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On the Linear Threshold Model for Diffusion of Innovations in Multiplex Social Networks

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Abstract— **Diffusion of innovations in social networks has been studied using the linear threshold model. These studies assume monoplex networks, where all connections are treated equally. To reflect the influence of different kinds of connections within social groups, we consider multiplex networks, which allow multiple layers of connections for a given set of nodes. We extend the linear threshold model to multiplex networks by designing protocols that combine signals from different layers. To analyze these protocols, we generalize the definition of live-edge models and reachability to the duplex setting. We introduce the live-edge tree and with it an algorithm to compute cascade centrality of individual nodes in a duplex network.**

I. INTRODUCTION

In multi-agent network models, nodes represent agents and edges represent sensing, communication or physical connections among agents. Typically, the network has a single layer of connections, where each connection refers to the same kind of sensing, communication or interaction. In real networks, however, there can be more than one mode of sensing, communication, or interaction among agents. For example, a group of friends or colleagues may be connected both through face-to-face interactions as well as through social media. Individuals in a crowd may be able to see people standing in front of them but may be able to hear people standing behind them. Here, we propose studying dynamics on *multiplex networks*, which allow multiple layers of connections for a given set of nodes [1].

Diffusion of innovations refers to the dynamic spread of an idea or activity. Young [2] discussed three approaches to modeling diffusion of innovation in networks: (i) dynamics in which agents adopt or reject an innovation deterministically by comparing the fraction of their neighbors that have adopted the innovation with a potentially random threshold; (ii) dynamics in which agents adopt or reject innovation probabilistically based on a coordination game played on the network; (iii) dynamics that allow network structure itself to evolve. The model from the first approach is sometimes referred to as the linear threshold model (LTM).

The LTM was first introduced in [3], [4] and its various applications including riot behavior, voting, and migration were discussed. Watts [5] used the LTM to explain cascades in random networks that are triggered by small initial shocks. Kempe et al. [6] and Lim et al. [7] studied diffusion of innovations in standard single-layer (monoplex) networks using a LTM with randomly drawn thresholds. Como et al. [8] studied LTM in large scale networks using a mean-field approximation and associated bifurcations. Lelarge [9] studied diffusion in a random network using the LTM and identified conditions for widespread adoption of innovation in the network.

The LTM is discrete-time network dynamics, in which a set of agents S_0 is initially active (has an innovation) and over time other agents become active (adopt an innovation) if a sufficiently large number of their neighbors are active (number is above a threshold). In [6] an equivalence is established between the LTM and a live-edge process, This live-edge process, referred to as live-edge model (LEM) in this paper, can be studied without temporal iteration. In the LEM, one edge among incoming edges is randomly selected for each agent, and connectivity of each agent with S_0 is examined.

The LEM is used to evaluate the *social influence* of a set of agents S_0, defined in [6] as the expected number of active agents after iterating over the initially active set S_0. In [7] the social influence of a single active agent is called *cascade centrality* and a closed-form expression is provided using the LEM. This expression requires enumerating every path between the single active agent and the rest of the agents in the network. Acemoglu et al. [10] studied the LTM for deterministic thresholds at each node; in this case, analysis of the LTM becomes very challenging and has limited analytic tractability. Yağan and Gligor [11] proposed a LTM for multiplex networks in which each node computes the weighted average fraction of its active neighbors in each layer and compares it to a randomly drawn threshold. They analyzed their model for random multiplex networks.

The problem of finding the set of k agents in a monoplex network that maximizes social influence in the LTM and in the alternative "independent cascade model" was proved to be NP-hard in [6], and approximations were used. Nguyen and Zheng [12] designed efficient algorithms to approximate social influence in the independent cascade model by casting the problem as statistical inference in a Bayesian network.

