e}_{N_{n,\ell}}^{(i)T} Q_{n,\ell}^T Q_{n,\ell} L_{n,\ell}\mathbf{v} + v_i - \alpha \mathbf{e}_{N_{n,\ell}}^{(i)T} X_{n,\ell} \mathbf{e}_{N_{n,\ell}}^{(1)} + \frac{\alpha}{2}\mathbf{e}_{N_{n,\ell}}^{(i)T} X_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(n+1)} + \frac{\alpha+\beta}{2}\mathbf{e}_{N_{n,\ell}}^{(i)T} X_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})} = 0. \quad \text{(B.9)}$$

But $\mathbf{e}_{N_{n,\ell}}^{(i)T} Q_{n,\ell}^T Q_{n,\ell} = \mathbf{e}_{N_{n,\ell}}^{(i)T}\left(I_{N_{n,\ell}} - \frac{1}{N_{n,\ell}}\mathbf{1}_{N_{n,\ell}}\mathbf{1}_{N_{n,\ell}}^T\right) = \mathbf{e}_{N_{n,\ell}}^{(i)T} - \frac{1}{N_{n,\ell}}\mathbf{1}_{N_{n,\ell}}^T$ by (2.3), and so by using (B.2) and (B.3), we find

$$\mathbf{e}_{N_{n,\ell}}^{(i)T} Q_{n,\ell}^T Q_{n,\ell} L_{n,\ell}\mathbf{v} = \frac{2}{N_{n,\ell}} v_1 - \frac{1}{N_{n,\ell}} v_{n+1} - \frac{1}{N_{n,\ell}} v_{N_{n,\ell}} + \begin{cases} v_i - v_{i-1} & \text{if } 1 < i \le N_{n,\ell},\ i \ne n+2, \\ v_{n+2} - v_1 & \text{if } i = n+2, \\ 0 & \text{if } i = 1. \end{cases}$$

Furthermore, we observe that

$$\mathbf{e}_{N_{n,\ell}}^{(i)T} X_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(1)} = x_{n,\ell\, i,1},$$

$$\mathbf{e}_{N_{n,\ell}}^{(i)T} X_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(n+1)} = x_{n,\ell\, i,n+1} \text{ and}$$

$$\mathbf{e}_{N_{n,\ell}}^{(i)T} X_{n,\ell}\mathbf{e}_{N_{n,\ell}}^{(N_{n,\ell})} = x_{n,\ell\, i,N_{n,\ell}}.$$

Substituting these expressions into (B.9) gives us

$$v_i = \frac{1}{2}v_{i-1} - \frac{1}{N_{n,\ell}}v_1 + \frac{1}{2N_{n,\ell}}v_{n+1} + \frac{1}{2N_{n,\ell}}v_{N_{n,\ell}} + \frac{\alpha}{2}x_{n,\ell\, i,1} - \frac{\alpha}{4}x_{n,\ell\, i,n+1} - \frac{\alpha+\beta}{4}x_{n,\ell\, i,N_{n,\ell}} \text{ if } 1 < i \leq N_{n,\ell},\ i \neq n+2, \quad \text{(B.10)}$$

$$v_{n+2} = \frac{1}{2}v_1 - \frac{1}{N_{n,\ell}}v_1 + \frac{1}{2N_{n,\ell}}v_{n+1} + \frac{1}{2N_{n,\ell}}v_{N_{n,\ell}} + \frac{\alpha}{2}x_{n,\ell\, n+2,1} - \frac{\alpha}{4}x_{n,\ell\, n+2,n+1} - \frac{\alpha+\beta}{4}x_{n,\ell\, n+2,N_{n,\ell}}, \text{ and} \quad \text{(B.11)}$$

$$v_1 = \frac{1}{N_{n,\ell}+2}v_{n+1} + \frac{1}{N_{n,\ell}+2}v_{N_{n,\ell}} + \frac{\alpha N_{n,\ell}}{N_{n,\ell}+2}x_{n,\ell\, 1,1} - \frac{\alpha N_{n,\ell}}{2\left(N_{n,\ell}+2\right)}x_{n,\ell\, 1,n+1} - \frac{(\alpha+\beta)\, N_{n,\ell}}{2\left(N_{n,\ell}+2\right)}x_{n,\ell\, 1,N_{n,\ell}}. \quad \text{(B.12)}$$

We can now recursively apply (B.10) n times, starting with $i = n+1$, to find

$$v_{n+1} = 2^{-n}v_1 - \frac{2v_1}{N_{n,\ell}}\sum_{k=1}^{n}2^{-k} + \frac{v_{n+1}}{N_{n,\ell}}\sum_{k=1}^{n}2^{-k} + \frac{v_{N_{n,\ell}}}{N_{n,\ell}}\sum_{k=1}^{n}2^{-k} + \alpha\sum_{k=1}^{n}x_{n,\ell\, n+2-k,1}2^{-k} - \frac{\alpha}{2}\sum_{k=1}^{n}x_{n,\ell\, n+2-k,n+1}2^{-k} - \frac{\alpha+\beta}{2}\sum_{k=1}^{n}x_{n,\ell\, n+2-k,N_{n,\ell}}2^{-k},$$

which simplifies (using (B.28)) to

$$\frac{N_{n,\ell}-1+2^{-n}}{N_{n,\ell}}v_{n+1} = \frac{\left[-2+\left(N_{n,\ell}+2\right)2^{-n}\right]}{N_{n,\ell}}v_1 + \frac{1-2^{-n}}{N_{n,\ell}}v_{N_{n,\ell}} + \alpha\sum_{k=1}^{n}x_{n,\ell\, n+2-k,1}2^{-k} - \frac{\alpha}{2}\sum_{k=1}^{n}x_{n,\ell\, n+2-k,n+1}2^{-k} - \frac{\alpha+\beta}{2}\sum_{k=1}^{n}x_{n,\ell\, n+2-k,N_{n,\ell}}2^{-k}. \quad \text{(B.13)}$$

Similarly, we can recursively apply (B.10) $\ell-1$ times, starting with $i = N_{\ell,n} = n+\ell+1$, to find

$$v_{N_{n,\ell}} = 2^{1-\ell}v_{n+2} - \frac{2v_1}{N_{n,\ell}}\sum_{k=1}^{\ell-1}2^{-k} + \frac{v_{n+1}}{N_{n,\ell}}\sum_{k=1}^{\ell-1}2^{-k} + \frac{v_{N_{n,\ell}}}{N_{n,\ell}}\sum_{k=1}^{\ell-1}2^{-k} + \alpha\sum_{k=1}^{\ell-1}x_{n,\ell\, N_{n,\ell}+1-k,1}2^{-k} - \frac{\alpha}{2}\sum_{k=1}^{\ell-1}x_{n,\ell\, N_{n,\ell}+1-k,n+1}2^{-k} - \frac{\alpha+\beta}{2}\sum_{k=1}^{\ell-1}x_{n,\ell\, N_{n,\ell}+1-k,N_{n,\ell}}2^{-k},$$

and then substitute in (B.11) to get

$$v_{N_{n,\ell}} = 2^{-\ell} v_1 - \frac{2v_1}{N_{n,\ell}} \sum_{k=1}^{\ell} 2^{-k} + \frac{v_{n+1}}{N_{n,\ell}} \sum_{k=1}^{\ell} 2^{-k} + \frac{v_{N_{n,\ell}}}{N_{n,\ell}} \sum_{k=1}^{\ell} 2^{-k} + \alpha \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,1} 2^{-k}$$
$$- \frac{\alpha}{2} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,n+1} 2^{-k} - \frac{\alpha+\beta}{2} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,N_{n,\ell}} 2^{-k}.$$

Using (B.28) again, we obtain

$$\frac{N_{n,\ell} - 1 + 2^{-\ell}}{N_{n,\ell}} v_{N_{n,\ell}} = \frac{\left[-2 + (N_{n,\ell}+2)\, 2^{-\ell}\right]}{N_{n,\ell}} v_1 + \frac{1 - 2^{-\ell}}{N_{n,\ell}} v_{n+1} + \alpha \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,1} 2^{-k}$$
$$- \frac{\alpha}{2} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,n+1} 2^{-k} - \frac{\alpha+\beta}{2} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,N_{n,\ell}} 2^{-k}. \quad \text{(B.14)}$$

Note that (B.14) reduces to (B.11) when $\ell = 1$.

