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Symmetry and Reduction for Coordinated Rigid Bodies

Heinz Hanßmann*

Institut für Reine und Angewandte Mathematik
RWTH Aachen, 52056 Aachen, Germany

Naomi Ehrich Leonard† and Troy R. Smith‡

Department of Mechanical and Aerospace Engineering
Princeton University, Princeton, NJ 08544, U.S.A.

Abstract

Motivated by interest in the collective behavior of autonomous agents, we lay foundations for a study of networks of rigid bodies and, specifically, the problem of aligning orientation and controlling relative position across the group. Our main result is the reduction of the (networked) system in the case that two individuals are coupled by control inputs that depend only on relative configuration. We use reduction theory based on semi-direct products, yielding Poisson spaces that enable efficient formulation of control laws. We apply these reduction results to satellite and underwater vehicle dynamics, proving stability of coordinated behaviors such as the case of two underwater vehicles moving at constant speed with their orientations stably aligned.

1 Introduction

In this paper we examine symmetry and reduction in rigid body networks for which coupling is provided by control inputs that depend only on the *relative* configuration (orientation and position) of the rigid bodies in the network. We have in mind systems of vehicles such as satellites, underwater vehicles and helicopters, and consider the problem of aligning the orientations of all vehicles in the group while they move as in Figure 1. Control algorithms for individual vehicles of this kind have been previously studied in [1, 15]. Here we focus on control of networks consisting of *two* rigid bodies, and we discuss at the end of the paper how these results may be extended to networks of larger numbers of rigid bodies. Our goal is to provide a comprehensive study of the coupled two-body system and contribute to laying the foundations for future work

*Partially supported by the Max Kade Foundation. Present address: Mathematisch Instituut, Universiteit Utrecht, Postbus 80010, 3508 TA Utrecht, The Netherlands.

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‡Present address: Control and Dynamical Systems, California Institute of Technology, Mail Stop 107 -81, 1200 E. California Blvd, Pasadena, CA, 91125, USA.

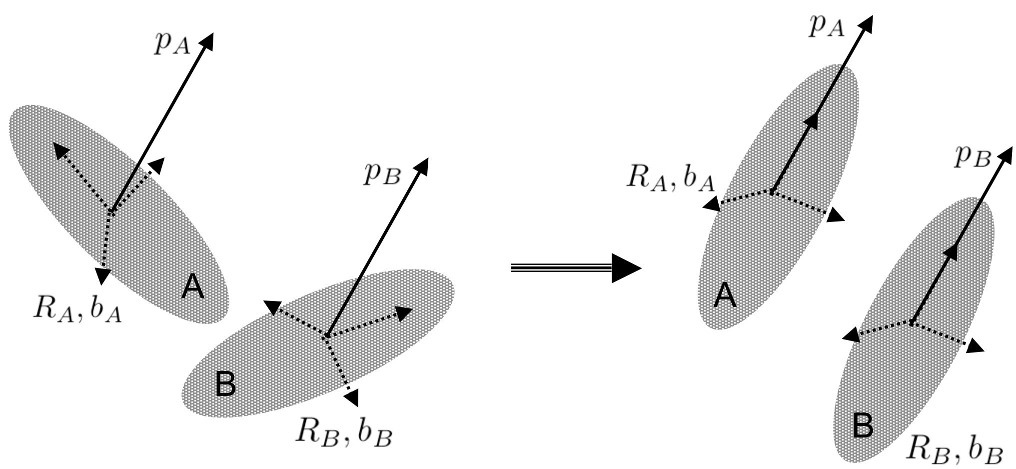

Figure 1: Schematic of the orientation problem for two underwater vehicles.

on coordinated control of relative orientation and position across networks in more general settings. Other recent work on the control of networked mechanical systems has recently appeared in [12] and [13].

The class of control inputs that we address to couple the rigid bodies derive from artificial potential functions. In contrast to the simplified setting presented in [10], our potential functions are interpreted as breaking certain symmetries, namely those associated with the relative configurations of the individual bodies. The most obvious symmetry present in a group of rigid bodies is associated with the invariance of the dynamics to the absolute position and orientation of the group. One may use this system symmetry to factor out the explicit position and orientation of each individual rigid body in space and only retain information on the relative positions and orientations within the group; this passage from the dynamics on the original phase space to simpler dynamics on the smaller (reduced) phase space is the essence of mechanical *reduction* [8].