In this paper, we introduce the weighted linear threshold model with thresholds chosen uniformly at random in $[0, 1]$ for fixed multiplex networks and define protocols to combine inputs from different layers. We specialize to duplex networks (two-layer multiplex networks) and derive tools to analyze cascade dynamics. We make the following contributions for duplex networks. First, we define and analyze two LTM protocols: Protocol OR and Protocol AND, which de-

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scribe a sensitive and a conservative response, respectively, to active neighbors. Second, we generalize the live-edge model and introduce the notion of reachability to duplex networks, and prove equivalence of the duplex LEM to the duplex LTM. Third, we introduce the *live-edge tree*, a representation of the network topology, to compute reachability in the duplex LEM. Fourth, we define and provide an algorithm to compute duplex cascade centrality.

In Section II, we define multiplex networks and graph-related properties. In Section III, we define the multiplex LTM and propose protocols for the duplex LTM. In Section IV, we generalize the LEM to duplex networks and define reachability corresponding to the duplex LTM protocols. In Section V, we prove equivalence of the duplex LTM and LEM. In Section VI, we generalize social influence and cascade centrality to duplex networks and illustrate with an example. We conclude in Section VII.

II. MULTIPLEX NETWORKS

A *multiplex network* $\mathcal{G}$ is a family of $m \in \mathbb{N}$ directed weighted graphs $G_1, ..., G_m$. Each graph $G_k = (V, E^k)$, $k = 1, \dots, m$, is referred to as a layer of the multiplex network. The agent set $V = 1, 2, 3, ..., n$ is the same in all layers. The edge set of layer k is $E^k \subseteq V \times V$ and can be different in different layers. Each edge $e^k_{u,v} \in E^k$, pointing from u to v in layer k, is assigned a weight $w^k_{u,v} \in \mathbb{R}^+$. Agent u is said to be an *in-neighbor* of agent v in layer k if $e^k_{u,v}$ exists. We denote the set of in-neighbors of v in layer k as N^k_v. For an agent v, we say that the weight of its in-neighbor u in layer k is the weight of the edge connecting them, i.e. $w^k_{u,v}$. We assume the weights of all in-neighbours of an agent sum up to 1, i.e. $\sum_{u \in N^k_v} w^k_{u,v} = 1$ for any agent v.

For undirected graphs, every edge is modeled with two opposing directed edges. For unweighted graphs, every edge $e^k_{u,v}$ can be assigned a weight $w^k_{u,v} = 1/d^k_v$, where d^k_v is the *in-degree* (the number of in-neighbors) of node v.

A *projection network* [1] of $\mathcal{G}$ is the graph proj$(\mathcal{G}) = (V, E)$ where $E = \cup^m_{k=1} E^k$.

III. THE LINEAR THRESHOLD MODEL

The linear threshold model (LTM) is described by a discrete-time dynamical system where the state of each agent at iteration t is either active (on) or inactive (off). At iteration $t = 0$, all agents are inactive except an initial active set S_0 called seeds. The active state spreads through the network following the rules introduced below. Once an agent is active, it remains active. Let S_t be the set of agents that are active by the end of iteration t. S_t reaches a steady state when $S_{t-1} = S_t$. For n agents, steady state is reached by $t \leq n$.

A. Monoplex LTM

In a monoplex network, each agent $v = 1, \dots, n$ chooses a threshold μ_v randomly and independently from a uniform distribution $U(0, 1)$. An inactive agent v becomes active at iteration t if the sum of weights of its active in-neighbors at $t-1$ exceeds μ_v, that is, if $\mu_v < \sum_{u \in N_v \cap S_{t-1}} w_{u,v}$.

B. Multiplex LTM

In a multiplex network with m layers, each agent v chooses a threshold μ^k_v in each layer k for $v = 1, \dots, n$ and $k = 1, \dots, m$. Each μ^k_v is randomly and independently drawn from $U(0, 1)$. Each agent might have different neighbors in different layers. If the sum of weights of active in-neighbors of v in layer k exceeds μ^k_v, we say agent i receives a positive input from layer k. Otherwise, the input is negative. The inputs that an agent receives can be conflicting, so an inactive agent needs a protocol to make one decision, either to become active or not. We propose two protocols for duplex (two-layer) networks:

Protocol OR: an inactive agent v becomes active at iteration t if it receives a positive input from *either* layer at $t-1$;

Protocol AND: an inactive agent v becomes active at iteration t if it receives positive inputs from *both* layers at $t-1$.