But now (B.12), (B.13) and (B.14) form a set of three of simultaneous linear equations in v_1, v_{n+1} and $v_{N_{n,\ell}}$. Their solution is given by

$$v_1 = \frac{\alpha}{N_{n,\ell}} \left(N_{n,\ell} - 2 + 2^{-n} + 2^{-\ell}\right) x_{n,\ell\, 1,1} - \frac{\alpha}{2N_{n,\ell}} \left(N_{n,\ell} - 2 + 2^{-n} + 2^{-\ell}\right) x_{n,\ell\, 1,n+1}$$
$$- \frac{\alpha+\beta}{2N_{n,\ell}} \left(N_{n,\ell} - 2 + 2^{-n} + 2^{-\ell}\right) x_{n,\ell\, 1,N_{n,\ell}} + \frac{\alpha}{N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\, n+2-k,1} 2^{-k}$$
$$+ \frac{\alpha}{N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,1} 2^{-k} - \frac{\alpha}{2N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\, n+2-k,n+1} 2^{-k}$$
$$- \frac{\alpha}{2N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,n+1} 2^{-k} - \frac{\alpha+\beta}{2N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\, n+2-k,N_{n,\ell}} 2^{-k}$$
$$- \frac{\alpha+\beta}{2N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,N_{n,\ell}} 2^{-k}, \quad \text{(B.15)}$$

$$v_{n+1} = \frac{\alpha}{N_{n,\ell}} \left(-2 + (N_{n,\ell}+1)\, 2^{-n} + 2^{-\ell}\right) x_{n,\ell\, 1,1}$$

$$
\begin{aligned}
&- \frac{\alpha}{2N_{n,\ell}} \left(-2 + (N_{n,\ell}+1)\, 2^{-n} + 2^{-\ell}\right) x_{n,\ell\,1,n+1} \\
- \frac{\alpha+\beta}{2N_{n,\ell}} &\left(-2 + (N_{n,\ell}+1)\, 2^{-n} + 2^{-\ell}\right) x_{n,\ell\,1,N_{n,\ell}} + \frac{\alpha\,(N_{n,\ell}+1)}{N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\,n+2-k,1} 2^{-k} \\
&+ \frac{\alpha}{N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,1} 2^{-k} - \frac{\alpha\,(N_{n,\ell}+1)}{2N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\,n+2-k,n+1} 2^{-k} \\
- \frac{\alpha}{2N_{n,\ell}} &\sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,n+1} 2^{-k} - \frac{(\alpha+\beta)\,(N_{n,\ell}+1)}{2N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\,n+2-k,N_{n,\ell}} 2^{-k} \\
&- \frac{\alpha+\beta}{2N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,N_{n,\ell}} 2^{-k}, \text{ and} \quad \text{(B.16)}
\end{aligned}
$$

$$
\begin{aligned}
v_{N_{n,\ell}} = \frac{\alpha}{N_{n,\ell}} &\left(-2 + 2^{-n} + (N_{n,\ell}+1)\, 2^{-\ell}\right) x_{n,\ell\,1,1} \\
&- \frac{\alpha}{2N_{n,\ell}} \left(-2 + 2^{-n} + (N_{n,\ell}+1)\, 2^{-\ell}\right) x_{n,\ell\,1,n+1} \\
- \frac{\alpha+\beta}{2N_{n,\ell}} &\left(-2 + 2^{-n} + (N_{n,\ell}+1)\, 2^{-\ell}\right) x_{n,\ell\,1,N_{n,\ell}} + \frac{\alpha}{N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\,n+2-k,1} 2^{-k} \\
&+ \frac{\alpha\,(N_{n,\ell}+1)}{N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,1} 2^{-k} - \frac{\alpha}{2N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\,n+2-k,n+1} 2^{-k} \\
- \frac{\alpha\,(N_{n,\ell}+1)}{2N_{n,\ell}} &\sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,n+1} 2^{-k} - \frac{\alpha+\beta}{2N_{n,\ell}} \sum_{k=1}^{n} x_{n,\ell\,n+2-k,N_{n,\ell}} 2^{-k} \\
&- \frac{(\alpha+\beta)\,(N_{n,\ell}+1)}{2N_{n,\ell}} \sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,N_{n,\ell}} 2^{-k}. \quad \text{(B.17)}
\end{aligned}
$$

Substituting (B.15), (B.16) and (B.17) into (B.8) then multiplying by $N_{n,\ell}$ (and using the definitions of α and β) gives us

$$
\begin{aligned}
N_{n,\ell}\, r(n,\ell+1) = N_{n,\ell} + 1 &+ \left(2^{2-n} - 2^{1-\ell}\right) x_{n,\ell\,1,1} + \left(2^{-\ell} - 2^{1-n}\right) x_{n,\ell\,1,n+1} \\
&+ (N_{n,\ell}+1) \left(2^{-\ell} - 2^{1-n}\right) x_{n,\ell\,1,N_{n,\ell}} + N_{n,\ell} x_{n,\ell\,n+1,n+1} + 4 \sum_{k=1}^{n} x_{n,\ell\,n+2-k,1} 2^{-k} \\
&- 2 \sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,1} 2^{-k} - 2 \sum_{k=1}^{n} x_{n,\ell\,n+2-k,n+1} 2^{-k} + \sum_{k=1}^{\ell} x_{n,\ell\,N_{n,\ell}+1-k,n+1} 2^{-k}
\end{aligned}
$$

$$- (2N_{n,\ell}+2)\sum_{k=1}^{n} x_{n,\ell\, n+2-k,N_{n,\ell}}2^{-k} + (N_{n,\ell}+1)\sum_{k=1}^{\ell} x_{n,\ell\, N_{n,\ell}+1-k,N_{n,\ell}}2^{-k}. \quad \text{(B.18)}$$

Now, by (4.8), we can write $x_{n,\ell\,k,j} = \frac{1}{2}x_{n,\ell\,k,k} + \frac{1}{2}x_{n,\ell\,j,j} - \frac{1}{2}r_{k,j}$. Furthermore, by Theorem 5.1 we know that

$$r_{k,j} = 2\,|k-j| \text{ if } 1 \le k,j \le n+1, \text{ or } n+2 \le k,j \le N_{n,\ell}, \text{ and} \quad \text{(B.19)}$$

$$r_{1,j} = 2\,(j-n-1) \text{ if } n+2 \le j \le N_{n,\ell}. \quad \text{(B.20)}$$

Finally, by the definition of $r(\cdot,\cdot)$, we can say that

$$r_{k,j} = r(k-1, j-n-1) \text{ if } 1 < k \le n+1 \text{ and } n+2 \le j \le N_{n,\ell}. \quad \text{(B.21)}$$

Therefore, we can make the following substitutions for the $x_{n,\ell\,k,j}$ terms in (B.18):

$$\begin{aligned}
x_{n,\ell\,1,n+1} &= \frac{1}{2}x_{n,\ell\,1,1} + \frac{1}{2}x_{n,\ell\,n+1,n+1} - n,\\
x_{n,\ell\,1,N_{n,\ell}} &= \frac{1}{2}x_{n,\ell\,1,1} + \frac{1}{2}x_{n,\ell\,N_{n,\ell},N_{n,\ell}} - \ell,\\
x_{n,\ell\,k,1} &= \frac{1}{2}x_{n,\ell\,1,1} + \frac{1}{2}x_{n,\ell\,k,k} - (k-1) \text{ if } 2 \le k \le n+1,\\
x_{n,\ell\,k,1} &= \frac{1}{2}x_{n,\ell\,1,1} + \frac{1}{2}x_{n,\ell\,k,k} - (k-n-1) \text{ if } n+2 \le k \le N_{n,\ell},\\
x_{n,\ell\,k,n+1} &= \frac{1}{2}x_{n,\ell\,n+1,n+1} + \frac{1}{2}x_{n,\ell\,k,k} - (n+1-k) \text{ if } 2 \le k \le n+1,\\
x_{n,\ell\,k,n+1} &= \frac{1}{2}x_{n,\ell\,n+1,n+1} + \frac{1}{2}x_{n,\ell\,k,k} - \frac{1}{2}r(n, k-n-1) \text{ if } n+2 \le k \le N_{n,\ell},\\
x_{n,\ell\,k,N_{n,\ell}} &= \frac{1}{2}x_{n,\ell\,N_{n,\ell},N_{n,\ell}} + \frac{1}{2}x_{n,\ell\,k,k} - \frac{1}{2}r(k-1, \ell) \text{ if } 2 \le k \le n+1, \text{ and}\\
x_{n,\ell\,k,N_{n,\ell}} &= \frac{1}{2}x_{n,\ell\,N_{n,\ell},N_{n,\ell}} + \frac{1}{2}x_{n,\ell\,k,k} - (N_{n,\ell}-k) \text{ if } n+2 \le k \le N_{n,\ell}.
\end{aligned}$$

This, along with the fact that $N_{n,\ell} = n+\ell+1$, gives us

$$
\begin{aligned}
(n+\ell+1)\,r(n,\ell+1) &= n+\ell+2-n\left(2^{-\ell}-2^{1-n}\right)-\ell\,(n+\ell+2)\left(2^{-\ell}-2^{1-n}\right)\\
&\quad-(4n+6)\sum_{k=1}^{n}2^{-k}+6\sum_{k=1}^{n}k2^{-k}+(n+3\ell+4)\sum_{k=1}^{\ell}2^{-k}-(n+\ell+4)\sum_{k=1}^{\ell}k2^{-k}\\
&\quad-\frac{1}{2}\sum_{k=1}^{\ell}r(n,\ell+1-k)2^{-k}+(n+\ell+2)\sum_{k=1}^{n}r(n+1-k,\ell)2^{-k}\\
&\quad+\left[(n+\ell-1)\left(2^{-1-\ell}-2^{-n}\right)+2\sum_{k=1}^{n}2^{-k}-\sum_{k=1}^{\ell}2^{-k}\right]x_{n,\ell\,1,1}\\
&\quad+\left[n+\ell+1+2^{-1-\ell}-2^{-n}-\sum_{k=1}^{n}2^{-k}+\frac{1}{2}\sum_{k=1}^{\ell}2^{-k}\right]x_{n,\ell\,n+1,n+1}\\
&\quad+(n+\ell+2)\left[\left(2^{-1-\ell}-2^{-n}\right)-\sum_{k=1}^{n}2^{-k}+\frac{1}{2}\sum_{k=1}^{\ell}2^{-k}\right]x_{n,\ell\,n+\ell+1,n+\ell+1}\\
&\quad-(n+\ell+1)\sum_{k=1}^{n}x_{n,\ell\,n+2-k,n+2-k}2^{-k}+\frac{n+\ell+1}{2}\sum_{k=1}^{\ell}x_{n,\ell\,n+\ell+2-k,n+\ell+2-k}2^{-k},
\end{aligned}
$$