The control law we present is formulated in terms of (artificial) potentials on the reduced phase space and this allows for achievement of any prescribed relative orientation (and position). In the analysis, the kinetic energy in the Hamiltonian remains in a general form and needs only be specified once the theory is applied to a particular kind of rigid body dynamics. In this paper we apply our results to both satellites and underwater vehicles. We prove conditions for stability of the controlled coupled dynamics in each case, paying special attention throughout the paper to the important case of control laws that align the (three-dimensional) orientations of the rigid bodies as in Figure 1.

There are a number of advantages for control design and analysis that come from the reduction derived here. These include the reduced dimensionality, and thus complexity, of the system, the simplification in the equations of motion for a given Hamiltonian and, importantly, the derivation of a Hamiltonian structure for the reduced system complete with Casimir functions which can be readily used in the study of stability and stabilization of the controlled network. In future work, interaction with the environment and/or prescription for a group objective can be re-interpreted as breaking the symmetries we use in this paper to reduce the phase space.

Our effort here is motivated by the problem of coordinated control of groups of rigid body systems and vehicles. The study of control laws for groups of autonomous vehicles has emerged as a challenging new research topic in recent years. There are currently few examples of, and yet many possible applications for, groups of highly autonomous agents that exhibit complex collective behavior in the engineered world. Researchers in this area are finding much inspiration from biological examples. Animal aggregations, such as schools of fish, are believed to use simple, local traffic rules at the individual level but exhibit remarkable capabilities at the group level.

Numerous researchers have attempted to model animal aggregation with each agent considered as a point mass (see, for example, [14, 19]). Likewise in engineering applications, group control problems are often formulated with point-mass vehicle models (e.g. [3, 4, 7]). Less often, the individuals are modelled as rigid bodies in three-dimensional physical space. Here, we consider a group in which individuals are modelled as rigid bodies from the outset. We note that a close biological analogue to this system, that of a school of fish, is believed to maintain group cohesion through each individual's desire to match the speed and direction of nearby fish, and to be surrounded by a certain amount of open space (see [16]).

We note that there is a natural connection between coordinated rigid body networks and multi-body dynamics problems; the former can be made to resemble the latter by imposing control inputs that artificially couple the individuals so that the network functions as one large multi-body system. Our aim in this paper is to realize this connection and develop a framework for analysis and design of coordinated rigid body networks in which we can make best use of tools and results from mechanics.

Problem Setup Throughout this paper we shall make much use of the modern theory of geometric mechanics (see [8] for a detailed and readable exposition)). In doing so, we shall often refer to the Lie groups $SE(3)$ and $SO(3)$; the former, the Euclidean group (or group of rigid motions) globally describes the position and orientation of a rigid body while the latter, the Special Orthogonal group (a subgroup of $SE(3)$), describes the orientation alone, mapping body coordinates into inertial coordinates. We shall also make use of the respective Lie algebras and their dual spaces $\mathfrak{se}(3)^*_-$ and $\mathfrak{so}(3)^*_-$, on which we choose the 'lower sign' Poisson structure (see [8], Chapter 9 for an exposition of Lie groups and their associated Lie algebras).

In our general problem, we consider N rigid bodies (vehicles), each with configuration space $SE(3)$ (or some subgroup of $SE(3)$) so that the configuration space for the network is $SE(3)^N = SE(3) \times \cdots \times SE(3)$. The phase space for the network can be described by the cotangent bundle $T^*SE(3)^N$, and the dynamics are Hamiltonian dynamics given a Hamiltonian function $H : T^*SE(3)^N \longrightarrow \mathbb{R}$.

We examine, in depth, the problem of $N = 2$ and discuss at the end of the paper extensions to $N > 2$. For $N = 2$ we consider two rigid bodies labelled A and B. An element in the system configuration space $SE(3) \times SE(3)$ is given by (R_A, b_A, R_B, b_B), where $R_k \in SO(3)$ and $b_k \in \mathbb{R}^3$ describe orientation and position of body k. A schematic of this problem setup is presented in Figure 1.