Protocol OR models agents that become active more readily, whereas Protocol AND models agents that are more conservative in their decisions to become active.

IV. THE LIVE-EDGE MODEL AND REACHABILITY

We review the live-edge model (LEM) proposed in [6] for monoplex directed weighted networks in Section IV-A. In Section IV-B we generalize the LEM to duplex networks and introduce two notions of reachability on the duplex LEM: Reachability OR and Reachability AND.

A. Monoplex LEM and Reachability

The LEM is defined as follows. Given a set of seeds S_0, each unseeded agent randomly selects one incoming edge among all of its incoming edges; an edge is selected with probability given by its weight. The selected edge is labeled as “live”, while the unselected edges are labeled as “blocked”. The seeds block all of their incoming edges. Every directed edge will thus be either live or blocked. Because the selection of edges can be done at the same time for every node, the LEM can be viewed as a static model. The LEM can alternatively be treated as an iterative process in the case the live edges are selected sequentially.

A *live-edge path* [6] is a directed path that consists only of live edges. If there exists a live-edge path from any of the seeds u to an unseeded agent v, we say v is *reachable* from u by a live-edge path.

B. Duplex LEM and Reachability

In a duplex network, each unseeded agent v randomly selects one incoming edge $e^1_{u,v}$ in layer 1 with probability $w^1_{u,v}$ and one incoming edge $e^2_{w,v}$ in layer 2 with probability $w^2_{w,v}$. The selected edges are labeled as “live”, while the rest are labeled as “blocked”. The seeds block all of their incoming edges in both layers. We will refer to such labeling process as *a selection of live edges*.

The challenge in generalizing the LEM to multiplex networks is to properly define reachability. Here we introduce the *live-edge tree* representation of a duplex network, which we will use to define two notions of reachability corresponding to the two duplex LTM protocols.

Definition 1. *Given a set of seeds S_0 and a selection of live edges, a* live-edge tree *associated with agent v is a tree that satisfies*

1) *Every node in the tree corresponds to an agent in the duplex network $\mathcal{G}$. The root corresponds to agent v;*
2) *For each parent p in the tree, p's left (respectively, right) child is the agent to which p's live edge in layer 1 (respectively, 2) is connected.*

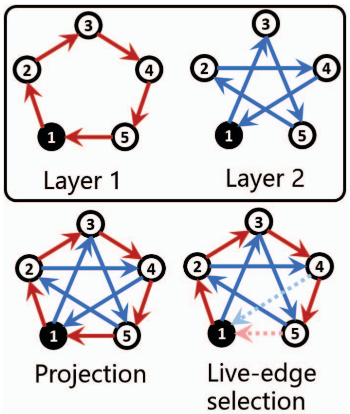

Fig. 1. A duplex network. Blocked edges are light dashed arrows.

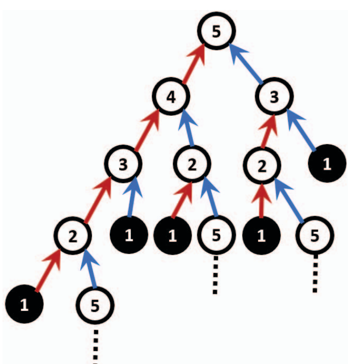

Fig. 2. The live-edge tree for agent 5 in the example duplex network.

Different nodes in the live-edge tree can be the same agent in the duplex network, and branches of the tree can have infinite length. Figure 1 shows an example duplex network with seed 1 and a selection of live edges. For this example, there is only one possible selection of live edges, since each unseeded agent has only one in-neighbour in each layer. Figure 2 shows the live-edge tree associated with 5. Some branches end with 1, the others come back to 5 again. The structure under 5 is the same as the structure shown in the figure, so we use dashed lines to show this repeated information. In this tree, some branches are infinite.