which (using (B.28) and (B.29)) simplifies to

$$
\begin{aligned}
(n+\ell+1)\,r(n,\ell+1) &= -4n+2\ell+4+\left(\ell^2+n\ell+2\ell-3\right)2^{1-n}+(\ell+4)\,2^{-\ell}\\
&\quad+(n+\ell+2)\sum_{k=1}^{n}r(n+1-k,\ell)2^{-k}-\frac{1}{2}\sum_{k=1}^{\ell}r(n,\ell+1-k)2^{-k}+\left(n+\ell+\frac{1}{2}\right)x_{n,\ell\,n+1,n+1}\\
&\quad+\left[(n+\ell+1)\left(2^{-1-\ell}-2^{-n}\right)+1\right]x_{n,\ell\,1,1}-\frac{n+\ell+2}{2}x_{n,\ell\,n+\ell+1,n+\ell+1}\\
&\quad-(n+\ell+1)\sum_{k=1}^{n}x_{n,\ell\,n+2-k,n+2-k}2^{-k}+\frac{n+\ell+1}{2}\sum_{k=1}^{\ell}x_{n,\ell\,n+\ell+2-k,n+\ell+2-k}2^{-k},
\end{aligned}
$$

or, by changing indices inside the sums,

$$
\begin{aligned}
(n+\ell+1)\,r(n,\ell+1) &= -4n+2\ell+4+\left(\ell^2+n\ell+2\ell-3\right)2^{1-n}+(\ell+4)\,2^{-\ell}\\
&\quad+\frac{n+\ell+2}{2}\sum_{k=1}^{n}r(k,\ell)2^{k-n}-\frac{1}{4}\sum_{k=1}^{\ell}r(n,k)2^{k-\ell}+\left(n+\ell+\frac{1}{2}\right)x_{n,\ell\,n+1,n+1}\\
&\quad+\left[(n+\ell+1)\left(2^{-1-\ell}-2^{-n}\right)+1\right]x_{n,\ell\,1,1}-\frac{n+\ell+2}{2}x_{n,\ell\,n+\ell+1,n+\ell+1}
\end{aligned}
$$

$$- \frac{n+\ell+1}{2} \sum_{k=1}^{n} x_{n,\ell\, k+1,k+1} 2^{k-n} + \frac{n+\ell+1}{4} \sum_{k=1}^{\ell} x_{n,\ell\, n+1+k,n+1+k} 2^{k-\ell}. \quad \text{(B.22)}$$

Now, by (5.20) from Lemma 5.7, we know that

$$x_{n,\ell\, i,i} = \frac{1}{N_{n,\ell}} \sum_{k=1}^{N_{n,\ell}} r_{i,k} - \frac{1}{N_{n,\ell}^2} \sum_{k=1}^{N_{n,\ell}-1} \sum_{j=k+1}^{N_{n,\ell}} r_{k,j}. \quad \text{(B.23)}$$

Using (B.19), (B.20) and (B.21), we can write the first sum in (B.23) as

$$\begin{aligned}
\sum_{k=1}^{n+\ell+1} r_{i,k} &= \begin{cases}
\displaystyle\sum_{k=2}^{n+1} 2(k-1) + \sum_{k=n+2}^{n+\ell+1} 2(k-n-1) & \text{if } i = 1, \\
\displaystyle\sum_{k=1}^{n+1} 2\,|k-i| + \sum_{k=n+2}^{n+\ell+1} r(i-1, k-n-1) & \text{if } 1 < i \le n+1, \\
\displaystyle 2(i-n-1) + \sum_{k=2}^{n+1} r(k-1, i-n-1) & \\
\qquad\qquad + \displaystyle\sum_{k=n+2}^{n+\ell+1} 2\,|i-k| & \text{if } n+2 \le i \le n+\ell+1,
\end{cases} \\
&= \begin{cases}
\displaystyle 2\sum_{k=1}^{n} k + 2\sum_{k=1}^{\ell} k & \text{if } i = 1, \\
\displaystyle 2\sum_{k=1}^{i-1} k + 2\sum_{k=1}^{n+1-i} k + \sum_{k=1}^{\ell} r(i-1, k) & \text{if } 1 < i \le n+1, \\
\displaystyle \sum_{k=1}^{n} r(k, i-n-1) + 2\sum_{k=1}^{i-n-1} k + 2\sum_{k=1}^{n+\ell+1-i} k & \text{if } n+2 \le i \le n+\ell+1.
\end{cases}
\end{aligned}$$

Therefore, by using (B.26) we obtain

$$\sum_{k=1}^{n+\ell+1} r_{i,k} = \begin{cases}
n^2 + \ell^2 + n + \ell & \text{if } i = 1, \\
n^2 + (3-2i)n + 2(i-1)^2 + \displaystyle\sum_{k=1}^{\ell} r(i-1, k) & \text{if } 1 < i \le n+1, \\
2n^2 + \ell^2 + 2n\ell + (4-4i)n + (3-2i)\ell & \\
\qquad + 2(i-1)^2 + \displaystyle\sum_{k=1}^{n} r(k, i-n-1) & \text{if } n+2 \le i \le n+\ell+1.
\end{cases} \quad \text{(B.24)}$$

We can note that the double sum in (B.23) is independent of i. Let

$$f := \sum_{k=1}^{N_{n,\ell}-1} \sum_{j=k+1}^{N_{n,\ell}} r_{k,j}.$$

Then, substituting (B.23) and (B.24) into (B.22) produces

$$\begin{aligned}
(n+\ell+1)\, r(n,\ell+1) &= -4n + 2\ell + 4 + \left(\ell^2 + n\ell + 2\ell - 3\right) 2^{1-n} + (\ell+4)\, 2^{-\ell} \\
&\quad + \frac{n+\ell+2}{2} \sum_{k=1}^{n} r(k,\ell) 2^{k-n} - \frac{1}{4} \sum_{k=1}^{\ell} r(n,k) 2^{k-\ell} \\
&\quad + \left[(n+\ell+1)\left(2^{-1-\ell} - 2^{-n}\right) + 1\right] \left[\frac{n^2+\ell^2+n+\ell}{n+\ell+1} - \frac{f}{(n+\ell+1)^2}\right] \\
&\quad + \left(n+\ell+\frac{1}{2}\right) \left[\frac{n^2+n}{n+\ell+1} + \frac{1}{n+\ell+1} \sum_{k=1}^{\ell} r(n,k) - \frac{f}{(n+\ell+1)^2}\right] \\
&\quad - \frac{n+\ell+2}{2} \left[\frac{\ell^2+\ell}{n+\ell+1} + \frac{1}{n+\ell+1} \sum_{k=1}^{n} r(k,\ell) - \frac{f}{(n+\ell+1)^2}\right] \\
&\quad - \frac{n+\ell+1}{2} \sum_{k=1}^{n} \left[\frac{n^2+(1-2k)n+2k^2}{n+\ell+1} + \frac{1}{n+\ell+1} \sum_{j=1}^{\ell} r(k,j) - \frac{f}{(n+\ell+1)^2}\right] 2^{k-n} \\
&\quad + \frac{n+\ell+1}{4} \sum_{k=1}^{\ell} \left[\frac{\ell^2+(1-2k)\ell+2k^2}{n+\ell+1} + \frac{1}{n+\ell+1} \sum_{j=1}^{n} r(j,k) - \frac{f}{(n+\ell+1)^2}\right] 2^{k-\ell},
\end{aligned}$$

which, by changing indices and using the results of Lemma B.1, simplifies to

$$\begin{aligned}
(n+\ell+1)\, r(n,\ell+1) &= \frac{-3n^2+3\ell^2-2n\ell-n+5\ell+2}{2(n+\ell+1)} + \left(\ell^2+2n\ell+2n+3\ell\right) 2^{-n} \\
&\quad + \left(n^2+n+2\right) 2^{-1-\ell} + \frac{1}{4} \sum_{k=1}^{\ell} \left(4 - \frac{2}{n+\ell+1} - 2^{k-\ell}\right) r(n,k) \\
&\quad - \frac{n+\ell+2}{2} \sum_{k=1}^{n} \left(\frac{1}{n+\ell+1} - 2^{k-n}\right) r(k,\ell) - \frac{1}{4} \sum_{k=1}^{n} \sum_{j=1}^{\ell} \left(2^{1+k-n} - 2^{j-\ell}\right) r(k,j).
\end{aligned} \tag{B.25}$$

Finally, dividing (B.25) through by $n+\ell+1$ produces our desired result. □

B.3 Finite series

The following series are either well-known or special cases of well-known series. The first two and the general cases of the third and fourth usually appear in any introductory mathematical text that covers series (e.g. §4.2 of [86]). The fifth is slightly more obscure.

Lemma B.1. *For integer values of* $n > 0$*,*

$$\textit{(i)} \quad \sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \tag{B.26}$$

$$\textit{(ii)} \quad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \tag{B.27}$$

$$\textit{(iii)} \quad \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}, \tag{B.28}$$

$$\textit{(iv)} \quad \sum_{k=1}^{n} k2^{-k} = 2 - (n+2)2^{-n}, \textit{ and} \tag{B.29}$$

$$\textit{(v)} \quad \sum_{k=1}^{n} k^2 2^{-k} = 6 - \left(n^2 + 4n + 6\right) 2^{-n}. \tag{B.30}$$

Proof. Equations (B.26) and (B.27) are special cases of Equation (6.2.1) in [50], while (B.28), (B.29) and (B.30) are special cases of Equation (6.9.1) in [50]. All are easily proved using induction. □

B.3.1 Finite series of binomial coefficients

Although there are many interpretations and uses of binomial coefficients, we will simply assume two basic facts about them, namely *Pascal's rule*;

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \; 1 \leq k \leq n-1, \tag{B.31}$$

and the *binomial formula*;

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},\ n \geq 0. \tag{B.32}$$

Pascal's rule follows easily from (2.6) while the binomial formula can be inductively proved using Pascal's rule. Equations (B.31) and (B.32) can also be found in standard introductory mathematics texts, such as §1.5–§1.6 in [86].

We can use Pascal's rule to derive some identities involving binomial coefficients. These identities include the two in the following lemma.