We consider in this paper three general settings motivated by applications. Each case describes a system that moves in three-dimensional space (but which can be specialized to motion in two dimensions). The key idea is that the dynamics of the network are invariant to the global position and/or orientation of the group as a whole; we interpret this as the dynamics being invariant under the continuous symmetry $G = SO(3)$, $G = SE(3) = SO(3) \times_\delta \mathbb{R}^3$ or $G = SO(3) \times_\delta (\mathbb{R}^3 \times \mathbb{R}^3)$ (where $\times_\delta$ denotes the semidirect product induced by the canonical action $\delta : SO(3) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ in the former case and by the diagonal action $\delta : SO(3) \times (\mathbb{R}^3 \times \mathbb{R}^3) \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$ in the latter case) as appropriate.

The most general case we consider is motivated by the problem of designing stable controllers to align the rigid bodies in the group and to fix their relative positions. The way to take advantage of the lack of dependence of the dynamics of the network on its global position and orientation is to reduce $T^*SE(3) \times \ldots \times T^*SE(3)$ by the (left) diagonal action of $SE(3)$. For $N = 2$, dividing out the 6-dimensional symmetry group $G = SE(3)$ from the 24-dimensional phase space $T^*SE(3) \times T^*SE(3)$, yields the 18-dimensional reduced phase space

$$T^*SE(3) \times T^*SE(3) \big/ SE(3) \;\cong\; (\mathfrak{se}(3)^*_- \times \mathfrak{se}(3)^*_-) \times SE(3) \;=\; \mathfrak{t}^*_-, \tag{1}$$

where the remaining copy of $SE(3)$ contains the relative position and orientation of one body with respect to the other. The isomorphism and the direct product on the right hand side refer to the manifold character of the sets involved and not to the group structure (in fact, the left hand side does not even come equipped with a group structure). We call this *Case C* and furthermore consider explicitly two simplifications of this setting, each interesting in its own right, where the concepts introduced are simpler to describe.

The first simplified setting is the case studied in [18] in which we not only abstract from the absolute positions of the two rigid bodies, but also from their relative position. We thus consider control inputs that depend only on the relative orientation of the individual bodies, thereby seeking orientation alignment of the two individuals but *not* relative position alignment. To this end we again reduce the left action of a Lie group G but instead of $G = SE(3)$, the semidirect product of $SO(3)$ with $\mathbb{R}^3$, this group G now becomes the 9-dimensional semidirect product of $SO(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$ (see [5] for more on this semidirect product). The additional copy of $\mathbb{R}^3$ in the symmetry group, as compared with Case C, comes from the invariance of the system to the relative position of the two bodies. Correspondingly, we do not end up with (1), but with the 15-dimensional reduced phase space

$$\mathfrak{t}_-^* \;=\; (\mathfrak{se}(3)_-^* \times \mathfrak{se}(3)_-^*) \times SO(3) \tag{2}$$

(again concentrating on the (Poisson) manifold character and not the group structure). We call this *Case B*.

Our final setting, which we have denoted *Case A*, is even simpler. Here we ignore the translational part of the problem altogether and only consider the rotational aspect. This particular case is relevant to the alignment problem for two rigid bodies (satellites, for example) where the phase space is the 12-dimensional cotangent bundle $T^*SO(3) \times T^*SO(3)$, and we assume control inputs that depend on relative orientation. The 3-dimensional symmetry group $G = SO(3)$ corresponds to the invariance of the controlled system with respect to the orientation of the group of rigid bodies as a whole and the 9-dimensional reduced phase space is

$$T^*SO(3) \times T^*SO(3) \big/ SO(3) \;\cong\; (\mathfrak{so}(3)_-^* \times \mathfrak{so}(3)_-^*) \times SO(3) \;=\; \mathfrak{t}_-^* \,. \tag{3}$$