Now we are ready to propose two reachability definitions:

Reachability OR: for a given selection of live edges and a set S_0, an agent v is reachable from S_0 by a selection of live edges if the live-edge tree associated with v has at least one finite branch, and every finite branch ends with a seed;

Reachability AND: for a given selection of live edges and a set S_0, an agent v is reachable from S_0 by a selection of live edges if all branches of the live-edge tree associated with v are finite, and every branch ends with a seed.

Following these definition, the live-edge tree in Figure 2 shows that 5 is reachable from 1 under Reachability OR, but not under Reachability AND. A branch is infinite if an agent reappears in the branch. This simple condition can verify an infinite branch in the algorithm.

V. Equivalence of LEM and LTM

The monoplex LEM was introduced in [6] and proved to be equivalent to the monoplex LTM in the sense that the probability distributions of agents being reachable from a set S_0 in the LEM are equal to the probability distributions of agents being active at steady state after iterating over the set S_0 in the LTM. Computing these probability distributions for the LTM is challenging because it requires solving over the temporal iterations. However, leveraging the equivalence, the probability distributions can be computed without temporal iteration using the LEM treated as a static model.

Recall that S_t is the set of active agents at the end of iteration t for the LTM. S_t reaches steady state when $S_t = S_{t-1}$, and this takes no longer than n iterations.

To prove the equivalence, the LEM can also be treated as an iterative process by revealing the reachabilities of live edges gradually, following [6]. From an initial set S'_0, check the reachability of the agents with at least one edge coming from S'_0. If an agent is determined to be reachable from S'_0, add it to S'_0 at the end of the iteration to get a new set S'_1. In the next iteration, follow the same procedure and get a sequence of sets $S'_0, S'_1, S'_2, ...$. The process ends at iteration t if $S'_t = S'_{t-1}$.

In this section, we first show the equivalence for the monoplex case whose proof can be found in [6]. We then prove the equivalence for the duplex case.

A. Monoplex Networks

Proposition 1. *[6] For a given set S_0, the probabilities of the following events regarding an arbitrary unseeded agent v are the same:*

1) *v is active at steady state by running the LTM under random thresholds given initial active set S_0;*
2) *v is reachable from set S_0 by live-edge paths under the random selection of live edges in the LEM.*

B. Duplex Networks: Protocol OR and Reachability OR

Lemma 1. *Given an initial active set S_0 and a selection of live edges, consider an agent i_0. Assume its live edge in layer 1 comes from agent i^1_1, and its live edge in layer 2 comes from agent i^2_1. Then i_0 is reachable from S_0 under Reachability OR if and only if either i^1_1 or i^2_1 is reachable from S_0 under Reachability OR.*

Proof. If i^1_1 is reachable from S_0 under Reachability OR, then there exists a finite branch in the live-edge tree associated with i^1_1. Denote the branch as $P_{i^1_1} = (i^1_1, i^1_2, i^1_3, ..., i^1_n)$, where $i^1_n \in S_0$. Then the live-edge tree associated with i_0

has a finite branch $P_{i_0}^1 = (i_0, i_1^1, i_2^1, ..., i_n^1)$, which means i_0 is reachable from S_0 under Reachability OR. Following similar analysis for i_1^2, we prove the "if" part.

If neither i_1^1 nor i_1^2 is reachable under Reachability OR, there is no finite branch in their live-edge trees that end with the seeds. Consequently, there is no finite branch in the live-edge tree associated with i_0. i_0 is not reachable from S_0 under Reachability OR. This proves the "only if" part. $\square$

In the following, we prove that the LTM under Protocol OR is equivalent to the LEM under Reachability OR.

Proposition 2. *For a given set S_0, the probabilities of the following events regarding an arbitrary unseeded agent v are the same:*

1) *v is active at steady state by running the LTM under Protocol OR given the initial active set S_0;*
2) *v is reachable from the set S_0 defined by Reachability OR by running the LEM.*

Proof. We prove by mathematical induction.