Lemma B.2. *For integer values of n, m and k, with $n > 0$, $m \geq 0$ and $0 \leq k \leq m$,*

(i) $$\sum_{i=1}^{n} \binom{m+i}{m+1} = \binom{n+m+1}{m+2}, \text{ and} \tag{B.33}$$

(ii) $$\sum_{i=1}^{n} \binom{m+i}{k+i} = \binom{n+m+1}{n+k} - \binom{m+1}{k}. \tag{B.34}$$

Proof. (i) We will proceed by induction. By direct evaluation, (B.33) holds for $n = 1$, as $\binom{m+1}{m+1} = 1 = \binom{m+2}{m+2}$. Suppose that it holds for $n = \ell$. Then

$$\begin{aligned}
\sum_{i=1}^{\ell+1} \binom{m+i}{m+1} &= \sum_{i=1}^{\ell} \binom{m+i}{m+1} + \binom{m+\ell+1}{m+1} \\
&= \binom{m+\ell+1}{m+2} + \binom{m+\ell+1}{m+1} \\
&= \binom{m+\ell+2}{m+2} \text{ by Pascal's rule,}
\end{aligned}$$

and so (B.33) then holds for $n = \ell + 1$ as well. Thus it holds for all $n > 0$. Note that all these binomial coefficients are well-defined for $m \geq 0$.

(ii) Once again, we will proceed by induction. By direct evaluation, (B.34) holds for $n = 1$, as $\binom{m+1}{k+1} = \binom{m+2}{k+1} - \binom{m+1}{k}$ by Pascal's rule. Suppose that it holds for

$n = \ell$. Then

$$\begin{aligned}\sum_{i=1}^{\ell+1} \binom{m+i}{k+i} &= \sum_{i=1}^{\ell} \binom{m+i}{k+i} + \binom{m+\ell+1}{k+\ell+1} \\ &= \binom{m+\ell+1}{k+\ell} + \binom{m+\ell+1}{k+\ell+1} - \binom{m+1}{k} \\ &= \binom{m+\ell+2}{k+\ell+1} - \binom{m+1}{k} \text{ by Pascal's rule,}\end{aligned}$$

and so (B.34) then holds for $n = \ell+1$ as well. Thus it holds for all $n > 0$. Note that all these binomial coefficients are well-defined for $m \geq 0$ and $0 \leq k \leq m$.

□

A special case of the binomial formula can be found by substituting $y = 1$ into (B.32), which gives

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i, \ n \geq 0. \tag{B.35}$$

Differentiating this expression with respect to x gives us

$$n\,(1+x)^{n-1} = \sum_{i=0}^{n} i\binom{n}{i} x^{i-1}, \ n \geq 1. \tag{B.36}$$

In the following results, we will make use of a few "well-known" series of binomial coefficients (for example, the first two can be found in Chapter 3 of [93] and all can be solved by Mathematica). Since they are not as standard as the basic facts stated above, we will include a brief proof of them for the sake of completeness.

Lemma B.3 (Standard sums of binomial coefficients)**.** *For integer values of n,*

(i) $$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = 2^{n-1}, \ n > 0, \tag{B.37}$$

(ii) $$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} = 2^{n-1}, \ n > 0, \tag{B.38}$$

(iii) $$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2i\binom{n}{2i} = n2^{n-2},\ n > 1,\ \textit{and} \tag{B.39}$$

(iv) $$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (2i+1)\binom{n}{2i+1} = n2^{n-2},\ n > 1. \tag{B.40}$$

Proof. Substituting $x = \pm 1$ into (B.35) gives us

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \text{ and}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

for any $n > 0$. Equations (B.37) and (B.38) can be found by taking the sum and difference of these two expressions and dividing by 2. Similarly, substituting $x = \pm 1$ into (B.36) gives us

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1} \text{ and}$$

$$\sum_{i=0}^{n} (-1)^{i-1} i\binom{n}{i} = 0$$

for any $n > 1$. Equations (B.39) and (B.40) can be found by taking the sum and difference of these two expressions and dividing by 2. □

We can now use the results from Lemma B.3 to derive some more specialised series. These are summarised in the following lemma. As a point of notation, we will assume that any sum not containing any terms (e.g. $\sum_{i=0}^{-1} a_i$) is equal to zero.

Lemma B.4 (Specialised sums of binomial coefficients)**.** *For integer values of* $p \geq 0$*,*

(i) $$\sum_{i=1}^{p+1} i\binom{2p+4}{2i+2} = p2^{2p+2} + 1, \tag{B.41}$$

(ii) $$\sum_{i=1}^{p} i\binom{2p+2}{2i+1} = p2^{2p}, \tag{B.42}$$

(iii) $$\sum_{i=1}^{p+1} i\binom{2p+4}{2i+1} = (p+1)\,2^{2p+2}, \tag{B.43}$$

(iv) $$\sum_{i=1}^{p} i\binom{2p+3}{2i+2} = (2p-1)\,2^{2p}+1, \tag{B.44}$$

(v) $$\sum_{i=1}^{p} i\binom{2p+3}{2i+1} = (2p+1)\,2^{2p}-p-1, \tag{B.45}$$

(vi) $$\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\binom{k+2}{2i+1} = p^2+\frac{1}{2}p, \tag{B.46}$$

(vii) $$\sum_{i=1}^{p}\sum_{k=2i-1}^{2p-1} i2^{-k}\binom{k+2}{2i+1} = p^2-\frac{1}{2}p, \textit{ and} \tag{B.47}$$

(viii) $$\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\binom{k+2}{2i+1} = p^2+\frac{3}{2}p+\frac{1}{2}. \tag{B.48}$$

Proof. (i) This follows by substituting $n = 2p+4$ into (B.39) and (B.37), taking the difference between the first expression and twice the second, removing the $i = 0$ and $i = 1$ terms, shifting indices by 1, and then dividing by 2. Note that by the conditions on (B.39) and (B.37), this is true for $p \geq 0$.

(ii) This follows by substituting $n = 2p+2$ into (B.40) and (B.38), taking the difference between these expressions and dividing by 2. Note that by the conditions on (B.40) and (B.38), this is true for $p \geq 0$.

(iii) This follows by substituting $p+1$ for p in (B.42).

(iv) This follows by substituting $p-1$ for p in (B.41), adding this to (B.42), and using Pascal's rule to say that

$$\binom{2p+2}{2i+1} + \binom{2p+2}{2i+2} = \binom{2p+3}{2i+2}.$$

(v) This follows by substituting $n = 2p+3$ into (B.40) and (B.38), taking the difference between these expressions, dividing by 2 and taking the final term

out of the sum. Note that by the conditions on (B.40) and (B.38), this is true for $p \geq 0$.

(vi) Let $s(p)$ represent the value of this sum, as a function of p. That is,

$$s(p) := \sum_{i=1}^{p} \sum_{k=2i-1}^{2p} i2^{-k} \binom{k+2}{2i+1}.$$

Then we can see that $s(0) = 0$ and furthermore,

$$\begin{aligned}
s(p+1) &= \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+2} i2^{-k} \binom{k+2}{2i+1} \\
&= \sum_{i=1}^{p} \sum_{k=2i-1}^{2p} i2^{-k} \binom{k+2}{2i+1} \\
&\quad + 2^{-2p-1} \sum_{i=1}^{p+1} i \binom{2p+3}{2i+1} + 2^{-2p-2} \sum_{i=1}^{p+1} i \binom{2p+4}{2i+1} \\
&= s(p) + 2^{-2p-1} (2p+1) 2^{2p} + 2^{-2p-2} (p+1) 2^{2p+2} \\
&\qquad\qquad \text{(by (B.45) and (B.43))} \\
&= s(p) + 2p + \frac{3}{2} \\
&= s(p) + 2(p+1) - \frac{1}{2}.
\end{aligned}$$

Thus, we can say that

$$\begin{aligned}
s(p) &= \sum_{k=1}^{p} \left(2k - \frac{1}{2} \right) \\
&= 2 \sum_{k=1}^{p} k - \frac{1}{2} \sum_{k=1}^{p} 1 \\
&= p(p+1) - \frac{1}{2} p \text{ (using (B.26))} \\
&= p^2 + \frac{1}{2} p.
\end{aligned}$$

(vii) We can see that

$$\sum_{i=1}^{p}\sum_{k=2i-1}^{2p-1} i2^{-k}\binom{k+2}{2i+1} = \sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\binom{k+2}{2i+1} - 2^{-2p}\sum_{i=1}^{p} i\binom{2p+2}{2i+1},$$

and then the result follows from (B.46) and (B.42).

(viii) This follows by substituting $p+1$ for p in (B.47).

□

In addition to these series evaluations, the following series manipulations will prove to be useful.