Overview In the first part of this paper we describe the reduction for Cases A, B and C. We do this *without* assuming the form of the kinetic energy of the rigid bodies of interest; we only assume the dependency of the coupling potentials that are realized by the control inputs. The reduction can therefore be seen as providing a framework for a class of problems. For example, Cases B and C which concern rigid bodies, each on $SE(3)$, can be applied to a groups of vehicles such as stratospheric balloons, underwater vehicles and helicopters. Case A, which concerns rigid bodies on $SO(3)$, is applicable to satellite cluster coordination, for example. The presentation of the reduction is organized as follows. In §2 we obtain the quotient spaces provided by the abstract theory. In §3 we review the reduction theory by means of semi-direct products and apply this to Cases A-C. This yields flat (Euclidean) Poisson spaces $\mathfrak{s}_-^* \cong \mathbb{R}^\nu$ which we treat in their own right in §4; we write down Casimirs, study how the abstract quotients of §2 are realized as submanifolds and examine the singular (lower-dimensional) symplectic leaves. These flat Poisson spaces described in §3 and §4 are preferable to the abstract reduced spaces described in §2 since they make it possible for us to efficiently formulate control laws.

In the second part of the paper we apply the reduction results to particular examples, i.e., we study the controlled dynamics for particular choices of kinetic energy. In §5 the dynamics are given by Hamiltonians that are obtained by adding a potential energy incorporating our control law to the (free) kinetic energy. The (relative) equilibria of these controlled dynamics are described in §6, while the stability properties of these are analysed in §7. To give an impression of the dynamics we performed some simulations that are summarised in §8. Extensions to $N > 2$ are presented in §9 and final remarks in §10.

2 Abstract reduction theory

As noted in the Introduction, we describe the absolute position of our network of rigid bodies as an element in a Lie group (either $SO(3)$ or $SE(3)$) and we make use of the fact that the dynamics of the network are invariant to changes in this position. Mathematically, changes in the absolute position can be described as the action of a Lie group G on the configuration space; this action can be lifted to the phase space through a cotangent lift ([8], Chapter 6), and the invariance of the dynamics to this lifted action may be considered to be a continuous symmetry. According to Noether's theorem, such continuous symmetries lead to conserved quantities; a concise formulation of this result and its implications can be given within Hamiltonian mechanics, as we shall now describe (following the exposition in [8]).

We consider a Hamiltonian system $(P, H, \{\cdot,\cdot\})$, where P is the phase space, H is the Hamilonian function $H : P \to \mathbb{R}$ and $\{\cdot,\cdot\}$ is the Poisson bracket on functions on P. If H is invariant under the proper Poisson action of some Lie group G, we call G a symmetry group, and we have our choice of two method of simplifying the system. In the first of these, Poisson reduction, we form the quotient space

$$P\big/_G \;=\; \big\{ \{ \, g(p) = p \mid g \in G \, \} \;\big|\; p \in P \, \big\}$$

and note that this space is again Poisson, even a Poisson manifold if the action of G is free [8]. The conserved quantities predicted by Noether's theorem are *Casimir functions* on this quotient space, i.e. functions that Poisson-commute with all other functions, in particular the Hamiltonian function. The system X_H canonically projects to a Hamiltonian system on the quotient space, the so-called reduced system. Like every Hamiltonian system on $P/_G$, it leaves the values of the Casimir functions fixed. Therefore, the reduced dynamics are restricted to a submanifold of the quotient space, defined by the Casimir functions.

A second way of doing reduction swaps the order of the steps and fixes the values of conserved quantities *before* passing to the quotient. Because of the intimate relation between the two procedures, going through the steps of each helps to efficiently work with the reduced dynamics in applications. For this reason we present here the latter, more abstract, reduction, often called *regular reduction* in the literature.

Assembling the conserved quantities in the unreduced space, P, leads to the momentum mapping

$$\mathbf{J} \,:\; P \;\longrightarrow\; \mathfrak{g}^*$$

with values in the dual of the Lie algebra $\mathfrak{g}$ of the symmetry group G. Given the value $\mu \in \mathfrak{g}^*$ of $\mathbf{J}$, the dynamics are restricted to the subset $\mathbf{J}^{-1}(\mu) \subseteq P$. In the important case that μ is a regular value of $\mathbf{J}$ the subset $\mathbf{J}^{-1}(\mu)$ is a (smooth) submanifold. Regular reduction aims to divide out the remaining symmetry.