First, we define some events regarding the LTM. Let $X_k := \mu_v^k < \sum_{u \in N_v^k \cap S_t} w_{u,v}^k$ and $Y_k := \mu_v^k \geq \sum_{u \in N_v^k \cap S_{t-1}} w_{u,v}^k$, for $k \in \{1, 2\}$.

In the LTM, if agent v has not become active at the end of iteration t, then we denote the probability that it becomes active in iteration $t+1$ as P_v^{t+1}. In this case, both μ_v^1 and μ_v^2 have not been exceeded at the end of iteration t. Then the probability that μ_v^1 is exceeded in iteration $t+1$ is $P_1 = P(X_1|Y_1, Y_2) = P(X_1|Y_1)$. The last equality holds because random variables μ_v^1 and μ_v^2 are independent. Similarly, the probability that μ_v^2 is exceeded in iteration $t+1$ is $P_2 = P(X_2|Y_2)$. Using the inclusion-exclusion principle, we have that $P_v^{t+1} = P_1 + P_2 - P_1 \times P_2$.

Next we define some events regarding the LEM. We denote $f_v^k(t)$ as the event that v's live edge in layer k comes from the reachable set by the end of iteration t. We denote $g_v^k(t)$ as v's live edge in layer k does not come from the reachable set by the end of iteration t. Then we let $X_k' = f_v^k(t+1)$ and $Y_k' = g_v^k(t)$, for $k \in \{1, 2\}$.

We look at the LEM as an iterative process. If agent v has not become reachable at the end of iteration t, then we denote the probability that it becomes reachable in iteration $t+1$ as $P_v^{'t+1}$. In this case, the probability that v's live edge in layer 1 comes from S_t' in iteration $t+1$ is $P_1' = P(X_1'|Y_1', Y_2') = P(X_1'|Y_1')$. Similarly, the probability that v's live edge in layer 2 comes from S_t' in iteration $t+1$ is $P_2' = P(X_2'|Y_2')$. Then the probability that either v's live edge in layer 1 comes from S_t' or v's live edge in layer 2 comes from S_t' in iteration $t+1$ is $P_v^{'t+1} = P_1' + P_2' - P_1' \times P_2'$. By Lemma 1, we conclude that $P_v^{'t+1}$ is the probability of reachability of node v under Reachability OR.

Similar to [6], we can see that $P_1 = P_1'$ and $P_2 = P_2'$, so we have $P_v^{t+1} = P_v^{'t+1}$. Thus, by induction over the iterations, we have proved that the probabilities of the two events are the same. $\square$

C. Duplex Network - Protocol AND and Reachability AND

Lemma 2. *Given an initial active set S_0 and a selection of live edges, consider an agent i_0. Assume its live edge in layer 1 (respectively, layer 2) comes from agent i_1^1 (respectively, i_1^2). Then i_0 is reachable from S_0 if and only if both i_1^1 and i_1^2 are reachable from S_0 under Reachability AND.*

Proof. If i_1^1 and i_1^2 are reachable from S_0 under Reachability AND, their live-edge trees do not have any infinite branch. In the live-edge tree associated with i_0, i_0 has two children: i_1^1 and i_1^2. Since the branches under i_1^1 and i_1^2 are all finite, the live-edge tree associated with i_0 has no infinite branch. Since all the leaves in the tree are the union of leaves of the live-edge trees associated with i_1^1 and i_1^2, all of them are seeds. We conclude that i_0 is reachable from S_0 under Reachability AND. If either i_1^1 or i_1^2 is not reachable under Reachability AND, then the infinite branch will result in an infinite branch in the live-edge tree associated with i_0, or a branch end with unseeded agent will result in a branch end with unseeded agent in the live-edge tree associated with i_0, which means i_0 is not reachable. $\square$

We next prove that the LTM under Protocol AND is equivalent to the LEM under Reachability AND.