Lemma B.5 (Equivalent binomial series)**.** *For integer values of $p \geq 0$ and $n \geq 0$,*

(i) $$\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\tbinom{n+k+3}{n+2i+2} = \sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k+1}\tbinom{n+k+2}{n+2i+1} - 2^{-2p-1}\sum_{i=1}^{p+1} i\tbinom{n+2p+4}{n+2i+2}, \quad \text{(B.49)}$$

and

(ii) $$\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\tbinom{n+k+3}{n+2i+2} = \sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k+1}\tbinom{n+k+2}{n+2i+1} - 2^{-2p}\sum_{i=1}^{p} i\tbinom{n+2p+3}{n+2i+2}. \quad \text{(B.50)}$$

Proof. (i) First, let us suppose that $p \geq 0$, $n \geq 0$ and i is an integer between 1 and $p+1$ (inclusive). Then, we can use Pascal's rule with $k > 2i-1$ to write

$$\binom{n+k+3}{n+2i+2} = \binom{n+k+2}{n+2i+2} + \binom{n+k+2}{n+2i+1},$$

while for $k = 2i-1$ we can say

$$\binom{n+k+3}{n+2i+2} = \binom{n+2i+2}{n+2i+2} = 1 = \binom{n+2i+1}{n+2i+1} = \binom{n+k+2}{n+2i+1}.$$

With these two facts, we can write

$$\sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+3}{n+2i+2} = \sum_{k=2i}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+2} + \sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+1}. \quad \text{(B.51)}$$

By shifting indices by 1, the first sum on the right becomes

$$\begin{aligned}\sum_{k=2i}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+2} &= \sum_{k=2i-1}^{2p} 2^{-k-1}\binom{n+k+3}{n+2i+2} \\ &= \frac{1}{2}\sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+3}{n+2i+2} - 2^{-2p-2}\binom{n+2p+4}{n+2i+2} \\ &= \frac{1}{2}\sum_{k=2i}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+2} + \frac{1}{2}\sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+1} \\ &\qquad - 2^{-2p-2}\binom{n+2p+4}{n+2i+2} \quad \text{(by (B.51))}.\end{aligned}$$

Thus

$$\sum_{k=2i}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+2} = \sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+1} - 2^{-2p-1}\binom{n+2p+4}{n+2i+2},$$

and so (B.51) becomes

$$\sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+3}{n+2i+2} = 2\sum_{k=2i-1}^{2p+1} 2^{-k}\binom{n+k+2}{n+2i+1} - 2^{-2p-1}\binom{n+2p+4}{n+2i+2}.$$

Substituting this expression into the left hand side of (B.49) produces the desired result.

(ii) Again, let us suppose that $p \geq 0$, $n \geq 0$ and i is an integer, now between 1 and p (inclusive). As above, we can use Pascal's rule to write

$$\sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+3}{n+2i+2} = \sum_{k=2i}^{2p} 2^{-k}\binom{n+k+2}{n+2i+2} + \sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+2}{n+2i+1}. \tag{B.52}$$

By shifting indices by 1, the first sum on the right becomes

$$\sum_{k=2i}^{2p} 2^{-k}\binom{n+k+2}{n+2i+2} = \sum_{k=2i-1}^{2p-1} 2^{-k-1}\binom{n+k+3}{n+2i+2}$$

$$= \frac{1}{2}\sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+3}{n+2i+2} - 2^{-2p-1}\binom{n+2p+3}{n+2i+2}$$
$$= \frac{1}{2}\sum_{k=2i}^{2p} 2^{-k}\binom{n+k+2}{n+2i+2} + \frac{1}{2}\sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+2}{n+2i+1}$$
$$- 2^{-2p-1}\binom{n+2p+3}{n+2i+2} \quad \text{(by (B.52))}.$$

Thus

$$\sum_{k=2i}^{2p} 2^{-k}\binom{n+k+2}{n+2i+2} = \sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+2}{n+2i+1} - 2^{-2p}\binom{n+2p+3}{n+2i+2},$$

and so (B.52) becomes

$$\sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+3}{n+2i+2} = 2\sum_{k=2i-1}^{2p} 2^{-k}\binom{n+k+2}{n+2i+1} - 2^{-2p}\binom{n+2p+3}{n+2i+2}.$$

Substituting this expression into the left hand side of (B.50) produces the desired result.

□

Now, we can use Lemmas B.4 and B.5 to evaluate two more complicated expressions which will be necessary for the proof of Theorem 5.3.

Lemma B.6. *Let p and n be non-negative integers, and let*

$$g(n,p) := \frac{4p^2+6p+2}{n+2p+2} + 4p + \left(4p^2+4np+4n+10p+6\right)2^{1-n} + 2^{-2p}$$
$$+ 2^{-n-2p}\sum_{i=1}^{p+1} i\left\{2\binom{n+2p+4}{n+2i+1} - \binom{n+2p+4}{n+2i+2} - (2n+4p+6)\binom{2p+4}{2i+1}\right\}$$
$$+ 2^{2-n}\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\left\{\frac{n+2p+1}{n+2p+2}\binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+3}{2i+1}\right\}$$
$$+ 2^{-2p}\sum_{i=1}^{p+1}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+4}{k+2i+2} - \frac{2n+4p+6}{n+2p+2}\binom{k+2p+3}{k+2i+1}\right\}. \quad \text{(B.53)}$$

Then $g(n,p) = 0 \;\; \forall\, n, p \geq 0$.

Proof. First, we can use (B.43) to simplify the third term in the first sum. In addition, the third term in the second sum can be written as $\binom{k+2}{2i+1} + \binom{k+2}{2i}$ using Pascal's rule for $k \geq 2i-1$. We can then apply (B.48) to the $\binom{k+2}{2i+1}$ term. This gives us

$$\begin{aligned} g(n,p) = \frac{4p^2+6p+2}{n+2p+2} + 4p - \left(2p^2+7p+5\right)2^{1-n} + 2^{-2p} \\ + 2^{-n-2p}\sum_{i=1}^{p+1} i\left\{2\binom{n+2p+4}{n+2i+1} - \binom{n+2p+4}{n+2i+2}\right\} \\ + 2^{2-n}\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\left\{\frac{n+2p+1}{n+2p+2}\binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+2}{2i}\right\} \\ + 2^{-2p}\sum_{i=1}^{p+1}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+4}{k+2i+2} - \frac{2n+4p+6}{n+2p+2}\binom{k+2p+3}{k+2i+1}\right\}. \end{aligned} \tag{B.54}$$

Next, we will consider the case when $p = 0$. Then our expression becomes

$$\begin{aligned} g(n,0) &= \frac{2}{n+2} - (5)\,2^{1-n} + 1 + 2^{-n}\left\{2\binom{n+4}{n+3} - \binom{n+4}{n+4}\right\} \\ &\quad + 2^{1-n}\left\{\frac{n+1}{n+2}\binom{n+3}{n+3} - \binom{n+3}{n+2} + \binom{3}{2}\right\} \\ &\quad + \sum_{k=1}^{n} 2^{-k}\left\{\binom{k+4}{k+4} - \frac{2n+6}{n+2}\binom{k+3}{k+3}\right\} \\ &= \frac{n+4}{n+2} + \left(-10 + 2\,(n+4) - 1 + \frac{2n+2}{n+2} - 2\,(n+3) + 6\right)2^{-n} \\ &\quad - \frac{n+4}{n+2}\sum_{k=1}^{n} 2^{-k} \\ &= \frac{n+4}{n+2}\left\{1 - 2^{-n} - \sum_{k=1}^{n} 2^{-k}\right\} \\ &= 0 \;\forall\, n \geq 0 \quad \text{by (B.28).} \end{aligned} \tag{B.55}$$

Thus, in the rest of the proof, we will assume that $p > 0$. Now, when $n = 0$, our

expression becomes

$$
\begin{aligned}
g(0,p) &= \frac{4p^2+6p+2}{2p+2} + 4p - 2\left(2p^2+7p+5\right) + 2^{-2p} \\
&\qquad + 2^{-2p}\sum_{i=1}^{p+1} i\left\{2\binom{2p+4}{2i+1} - \binom{2p+4}{2i+2}\right\} \\
&\qquad\qquad + 4\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\left\{\frac{2p+1}{2p+2}\binom{k+2}{2i+1} - \binom{k+2}{2i} + \binom{k+2}{2i}\right\} \\
&= -4p^2 - 8p - 9 + 2^{-2p} + 2^{1-2p}\sum_{i=1}^{p+1} i\binom{2p+4}{2i+1} - 2^{-2p}\sum_{i=1}^{p+1} i\binom{2p+4}{2i+2} \\
&\qquad + \frac{4p+2}{p+1}\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\binom{k+2}{2i+1} \\
&= 0 \quad \text{by (B.43), (B.41) and (B.48).}
\end{aligned} \tag{B.56}
$$

Next, let us consider $g(n+1,p)$. Substituting $n+1$ in for n in (B.54) gives us

$$
\begin{aligned}
g(n+1,p) &= \frac{4p^2+6p+2}{n+2p+3} + 4p - \left(2p^2+7p+5\right)2^{-n} + 2^{-2p} \\
&\qquad + 2^{-n-2p-1}\sum_{i=1}^{p+1} i\left\{2\binom{n+2p+5}{n+2i+2} - \binom{n+2p+5}{n+2i+3}\right\} \\
&\quad + 2^{1-n}\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\left\{\frac{n+2p+2}{n+2p+3}\binom{n+k+3}{n+2i+2} - \binom{n+k+3}{n+2i+1} + \binom{k+2}{2i}\right\} \\
&\qquad + 2^{-2p}\sum_{i=1}^{p+1}\sum_{k=1}^{n+1} i2^{-k}\left\{\binom{k+2p+4}{k+2i+2} - \frac{2n+4p+8}{n+2p+3}\binom{k+2p+3}{k+2i+1}\right\}.
\end{aligned}
$$

We can take the $k = n+1$ terms out of the final sum and apply (B.31) and (B.49) to produce

$$
\begin{aligned}
g(n+1,p) &= \frac{4p^2+6p+2}{n+2p+3} + 4p - \left(2p^2+7p+5\right)2^{-n} + 2^{-2p} \\
&\qquad + 2^{-n-2p-1}\sum_{i=1}^{p+1} i\left\{2\binom{n+2p+4}{n+2i+1} - 2\binom{n+2p+4}{n+2i+2}\right\}
\end{aligned}
$$

$$+ 2^{1-n} \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+1} i 2^{-k} \left\{ \frac{n+2p+1}{n+2p+3} \binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+2}{2i} \right\}$$
$$+ 2^{-2p} \sum_{i=1}^{p+1} \sum_{k=1}^{n} i 2^{-k} \left\{ \binom{k+2p+4}{k+2i+2} - \frac{2n+4p+8}{n+2p+3} \binom{k+2p+3}{k+2i+1} \right\}. \quad \text{(B.57)}$$