The action of the whole group G would in general alter the value of the momentum mapping. An important case is when $\mathbf{J}$ is equivariant with respect to the given action of G on P, Φ_g, and the *coadjoint action* $\mathrm{Ad}^*_{g^{-1}}$ of G on $\mathfrak{g}^*$, i.e.,

$$\mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J} \;=\; \mathbf{J} \circ \Phi_g \qquad \text{for all } g \in G.$$

In this case, we have that the *isotropy subgroup*

$$G_\mu \;=\; \Big\{ \, g \in G \;\Big|\; \mathrm{Ad}^*_{g^{-1}}(\mu) = \mu \, \Big\}$$

does leave $\mathbf{J}^{-1}(\mu)$ invariant and hence defines a free action, i.e. one that does not have any fixed points. (This is true provided that the action of G on P is free and μ is a regular value of the momentum mapping). If P is a symplectic manifold, then the quotient

$$P_\mu \;=\; \mathbf{J}^{-1}(\mu)\big/G_\mu$$

is (in a canonical way) again a symplectic manifold. The two ways of reduction are related through

$$P\big/G \;=\; \bigcup_{\mu\in\mathfrak{g}^*} P_\mu \tag{4}$$

with the corresponding identifications; in fact the P_μ constitute the *symplectic leaves* of this Poisson manifold.

In the important case that the phase space P is given by the cotangent bundle T^*G of the symmetry group G (acting by left translation) – for $G = SO(3)$ this is the situation of the free rigid body and for $G = SE(3)$ that of a free underwater vehicle – equation (4) expresses that $\mathfrak{g}^*$ is foliated by the coadjoint orbits, i.e. the orbits of the coadjoint action $\mathrm{Ad}^*_{g^{-1}}$ [8]. We now examine the abstract reduction theory in the three cases of interest in this paper.

2.1 Case A

The phase space of two rigid bodies in three-dimensional space,

$$P \;=\; T^*(SO(3)\times SO(3)) \;\cong\; T^*SO(3)\times T^*SO(3)$$

is a symplectic manifold, and we define the action of the group $G = SO(3)$ on this space through the lift of the diagonal action

$$\begin{array}{ccc} G\times(SO(3)\times SO(3)) & \longrightarrow & SO(3)\times SO(3) \\ (R, R_A, R_B) & \mapsto & (RR_A, RR_B). \end{array} \tag{5}$$

This corresponds to rotating bodies A and B, whose orientations with respect to an inertial frame are specified by R_A and R_B, respectively, by the same (fixed) rotation R. A fixed value $\mu \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$ of the momentum mapping $\mathbf{J} : P \longrightarrow \mathbb{R}^3$ corresponds to the sum

$$\mu \;=\; R_A\pi_A + R_B\pi_B \tag{6}$$

of the two angular momenta of the two bodies, measured in the inertial frame, where $\pi_A, \pi_B \in \mathfrak{so}(3)^*$ are the individual angular momenta in the respective body frames of the two vehicles. Hence, the isotropy subgroup G_μ is isomorphic to S^1 and consists of all rotations about μ (provided this vector does not vanish). Since $\mathbf{J}^{-1}(\mu)$ is 9-dimensional and $\dim S^1 = 1$, we obtain 8-dimensional symplectic leaves P_μ. Correspondingly, the squared length

$$\mu^2 \;=\; \pi_A^2 \,+\, 2\pi_B^T(R_B^T R_A)\pi_A \,+\, \pi_B^2 \tag{7}$$

is the Casimir function on the 9-dimensional Poisson manifold (3). The rank of the reduced Poisson bracket is 8; we have reduced from 6 to 4 degrees of freedom.