Proposition 3. *For a given set S_0, the probabilities of the following events regarding an arbitrary unseeded agent v are the same:*

1) *v is active at steady state by running the LTM under Protocol AND given the initial active set S_0;*
2) *v is reachable from the set S_0 defined by Reachability AND by running the LEM.*

Proof. We prove by mathematical induction.

In addition to X_k and Y_k in the previous proof, we let $Z_k := \mu_v^k < \sum_{u \in N_v^k \cap S_{t-1}} w_{u,v}^k$, for $k \in \{1, 2\}$.

In the LTM, if agent v has not become active at the end of iteration t, then we denote the probability that it becomes active in iteration $t+1$ as P_v^{t+1}. If v becomes active at the end of iteration $t+1$, X_1 and X_2 must both be true. If v is inactive at the end of iteration t, then at least one of the thresholds is not exceeded, for which there are three possibilities. The three corresponding probabilities are $P_1 = P(X_1, X_2|Y_1, Y_2)$, $P_2 = P(X_1, X_2|Z_1, Y_2)$ and $P_3 = P(X_1, X_2|Y_1, Z_2)$. As they are independent of one another, we have $P_v^{t+1} = P_1 + P_2 + P_3$.

In addition to X_k' and Y_k' in the previous proof, we let $Z_k' = f_v^k(t)$, for $k \in \{1, 2\}$.

We look at the LEM as an iterative process. If agent v has not become reachable by the end of iteration t, then we denote the probability that it becomes reachable in iteration $t+1$ as $P_v^{'t+1}$. By Lemma 2, if v is reachable by the end of iteration $t+1$, v's live edges in both layers must come from the reachable set by the end of iteration $t+1$. Moreover, at the end of iteration t at least one of v's live edges has not come from the reachable set, for which there are three possible cases. The probabilities of the three cases are $P_1' = P(X_1', X_2'|Y_1', Y_2')$, $P_2' = P(X_1', X_2'|Z_1', Y_2')$ and

$P_3' = P(X_1', X_2'|Y_1', Z_2')$. As they are independent of one another, we have $P_v'^{t+1} = P_1' + P_2' + P_3'$.

Similar to [6], we can show that $P(X_1|Y_1) = P(X_1'|Y_1')$ and $P(X_2|Y_2) = P(X_2'|Y_2')$ then we claim $P_i = P_i', i = 1, 2, 3$:

$$\begin{aligned} P_1 &= P(X_1, X_2, Y_1, Y_2)/P(Y_1, Y_2) \\ &= P(X_1, Y_1)/P(Y_1) \times P(X_2, Y_2)/P(Y_2) \\ &= P(X_1', Y_1')/P(Y_1') \times P(X_2', Y_2')/P(Y_2') \\ &= P(X_1', X_2', Y_1', Y_2')/P(Y_1', Y_2') = P_1'; \end{aligned}$$

$$\begin{aligned} P_2 &= P(X_1, X_2, Z_1, Y_2)/P(Z_1, Y_2) \\ &= P(X_1, Z_1)/P(Z_1) \times P(X_2, Y_2)/P(Y_2) \\ &= 1 \times P(X_2, Y_2)/P(Y_2) \\ &= 1 \times P(X_2', Y_2')/P(Y_2') \\ &= P(X_1', Z_1')/P(Z_1') \times P(X_2', Y_2')/P(Y_2') \\ &= P(X_1', X_2', Z_1', Y_2')/P(Z_1', Y_2') = P_2'; \end{aligned}$$

$$\begin{aligned} P_3 &= P(X_1, X_2, Y_1, Z_2)/P(Y_1, Z_2) \\ &= P(X_1, Y_1)/P(Y_1) \times P(X_2, Z_2)/P(Z_2) \\ &= P(X_1, Y_1)/P(Y_1) \times 1 \\ &= P(X_1', Y_1')/P(Y_1') \times 1 \\ &= P(X_1', Y_1')/P(Y_1') \times P(X_2', Y_2')/P(Z_2') \\ &= P(X_1', X_2', Y_1', Z_2')/P(Y_1', Z_2') = P_3'. \end{aligned}$$

Then we can show $P_v^{t+1} = P_v'^{t+1}$. Thus, by induction over the iterations, we see that the probabilities of the two events are the same. □

VI. Social Influence and Cascade Centrality

A. Monoplex Social Influence and Cascade Centrality

The *social influence* of a set of agents S_0 is defined as the expected number of active agents at the steady state of the LTM given that S_0 is the initial active set [6]. This measure of social influence for a single agent as the set S_0 is called *cascade centrality* in [7].