Now, let us define a new function, $a(n,p)$, as

$$a(n,p) := (n+2p+3)\, g(n+1,p) - (n+2p+2)\, g(n,p). \quad \text{(B.58)}$$

Then, from (B.54) and (B.57), we obtain

$$a(n,p) = 4p + \left(4p^3 + 2np^2 + 16p^2 + 7np + 5n + 17p + 5\right) 2^{-n} + 2^{-2p}$$
$$+ 2^{-n-2p} \sum_{i=1}^{p+1} i \left\{ -(n+2p+1) \binom{n+2p+4}{n+2i+1} - \binom{n+2p+4}{n+2i+2} \right\}$$
$$+ (n+2p+1)\, 2^{1-n} \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+1} i 2^{-k} \left\{ -\binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i} \right\}$$
$$+ 2^{-2p} \sum_{i=1}^{p+1} \sum_{k=1}^{n} i 2^{-k} \left\{ \binom{k+2p+4}{k+2i+2} - 2\binom{k+2p+3}{k+2i+1} \right\}. \quad \text{(B.59)}$$

When $n = 0$, this expression becomes

$$a(0,p) = 4p + \left(4p^3 + 16p^2 + 17p + 5\right) + 2^{-2p} - (2p+1)\, 2^{-2p} \sum_{i=1}^{p+1} i \binom{2p+4}{2i+1}$$
$$- 2^{-2p} \sum_{i=1}^{p+1} i \binom{2p+4}{2i+2} - (4p+2) \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+1} i 2^{-k} \binom{k+2}{2i+1}$$
$$= 0 \quad \text{by (B.43), (B.41) and (B.48).} \quad \text{(B.60)}$$

In a similar manner as before, we will next consider $a(n+1,p)$. Substituting $n+1$ in for n in (B.59) produces

$$a(n+1,p) = 4p + \left(4p^3 + 2np^2 + 18p^2 + 7np + 5n + 24p + 10\right) 2^{-n-1} + 2^{-2p}$$

$$+ 2^{-n-2p-1} \sum_{i=1}^{p+1} i \left\{ - (n+2p+2) \binom{n+2p+5}{n+2i+2} - \binom{n+2p+5}{n+2i+3} \right\}$$
$$+ (n+2p+2)\, 2^{-n} \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+1} i 2^{-k} \left\{ - \binom{n+k+3}{n+2i+2} + \binom{n+k+3}{n+2i+1} - \binom{k+2}{2i} \right\}$$
$$+ 2^{-2p} \sum_{i=1}^{p+1} \sum_{k=1}^{n+1} i 2^{-k} \left\{ \binom{k+2p+4}{k+2i+2} - 2 \binom{k+2p+3}{k+2i+1} \right\}.$$

As before, we can take the $k = n+1$ terms out of the final sum and apply (B.31) and (B.49) to obtain

$$a(n+1,p) = 4p + \left(4p^3 + 2np^2 + 18p^2 + 7np + 5n + 24p + 10\right) 2^{-n-1} + 2^{-2p}$$
$$+ 2^{-n-2p-1} \sum_{i=1}^{p+1} i \left\{ - (n+2p+2) \binom{n+2p+4}{n+2i+1} - 2 \binom{n+2p+4}{n+2i+2} \right\}$$
$$+ (n+2p+2)\, 2^{-n} \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+1} i 2^{-k} \left\{ - \binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i} \right\}$$
$$+ 2^{-2p} \sum_{i=1}^{p+1} \sum_{k=1}^{n} i 2^{-k} \left\{ \binom{k+2p+4}{k+2i+2} - 2 \binom{k+2p+3}{k+2i+1} \right\}. \quad \text{(B.61)}$$

Once again, we will define a new function, $b(n,p)$, as

$$b(n,p) := \frac{a(n+1,p) - a(n,p)}{n+2p}. \quad \text{(B.62)}$$

Note that $b(n,p)$ is well-defined since its denominator is positive for all $p > 0$ and $n \geq 0$. Then, from (B.59) and (B.61), we obtain

$$b(n,p) = - \left(2p^2 + 7p + 5\right) 2^{-n-1} + 2^{-n-2p-1} \sum_{i=1}^{p+1} i \binom{n+2p+4}{n+2i+1}$$
$$- 2^{-n} \sum_{i=1}^{p+1} \sum_{k=2i-1}^{2p+1} i 2^{-k} \left\{ - \binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i} \right\}. \quad \text{(B.63)}$$

When $n = 0$, this expression becomes

$$b(0,p) = -p^2 - \frac{7}{2}p - \frac{5}{2} + 2^{-2p-1}\sum_{i=1}^{p+1} i\binom{2p+4}{2i+1} + \sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\binom{k+2}{2i+1}$$
$$= 0 \text{ by (B.43) and (B.48).} \tag{B.64}$$

Finally, we will follow our previous procedure once more and consider $b(n+1,p)$. Substituting $n+1$ in for n in (B.63) and using (B.31) and (B.49) produces

$$b(n+1,p) = -\left(2p^2+7p+5\right)2^{-n-2} + 2^{-n-2p-2}\sum_{i=1}^{p+1} i\binom{n+2p+4}{n+2i+1}$$
$$- 2^{-n-1}\sum_{i=1}^{p+1}\sum_{k=2i-1}^{2p+1} i2^{-k}\left\{-\binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i}\right\}$$
$$= \frac{1}{2}b(n,p) \text{ by (B.63).} \tag{B.65}$$

Hence, from (B.64) and (B.65) we conclude that

$$b(n,p) = 0 \ \ \forall\, n \geq 0,\ p > 0.$$

Substituting this result into (B.62) tells us that

$$a(n+1,p) = a(n,p) \ \ \forall\, n \geq 0,\ p > 0,$$

which, along with (B.60), allows us to conclude that

$$a(n,p) = 0 \ \ \forall\, n \geq 0,\ p > 0.$$

Finally, we can substitute this result into (B.58) to find that

$$g(n+1,p) = \frac{n+2p+2}{n+2p+3}g(n,p) \ \ \forall\, n \geq 0,\ p > 0,$$

which, along with (B.56), allows us to conclude that

$$g(n,p) = 0 \;\; \forall\, n \geq 0,\; p > 0. \tag{B.66}$$

□

Lemma B.7. *Let p and n be non-negative integers, and let*

$$\begin{aligned} h(n,p) := \frac{4p^2+2p}{n+2p+1} + 4p - 2 + \left(4p^2+4np+2n+6p+2\right)2^{1-n} + 2^{1-2p} \\ - \left(4p^2+2np+2n+6p+2\right)2^{1-n-2p} \\ + 2^{1-n-2p}\sum_{i=1}^{p} i\left\{2\binom{n+2p+3}{n+2i+1} - \binom{n+2p+3}{n+2i+2} - (2n+4p+4)\binom{2p+3}{2i+1}\right\} \\ + 2^{2-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{\frac{n+2p}{n+2p+1}\binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+3}{2i+1}\right\} \\ + 2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2} - \frac{2n+4p+4}{n+2p+1}\binom{k+2p+2}{k+2i+1}\right\}. \end{aligned} \tag{B.67}$$

Then $h(n,p) = 0 \;\; \forall\, n,p \geq 0$.

Proof. First, we can use (B.45) to simplify the third term in the first sum. In addition, the third term in the second sum can be written as $\binom{k+2}{2i+1} + \binom{k+2}{2i}$ using Pascal's rule for $k \geq 2i-1$. We can then apply (B.46) to the $\binom{k+2}{2i+1}$ term. This gives us

$$\begin{aligned} h(n,p) = \frac{4p^2+2p}{n+2p+1} + 4p - 2 - \left(2p^2+5p+2\right)2^{1-n} + (p+1)\,2^{2-n-2p} + 2^{1-2p} \\ + 2^{1-n-2p}\sum_{i=1}^{p} i\left\{2\binom{n+2p+3}{n+2i+1} - \binom{n+2p+3}{n+2i+2}\right\} \\ + 2^{2-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{\frac{n+2p}{n+2p+1}\binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+2}{2i}\right\} \\ + 2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2} - \frac{2n+4p+4}{n+2p+1}\binom{k+2p+2}{k+2i+1}\right\}. \end{aligned} \tag{B.68}$$

Next, we will consider the case when $p = 0$. Then our expression becomes

$$\begin{aligned} h(n,0) &= 0 - 2 - 2^{2-n} + 2^{2-n} + 2 \\ &= 0 \;\forall\; n \geq 0. \end{aligned} \tag{B.69}$$

Thus, in the rest of the proof, we will assume that $p > 0$. Now, when $n = 0$, our expression becomes

$$\begin{aligned} h(0,p) = {} & \frac{4p^2+2p}{2p+1} + 4p - 2 - 2\left(2p^2+5p+2\right) + (p+1)\,2^{2-2p} + 2^{1-2p} \\ & + 2^{1-2p}\sum_{i=1}^{p} i\left\{2\binom{2p+3}{2i+1} - \binom{2p+3}{2i+2}\right\} \\ & + 4\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{\frac{2p}{2p+1}\binom{k+2}{2i+1} - \binom{k+2}{2i} + \binom{k+2}{2i}\right\} \\ = {} & -4p^2 - 4p - 6 + (2p+3)\,2^{1-2p} + 2^{2-2p}\sum_{i=1}^{p} i\binom{2p+3}{2i+1} \\ & - 2^{1-2p}\sum_{i=1}^{p} i\binom{2p+3}{2i+2} + \frac{8p}{2p+1}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\binom{k+2}{2i+1} \\ = {} & 0 \;\text{ by (B.45), (B.44) and (B.46).} \end{aligned} \tag{B.70}$$