2.2 Case B

The phase space of two rigid bodies with translational dynamics in three-dimensional space,

$$P \;=\; T^*(SE(3)\times SE(3)) \;\cong\; T^*SE(3)\times T^*SE(3),$$

is again a symplectic manifold, while the symmetry group G of interest in this case, as described in §1, is given by the semi-direct product of $SO(3)$ with $\mathbb{R}^3\times\mathbb{R}^3$. The action of G we choose is the lift of

$$\begin{array}{ccc} G\times(SE(3)\times SE(3)) & \longrightarrow & SE(3)\times SE(3) \\ (R,a,b,R_A,b_A,R_B,b_B) & \mapsto & (RR_A,Rb_A+a,RR_B,Rb_B+b) \end{array} \tag{8}$$

and corresponds to rotating bodies A and B by the same rotation R while translating them with two (possibly different) vectors a and b. A fixed value $\mu\in\mathfrak{g}^*\cong\mathbb{R}^9$ of the momentum mapping $\mathbf{J}:P\longrightarrow\mathbb{R}^9$ corresponds to the nine components of the three vectors

$$R_Ap_A,\quad R_Bp_B,\quad R_A\pi_A+b_A\times R_Ap_A\;+\;R_B\pi_B+b_B\times R_Bp_B$$

where p_A, p_B denote the linear momenta of the vehicles A, B, respectively, measured in the respective body frames. For regular values μ, the isotropy subgroup G_μ is isomorphic to $S^1\times S^1\times S^1$ and consists of all rotations about these three vectors. Since $\mathbf{J}^{-1}(\mu)$ is 15-dimensional and $\dim(S^1\times S^1\times S^1)=3$, we obtain 12-dimensional symplectic leaves. Correspondingly, we have on the 15-dimensional Poisson manifold (2) the three Casimir functions

$$p_A^2,\quad p_B^2\quad\text{and}\quad p_B^T(R_B^TR_A)p_A. \tag{9}$$

The rank of the reduced Poisson bracket is 12; we have reduced from 12 to 6 degrees of freedom.

2.3 Case C

The phase space is again

$$P \;=\; T^*(SE(3)\times SE(3)) \;\cong\; T^*SE(3)\times T^*SE(3)$$

and we choose the symmetry group G, the semi-direct product $SE(3)$ of $SO(3)$ with $\mathbb{R}^3$, to act through the lift of the diagonal action

$$\begin{array}{ccc} SE(3)\times(SE(3)\times SE(3)) & \longrightarrow & SE(3)\times SE(3) \\ (R,b,R_A,b_A,R_B,b_B) & \mapsto & (RR_A,Rb_A+b,RR_B,Rb_B+b). \end{array} \tag{10}$$

This corresponds to rotating bodies A and B by the same rotation R while translating them with the same vector b. A fixed value $\mu\in\mathfrak{se}(3)^*\cong\mathbb{R}^6$ of the momentum mapping corresponds to the six components of the two vectors

$$R_A\pi_A+b_A\times R_Ap_A\;+\;R_B\pi_B+b_B\times R_Bp_B\quad\text{and}\quad R_Ap_A\;+\;R_Bp_B.$$

Hence, the isotropy subgroup G_μ is isomorphic to $S^1 \times S^1$ and consists of all rotations about these two vectors. Since $\mathbf{J}^{-1}(\mu)$ is 18-dimensional and $\dim(S^1 \times S^1) = 2$, we obtain 16-dimensional symplectic leaves. On the 18-dimensional Poisson manifold (1), we have the two Casimir functions

$$\begin{aligned} &\pi_A \cdot p_A \; + \; p_B^T(R_B^T R_A)\pi_A \; + \; \pi_B^T(R_B^T R_A)p_A \\ + \; &\pi_B \cdot p_B \; + \; R_B^T(b_B - b_A)\cdot(p_B \times (R_B^T R_A)p_A) \\ &\text{and} \quad p_A^2 \; + \; 2p_B^T(R_B^T R_A)p_A \; + \; p_B^2. \end{aligned} \tag{11}$$

Fixing the values of these two Casimirs yields the 16-dimensional symplectic leaves of (1). Hence, the rank of the reduced Poisson bracket is 16; we have reduced from 12 to 8 degrees of freedom.

3 Semi-direct product reduction

While regular reduction yields symplectic manifolds, it does not provide an algorithm to obtain a realization of the quotients $P_\mu = \mathbf{J}^{-1}(\mu)/_{G_\mu}$ as submanifolds of some flat (i.e. linear) Poisson space. The advantage of such an embedding into $\mathbb{R}^\nu$ is that further computations may be done using the ν linear coordinates, and these contribute to efficient formulation of control laws. In the present case this can be achieved by means of semi-direct product reduction (see [9]).