B. Duplex Social Influence and Cascade Centrality

Duplex social influence and *duplex cascade centrality* can be defined analogously for each protocol of the duplex LTM. We define duplex cascade centrality under Protocol OR and under Protocol AND.

Definition 2. *The* duplex cascade centrality of agent v under Protocol OR *is the expected number of active agents at steady state of the duplex LTM under Protocol OR, given v is the only seed in the network.*

Definition 3. *The* duplex cascade centrality of agent v under Protocol AND *is the expected number of active agents at steady state of the duplex LTM under Protocol AND, given v is the only seed in the network.*

The expected number of active agents in the network is the sum of the probabilities of being active over the agents in the network. Calculating this probability distribution with the LTM requires doing simulations under different combinations of threshold values of all agents. However, by Propositions 2 and 3, the LEM gives us a way to calculate cascade centralities from the duplex network structure.

C. Algorithm for Duplex Cascade Centralities

The following algorithm is not intended to be efficient, but it serves as a way to accurately calculate duplex cascade centralities by leveraging the LEM.

Algorithm 1. *Calculate duplex cascade centralities*

1) *Find the D different selections of live edges, where $D = \prod_{j \in V \setminus \{i\}} d_j^1 \prod_{j \in V \setminus \{i\}} d_j^2$.*
2) *For each selection, construct the live-edge tree for each unseeded agent. Store reachability results under the two reachability definitions for each unseeded agent.*
3) *If agent j is reachable N_j^{OR} times under Reachability OR and N_j^{AND} times under Reachability AND, then the duplex cascade centralities of agent i are*
   $C_i^{OR} = 1 + \sum_{j \in V \setminus \{i\}} N_j^{OR}/D$,
   $C_i^{AND} = 1 + \sum_{j \in V \setminus \{i\}} N_j^{AND}/D$

Theorem 1. *The C_i^{OR} and C_i^{AND} computed by Algorithm 1 are the duplex cascade centrality of agent i under Protocol OR and Protocol AND, respectively.*

Proof. If follows directly from the equivalence of the LTM and the LEM. □

Using Algorithm 1, the two centralities can be calculated at the same time, whereas if we conduct simulations by the LTM, we must simulate the two protocols separately. From the live-edge tree associated with 5 in the example of Figure 2, we actually obtain the live-edge trees of all unseeded agents as they are part of this tree. More generally, we might not need to construct live-edge trees for all unseeded agents. For the example $C_1^{\text{OR}} = 5$ and $C_1^{\text{AND}} = 1$: if agent 1 is the seed, all agents are expected to be active at steady state of the LTM under Protocol OR and only agent 1 would be active under Protocol AND.

TABLE I
Comparison of Different Networks From Figure 3

Network	P_1	P_2	P_3	P_4	P_5	Cascade Centrality of 1
Duplex (OR)	1	1	1	$\frac{7}{8}$	$\frac{7}{8}$	4.75
Duplex (AND)	1	0	0	0	0	1
Layer 1	1	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{4}$	3
Layer 2	1	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	2
Projection	1	$\frac{16}{27}$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{13}{27}$	3.11

D. Ordering of probabilities

Let $\mathcal{G}$ be a duplex network with graphs G_1 and G_2 as its two layers. Given an initial active set S_0, we consider the probability of an unseeded agent v being active at steady state under Protocol OR (P_v^{OR}) and under Protocol AND (P_v^{AND}). The probabilities of v being active at steady state in monoplex networks G_1 and G_2 separately are denoted by P_v^1 and P_v^2, respectively.