Next, let us consider $h(n+1,p)$. Substituting $n+1$ in for n in (B.68) gives us

$$\begin{aligned} h(n+1,p) = {} & \frac{4p^2+2p}{n+2p+2} + 4p - 2 - \left(2p^2+5p+2\right)2^{-n} + (p+1)\,2^{1-n-2p} + 2^{1-2p} \\ & + 2^{-n-2p}\sum_{i=1}^{p} i\left\{2\binom{n+2p+4}{n+2i+2} - \binom{n+2p+4}{n+2i+3}\right\} \\ & + 2^{1-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{\frac{n+2p+1}{n+2p+2}\binom{n+k+3}{n+2i+2} - \binom{n+k+3}{n+2i+1} + \binom{k+2}{2i}\right\} \\ & + 2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n+1} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2} - \frac{2n+4p+6}{n+2p+2}\binom{k+2p+2}{k+2i+1}\right\}. \end{aligned}$$

We can take the $k = n+1$ terms out of the final sum and apply (B.31) and (B.50)

to produce

$$h(n+1,p) = \frac{4p^2+2p}{n+2p+2} + 4p - 2 - \left(2p^2+5p+2\right)2^{-n} + (p+1)\,2^{1-n-2p} + 2^{1-2p}$$
$$+ 2^{-n-2p}\sum_{i=1}^{p} i\left\{2\binom{n+2p+3}{n+2i+1} - 2\binom{n+2p+3}{n+2i+2}\right\}$$
$$+ 2^{1-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{\frac{n+2p}{n+2p+2}\binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+2}{2i}\right\}$$
$$+ 2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2} - \frac{2n+4p+6}{n+2p+2}\binom{k+2p+2}{k+2i+1}\right\}. \quad \text{(B.71)}$$

Next, we will define a new function, $c(n,p)$ as

$$c(n,p) := (n+2p+2)\,h(n+1,p) - (n+2p+1)\,h(n,p). \quad \text{(B.72)}$$

Then, from (B.68) and (B.71), we obtain

$$c(n,p) = 4p - 2 + \left(4p^3+2np^2+10p^2+5np+2n+4p\right)2^{-n}$$
$$- \left(2p^2+np+n+2p\right)2^{1-n-2p}$$
$$+ 2^{1-2p} + 2^{-n-2p}\sum_{i=1}^{p} i\left\{-(2n+4p)\binom{n+2p+3}{n+2i+1} - 2\binom{n+2p+3}{n+2i+2}\right\}$$
$$+ (n+2p)\,2^{1-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{-\binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i}\right\}$$
$$+ 2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2} - 2\binom{k+2p+2}{k+2i+1}\right\}. \quad \text{(B.73)}$$

When $n=0$, this expression becomes

$$c(0,p) = 4p - 2 + \left(4p^3+10p^2+4p\right) - \left(2p^2+2p\right)2^{1-2p} + 2^{1-2p}$$
$$- p2^{2-2p}\sum_{i=1}^{p} i\binom{2p+3}{2i+1} - 2^{1-2p}\sum_{i=1}^{p} i\binom{2p+3}{2i+2} - 4p\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\binom{k+2}{2i+1}$$

$$= 0 \text{ by (B.45), (B.44) and (B.46).} \tag{B.74}$$

In a similar manner as before, we will next consider $c(n+1,p)$. Substituting $n+1$ in for n in (B.73) produces

$$\begin{aligned}
c(n+1,p) = 4p-2+\left(4p^3+2np^2+12p^2+5np+2n+9p+2\right)2^{-n-1}+2^{1-2p} \\
-\left(2p^2+np+n+3p+1\right)2^{-n-2p} \\
+2^{-n-2p-1}\sum_{i=1}^{p} i\left\{-(2n+4p+2)\binom{n+2p+4}{n+2i+2}-2\binom{n+2p+4}{n+2i+3}\right\} \\
+(n+2p+1)\,2^{-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{-\binom{n+k+3}{n+2i+2}+\binom{n+k+3}{n+2i+1}-\binom{k+2}{2i}\right\} \\
+2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n+1} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2}-2\binom{k+2p+2}{k+2i+1}\right\}.
\end{aligned}$$

As before, we can take the $k=n+1$ terms out of the final sum and apply (B.31) and (B.50) to obtain

$$\begin{aligned}
c(n+1,p) = 4p-2+\left(4p^3+2np^2+12p^2+5np+2n+9p+2\right)2^{-n-1}+2^{1-2p} \\
-\left(2p^2+np+n+3p+1\right)2^{-n-2p} \\
+2^{-n-2p}\sum_{i=1}^{p} i\left\{-(n+2p+1)\binom{n+2p+3}{n+2i+1}-2\binom{n+2p+3}{n+2i+2}\right\} \\
+(n+2p+1)\,2^{-n}\sum_{i=1}^{p}\sum_{k=2i-1}^{2p} i2^{-k}\left\{-\binom{n+k+2}{n+2i+1}+\binom{n+k+2}{n+2i}-\binom{k+2}{2i}\right\} \\
+2^{1-2p}\sum_{i=1}^{p}\sum_{k=1}^{n} i2^{-k}\left\{\binom{k+2p+3}{k+2i+2}-2\binom{k+2p+2}{k+2i+1}\right\}.
\end{aligned} \tag{B.75}$$

Once again, we will define a new function, $d(n,p)$ as

$$d(n,p) := \frac{c(n+1,p)-c(n,p)}{n+2p-1}. \tag{B.76}$$

Note that $d(n,p)$ is well-defined since its denominator is positive for all $p > 0$ and $n \geq 0$. Then, from (B.73) and (B.75), we obtain

$$\begin{aligned} d(n,p) = &- \left(2p^2+5p+2\right) 2^{-n-1} + (p+1)\, 2^{-n-2p} + 2^{-n-2p} \sum_{i=1}^{p} i \binom{n+2p+3}{n+2i+1} \\ &- 2^{-n} \sum_{i=1}^{p} \sum_{k=2i-1}^{2p} i 2^{-k} \left\{ -\binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i} \right\}. \end{aligned} \tag{B.77}$$

When $n = 0$, this expression becomes

$$\begin{aligned} d(0,p) &= -p^2 - \frac{5}{2}p - 1 + (p+1)\, 2^{-2p} + 2^{-2p} \sum_{i=1}^{p} i \binom{2p+3}{2i+1} + \sum_{i=1}^{p} \sum_{k=2i-1}^{2p} i 2^{-k} \binom{k+2}{2i+1} \\ &= 0 \quad \text{by (B.45) and (B.46).} \end{aligned} \tag{B.78}$$

Finally, we will follow our previous procedure once more and consider $d(n+1,p)$. Substituting $n+1$ in for n in (B.77) and using (B.31) and (B.50) produces

$$\begin{aligned} d(n+1,p) = &- \left(2p^2+5p+2\right) 2^{-n-2} + (p+1)\, 2^{-n-2p-1} + 2^{-n-2p-1} \sum_{i=1}^{p} i \binom{n+2p+3}{n+2i+1} \\ &- 2^{-n-1} \sum_{i=1}^{p} \sum_{k=2i-1}^{2p} i 2^{-k} \left\{ -\binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} - \binom{k+2}{2i} \right\} \\ = &\frac{1}{2} d(n,p) \text{ by (B.77).} \end{aligned} \tag{B.79}$$

Hence, from (B.78) and (B.79) we conclude that

$$d(n,p) = 0 \;\; \forall\, n \geq 0,\; p > 0.$$

Substituting this result into (B.76) tells us that

$$c(n+1,p) = c(n,p) \;\; \forall\, n \geq 0,\; p > 0,$$

which, along with (B.74), allows us to conclude that

$$c(n,p) = 0 \ \ \forall\, n \geq 0,\ p > 0.$$

Finally, we can substitute this result into (B.72) to find that

$$h(n+1,p) = \frac{n+2p+1}{n+2p+2} h(n,p) \ \ \forall\, n \geq 0,\ p > 0,$$

which, along with (B.70), allows us to conclude that

$$h(n,p) = 0 \ \ \forall\, n \geq 0,\ p > 0. \tag{B.80}$$

□

Our final result covers some simplification required for the proof of Theorem 5.3.

Lemma B.8. *Suppose n and ℓ are positive integers, and let*

$$\begin{aligned} s(n,\ell) := \frac{-3n^2+3\ell^2-2n\ell-n+5\ell+2}{2(n+\ell+1)^2} + \frac{\ell^2+2n\ell+2n+3\ell}{n+\ell+1}2^{-n} + \frac{n^2+n+2}{2(n+\ell+1)}2^{-\ell} \\ + \frac{1}{2(n+\ell+1)}s_1(n,\ell) + \frac{1}{n+\ell+1}s_2(n,\ell) - \frac{n+\ell+2}{n+\ell+1}s_3(n,\ell) \\ - \frac{n+\ell+2}{n+\ell+1}s_4(n,\ell) - \frac{1}{2(n+\ell+1)}s_5(n,\ell) - \frac{1}{n+\ell+1}s_6(n,\ell), \text{ where} \end{aligned}$$

$$s_1(n,\ell) := \sum_{k=1}^{\ell} \left(4 - \frac{2}{n+\ell+1} - 2^{k-\ell}\right)(n-k),$$

$$s_2(n,\ell) := \sum_{k=1}^{\ell} \left(4 - \frac{2}{n+\ell+1} - 2^{k-\ell}\right) 2^{1-n-k} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} i \binom{n+k+2}{n+2i+1},$$

$$s_3(n,\ell) := \sum_{k=1}^{n} \left(\frac{1}{n+\ell+1} - 2^{k-n}\right)(k-\ell),$$

$$s_4(n,\ell) := \sum_{k=1}^{n} \left(\frac{1}{n+\ell+1} - 2^{k-n} \right) 2^{2-k-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} i \binom{k+\ell+2}{k+2i+1},$$

$$s_5(n,\ell) := \sum_{k=1}^{n} \sum_{j=1}^{\ell} \left(2^{1+k-n} - 2^{j-\ell} \right) (k-j), \text{ and}$$

$$s_6(n,\ell) := \sum_{k=1}^{n} \sum_{j=1}^{\ell} \left(2^{1+k-n} - 2^{j-\ell} \right) 2^{1-k-j} \sum_{i=1}^{\lfloor \frac{j+1}{2} \rfloor} i \binom{k+j+2}{k+2i+1}.$$