Indeed, since the phase space P is the cotangent bundle of some Lie group F, and the symmetry group G is a subgroup of F, the quotient $P/_G$ can be embedded in the flat Poisson space

$$\mathfrak{s}_-^* \;=\; \mathfrak{f}_-^* \times_{\rho'} V_-^*$$

where $\mathfrak{f}^*$ is the dual of the Lie algebra of F and V is a vector space; the minus sign indices indicate the lower sign choice in the Poisson brackets (12) given below. To define this semi-direct product one needs a dummy dependent variable $a \in V_+^*$ for the system and a left representation

$$\rho \;:\; F \;\longrightarrow\; \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the general linear group. This representation defines the semi-direct product $S = F \times_\rho V$ in such a way that the symmetry group G can be identified with the isotropy subgroup

$$F_a \;=\; \left\{ \, f \in F \;\middle|\; \rho^*(f)a = a \, \right\}$$

where ρ^* is the associated right representation of F on V^*. Further, $\rho' : \mathfrak{f} \longrightarrow V$ is the induced Lie algebra representation. The vector space V acts on itself by vector addition. This allows us to use regular reduction on T^*S. The reduction of the S-symmetry on T^*S provided by the lift of the (left) action of S on itself is the well-known Lie-Poisson reduction leading to

$$T^*S/_S \;=\; \mathfrak{s}_-^*$$

with lower sign choice in the Poisson bracket on $\mathfrak{s}^*_-$

$$\{W,Q\}_\pm(\ell,\alpha) \;=\; \pm\left\langle \ell, \left[\frac{\delta W}{\delta \ell}, \frac{\delta Q}{\delta \ell}\right]\right\rangle \pm \left\langle \alpha, \rho'\left(\frac{\delta W}{\delta \ell}\right)\cdot\frac{\delta Q}{\delta \alpha} - \rho'\left(\frac{\delta Q}{\delta \ell}\right)\cdot\frac{\delta W}{\delta \alpha}\right\rangle . \tag{12}$$

To obtain the upper sign one would have to reduce the right action of S on itself. The symplectic leaves of this Poisson manifold are the quotients

$$\mathbf{J}^{-1}(\mu,a)\Big/S_{(\mu,a)} \;=\; \mathbf{J}^{-1}(\mu)\times V\times\{a\}\Big/F_{(\mu,a)}\times V$$

where

$$F_{(\mu,a)} \;=\; \left\{\, g\in F_a \;\middle|\; \mathrm{Ad}^*_{g^{-1}}\mu=\mu \,\right\} \;=\; G_\mu .$$

The vector space V cancels and we have embedded

$$T^*F\Big/G \;=\; \bigcup_{\mu\in\mathfrak{f}^*_+}\mathbf{J}^{-1}(\mu)\Big/G_\mu$$

in the flat Poisson space $\mathfrak{s}^*_-$ by choosing for the dummy variable $a\in V^*_+$ some fixed value. This is called semi-direct product reduction and is summarized in the following theorem.

Theorem 3.1 (Marsden et al [9]) *Let $H_a : T^*F\to\mathbb{R}$ be a Hamiltonian depending smoothly on a parameter $a\in V^*$, and left invariant under the action on T^*F of the stabiliser F_a, defined as*

$$F_a \;=\; \left\{\, f\in F \;\middle|\; \rho^*(f)a=a \,\right\} \tag{13}$$

where ρ is a left representation of F on the vector space V, and ρ^* is the associated right representation of F on V^*. The family of Hamiltonians $\{H_a \mid a\in V^*\}$ induces a Hamiltonian function H on the space $\mathfrak{s}^*_-$, defined by $H((T_eL_f)^*\alpha_f,\rho^*(f)a)=H_a(\alpha_f)$, thus yielding Lie-Poisson equations on $\mathfrak{s}^*_-$.

Again we explicitly formulate the implications of this procedure for the cases of interest in this paper.