Corollary 1. *Under the above setting, we have*

$$P_v^{AND} \leq P_v^1 \leq P_v^{OR}$$
$$P_v^{AND} \leq P_v^2 \leq P_v^{OR}$$

Proof. Under a selection of live edges in the duplex network, if v is reachable under Reachability AND, all branches are finite and end with the seeds. In particular, this holds true for the leftmost branch, which only consists of edges in G_1. In G_1, this selection of edges forms a live-edge path. Thus, v is reachable in G_1 as a monoplex network. Considering all possible selections of live edges, we conclude that whenever v is reachable under Reachability AND, v is reachable in G_1 as a monoplex network. Using the equivalence of the LTM and LEM, $P_v^{\text{AND}} \leq P_v^1$.

Under a selection of live edges, if v is reachable in G_1 as a monoplex network, then a live-edge path in G_1 is formed. Considering the live-edge tree of v for the duplex network, the leftmost branch only consists of edges in G_1 and it is exactly the live-edge path. Thus, this branch is finite and ends with the seeds and v is reachable under Reachability OR. Considering all possible selections of live edges, we conclude that whenever v is reachable in G_1 as a monoplex network, v is reachable under Reachability OR. Using the equivalence of the LTM and LEM, $P_v^1 \leq P_v^{\text{OR}}$.

The inequality for layer 2 is proved similarly. □

E. Example

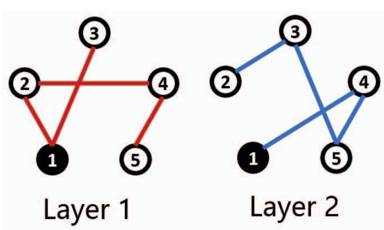

Fig. 3. Example duplex network

Figure 3 shows a duplex network with undirected graphs. We calculate cascade centrality of 1 in five cases: duplex cascade centrality under protocol OR, duplex cascade centrality under protocol AND, layer 1 as a monoplex network, layer 2 as a monoplex network and the projection network as a monoplex network. The results are shown in Table I, where the middle columns are the probabilities of agents becoming active. We can see that for each agent, the probabilities follow the results of Corollary 1.

VII. Final Remarks

We have generalized the linear threshold model with randomly selected thresholds to study diffusion of innovations in multiplex networks, deriving tools to compute social influence and cascade centralities in duplex (two-layer) networks. The LTM for duplex networks is more complicated than it is for monoplex networks as in [6], [7]. Similar to the monoplex case, the live-edge model is leveraged in the duplex case. However, the latter is inherently more complicated because a decision in the duplex case depends on selections in both layers, and we cannot simply analyze live-edge paths in each layer independently. In monoplex networks, the set of selections of live edges increases exponentially with number of agents, but formation of a live-edge path does not depend on selections of agents not in the path and different live-edge paths form independently. Thus, a closed form expression for cascade centrality can be provided for a monoplex network. In duplex networks, these properties do not hold.

We consider directed weighted networks, which is more general than previous research. Our approach does not require assumptions such as the connectedness of the networks. If we add some assumptions on the network, we may be able to give specialized algorithms. For instance, if the graphs in both layers are undirected and connected, we can prove that the finite branches in live-edge trees always end with seeds. Since infinite branches are caused by cycles in the projection network, checking for infinite branches can be implemented by checking for cycles in the selection of live edges and checking for finite branches can be implemented by checking for connectivity in the selection of live edges.

For multiplex networks with more than two layers, we can generalize our protocols by introducing another interlayer threshold parameter μ_{inter}. Then the protocol is stated as follows: if the portion of positive inputs from all layers of inactive agent v exceeds μ_{inter} at $t-1$, then v becomes active at t. In duplex networks, $\mu_{\text{inter}} \in [0, 0.5)$ corresponds to Protocol OR, and $\mu_{\text{inter}} \in [0.5, 1)$ corresponds to Protocol AND.

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