Then

$$\begin{aligned} s(n,\ell) = 2\left(n-\ell-1\right) + 2^{2-n-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} i \binom{n+\ell+3}{n+2i+1} + \frac{1}{n+\ell+1} \Bigg[\frac{\ell^2+\ell}{n+\ell+1} + 2\ell \\ - 2 + 2^{1-\ell} + \left(\ell^2 + 2n\ell + 2n + 3\ell + 2\right) 2^{1-n} \\ + 2^{1-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=1}^{n} i 2^{-k} \left\{ \binom{k+\ell+3}{k+2i+2} - \frac{2n+2\ell+4}{n+\ell+1} \binom{k+\ell+2}{k+2i+1} \right\} \\ + 2^{1-n-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} i \left\{ 2\binom{n+\ell+3}{n+2i+1} - \binom{n+\ell+3}{n+2i+2} - (2n+2\ell+4) \binom{\ell+3}{2i+1} \right\} \\ + 2^{2-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=2i-1}^{\ell} i 2^{-k} \left\{ \frac{n+\ell}{n+\ell+1} \binom{n+k+2}{n+2i+1} - \binom{n+k+2}{n+2i} + \binom{k+3}{2i+1} \right\} \Bigg]. \end{aligned} \tag{B.81}$$

Proof. Examining each of $s_1(n,\ell), s_2(n,\ell), s_3(n,\ell), s_4(n,\ell), s_5(n,\ell)$ and $s_6(n,\ell)$ in turn, we find that

$$\begin{aligned} s_1(n,\ell) &= \frac{4n^2\ell + 4n\ell^2 + 2n\ell}{n+\ell+1} - \frac{4n+4\ell+2}{n+\ell+1} \sum_{k=1}^{\ell} k - 2\sum_{k=1}^{\ell} \left(n+k-\ell-1\right) 2^{-k} \\ &= \frac{-2\ell^3 + 4n^2\ell + 2n\ell^2 - 3\ell^2 - \ell}{n+\ell+1} - 2\left(n-\ell-1\right) \sum_{k=1}^{\ell} 2^{-k} \\ &\qquad - 2\sum_{k=1}^{\ell} k 2^{-k} \text{ (using (B.26))} \end{aligned}$$

$$= \frac{-2\ell^3 + 4n^2\ell + 2n\ell^2 - 2n^2 - \ell^2 - 4n - \ell - 2}{n + \ell + 1}$$
$$+ (n+1)\, 2^{1-\ell} \text{ (using (B.28) and (B.29))},$$

$$s_2(n,\ell) = \left(4 - \frac{2}{n+\ell+1}\right) 2^{1-n} \sum_{k=1}^{\ell} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} i 2^{-k} \binom{n+k+2}{n+2i+1}$$
$$- 2^{1-n-\ell} \sum_{k=1}^{\ell} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} i \binom{n+k+2}{n+2i+1}$$
$$= \left(4 - \frac{2}{n+\ell+1}\right) 2^{1-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=2i-1}^{\ell} i 2^{-k} \binom{n+k+2}{n+2i+1}$$
$$- 2^{1-n-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} i \sum_{k=2i-1}^{\ell} \binom{n+k+2}{n+2i+1} \text{ (by exchanging indices)}$$
$$= \left(4 - \frac{2}{n+\ell+1}\right) 2^{1-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=2i-1}^{\ell} i 2^{-k} \binom{n+k+2}{n+2i+1}$$
$$- 2^{1-n-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} i \sum_{k=1}^{\ell-2i+2} \binom{n+2i+k}{n+2i+1}$$
$$= \left(4 - \frac{2}{n+\ell+1}\right) 2^{1-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=2i-1}^{\ell} i 2^{-k} \binom{n+k+2}{n+2i+1}$$
$$- 2^{1-n-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} i \binom{n+\ell+3}{n+2i+2} \text{ (using (B.33))},$$

$$s_3(n,\ell) = -\frac{n\ell}{n+\ell+1} + \frac{1}{n+\ell+1} \sum_{k=1}^{n} k - 2 \sum_{k=1}^{n} (n-k-\ell+1)\, 2^{-k}$$
$$= \frac{n^2 - 2n\ell + n}{2\,(n+\ell+1)} - 2\,(n-\ell+1) \sum_{k=1}^{n} 2^{-k} + 2 \sum_{k=1}^{n} k 2^{-k} \text{ (using (B.26))}$$
$$= \frac{-3n^2 + 4\ell^2 - 2n\ell + n + 8\ell + 4}{2\,(n+\ell+1)} - (\ell+1)\, 2^{1-n} \text{ (using (B.28) and (B.29))},$$

$$
\begin{aligned}
s_4(n,\ell) &= \frac{1}{n+\ell+1} 2^{2-\ell} \sum_{k=1}^{n} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} i 2^{-k} \binom{k+\ell+2}{k+2i+1} - 2^{2-n-\ell} \sum_{k=1}^{n} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} i \binom{k+\ell+2}{k+2i+1} \\
&= \frac{1}{n+\ell+1} 2^{2-\ell} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} \sum_{k=1}^{n} i 2^{-k} \binom{k+\ell+2}{k+2i+1} - 2^{2-n-\ell} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} i \sum_{k=1}^{n} \binom{k+\ell+2}{k+2i+1} \\
&\qquad \text{(by exchanging indices)} \\
&= \frac{1}{n+\ell+1} 2^{2-\ell} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} \sum_{k=1}^{n} i 2^{-k} \binom{k+\ell+2}{k+2i+1} - 2^{2-n-\ell} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} i \binom{n+\ell+3}{n+2i+1} \\
&\qquad + 2^{2-n-\ell} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} i \binom{\ell+3}{2i+1} \text{ (using (B.34)),}
\end{aligned}
$$

$$
\begin{aligned}
s_5(n,\ell) &= 4 \sum_{k=1}^{n} \sum_{j=1}^{\ell} (n+1-k-j)\, 2^{-k} - 2 \sum_{k=1}^{n} \sum_{j=1}^{\ell} (k+j-\ell-1)\, 2^{-j} \\
&= 4\,(n+1)\,\ell \sum_{k=1}^{n} 2^{-k} - 4\ell \sum_{k=1}^{n} k 2^{-k} - 4 \left(\sum_{j=1}^{\ell} j \right) \left(\sum_{k=1}^{n} 2^{-k} \right) \\
&\qquad + 2\,(\ell+1)\,n \sum_{j=1}^{\ell} 2^{-j} - 2n \sum_{j=1}^{\ell} j 2^{-j} - 2 \left(\sum_{k=1}^{n} k \right) \left(\sum_{j=1}^{\ell} 2^{-j} \right) \\
&= -n^2 - 2\ell^2 + 6n\ell - 3n - 6\ell + \left(\ell^2 + 3\ell\right) 2^{1-n} + \left(n^2 + 3n\right) 2^{-\ell} \\
&\qquad \text{(using (B.26), (B.28) and (B.29)), and}
\end{aligned}
$$

$$
\begin{aligned}
s_6(n,\ell) &= 2^{2-n} \sum_{k=1}^{n} \sum_{j=1}^{\ell} \sum_{i=1}^{\left\lfloor \frac{j+1}{2} \right\rfloor} i 2^{-j} \binom{k+j+2}{k+2i+1} - 2^{1-\ell} \sum_{k=1}^{n} \sum_{j=1}^{\ell} \sum_{i=1}^{\left\lfloor \frac{j+1}{2} \right\rfloor} i 2^{-k} \binom{k+j+2}{k+2i+1} \\
&= 2^{2-n} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} \sum_{j=2i-1}^{\ell} i 2^{-j} \sum_{k=1}^{n} \binom{k+j+2}{k+2i+1} \\
&\qquad - 2^{1-\ell} \sum_{i=1}^{\left\lfloor \frac{\ell+1}{2} \right\rfloor} \sum_{k=1}^{n} i 2^{-k} \sum_{j=2i-1}^{\ell} \binom{k+j+2}{k+2i+1} \text{ (by exchanging indices)}
\end{aligned}
$$

$$= 2^{2-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{j=2i-1}^{\ell} i 2^{-j} \binom{n+j+3}{n+2i+1} - 2^{2-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{j=2i-1}^{\ell} i 2^{-j} \binom{j+3}{2i+1}$$
$$- 2^{1-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=1}^{n} i 2^{-k} \sum_{j=1}^{\ell-2i+2} \binom{k+j+2i}{k+2i+1} \text{ (using (B.34))}$$
$$= 2^{2-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=2i-1}^{\ell} i 2^{-k} \left\{ \binom{n+k+2}{n+2i+1} + \binom{n+k+2}{n+2i} \right\}$$
$$- 2^{2-n} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=2i-1}^{\ell} i 2^{-k} \binom{k+3}{2i+1} - 2^{1-\ell} \sum_{i=1}^{\lfloor \frac{\ell+1}{2} \rfloor} \sum_{k=1}^{n} i 2^{-k} \binom{k+\ell+3}{k+2i+2}$$
(using (B.31) and (B.33)).

Substituting our simplified expressions for s_1, s_2, s_3, s_4, s_5 and s_6 into the definition of $s(n, \ell)$ gives us our desired result. □

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