3.1 Case A

Here $F=SO(3)\times SO(3)$ and we shall consider the symmetry group $G=SO(3)$. Define

$$F_K \;=\; \left\{\, (R_A,R_B)\in F \;\middle|\; R_A^T K R_B = K \,\right\}, \tag{14}$$

where $K\in\mathbb{R}^{3\times 3}$ is a dummy variable. We shall eventually fix $K=I_3$, the 3×3 identity matrix, whence F_{I_3} becomes $\left\{(R,R)\,\middle|\, R\in SO(3)\right\}$, and we recover the diagonal action (5) of the symmetry group $G=SO(3)$. As we shall see in §5.1, this choice of G may arise, for example, in the addition of a potential function which is added to stabilise the equilibrium $R_A = KR_B$; the choice $K = I_3$ corresponds to the two vehicles, A and B, having orientations in alignment.

We may now define a representation $\rho : SO(3) \times SO(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{3\times 3})$ by $\rho(R_A, R_B)K = R_A K (R_B)^T$ where $\mathrm{GL}(\mathbb{R}^{3\times 3})$ is the general linear group of the vector space $\mathbb{R}^{3\times 3}$ of three by three matrices. Given two matrices $Y \in \mathbb{R}^{n_1\times m_1}$, $Z \in \mathbb{R}^{n_2\times m_2}$, we define their *Kronecker Product* [17], denoted by $Y \otimes Z \in \mathbb{R}^{n_1 n_2\times m_1 m_2}$, as

$$Y \otimes Z \;=\; \left[\begin{array}{ccc} y_{11}Z & \cdots & y_{1m_1}Z \\ \vdots & \ddots & \vdots \\ y_{n_1 1}Z & \cdots & y_{n_1 m_1}Z \end{array}\right]. \tag{15}$$

With this definition it is easy to check that there is a direct correspondence between the elements of $R_A K R_B^T \in \mathbb{R}^{3\times 3}$ and those of $(R_A \otimes R_B)\widetilde{K} \in \mathbb{R}^9$, where, using the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, the latter is given by $\widetilde{K} = (e_1^T K, e_2^T K, e_3^T K)^T \in \mathbb{R}^9$. We can thus equivalently define $\rho : SO(3) \times SO(3) \longrightarrow \mathrm{GL}(\mathbb{R}^9)$ by $\rho(R_A, R_B)\widetilde{K} = (R_A \otimes R_B)\widetilde{K}$, where $\mathrm{GL}(\mathbb{R}^9)$ denotes the set of invertible linear mappings on $\mathbb{R}^9$.

Given this parameter space $V = \mathbb{R}^{3\times 3} \cong \mathbb{R}^9$, we may write the entire 15-dimensional space $S = F \times_\rho V = (SO(3) \times SO(3)) \times_\rho \mathbb{R}^9$ as a semi-direct product where group multiplication is given by

$$\begin{aligned} ((\bar{R}_A, \bar{R}_B), \bar{\widetilde{K}})((R_A, R_B), \widetilde{K}) &= ((\bar{R}_A, \bar{R}_B)(R_A, R_B), \rho(\bar{R}_A, \bar{R}_B)\widetilde{K} + \bar{\widetilde{K}}) \\ &= ((\bar{R}_A R_A, \bar{R}_B R_B, (\bar{R}_A \otimes \bar{R}_B)\widetilde{K} + \bar{\widetilde{K}}). \end{aligned}$$

Using this correspondence, an element in the Lie group S can be represented by a 16×16 matrix of the form

$$\left[\begin{array}{cccc} R_A & 0 & 0 & 0 \\ 0 & R_B & 0 & 0 \\ 0 & 0 & R_A \otimes R_B & \widetilde{K} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and the group action is given simply by matrix multiplication. Let ρ^* be the associated right representation of $SO(3) \times SO(3)$ on $\mathbb{R}^9$, given by

$$\rho^*(R_A, R_B)\widetilde{K} \;=\; (R_A^T \otimes R_B^T)\widetilde{K}. \tag{16}$$

With this definition, G may be identified with the isotropy subgroup

$$F_{\widetilde{K}} \;=\; \left\{ (R_A, R_B) \in F \;\middle|\; \rho^*(R_A, R_B)\widetilde{K} = \widetilde{K} \right\}. \tag{17}$$

By Theorem 3.1 the Hamiltonian dynamics on $T^*(SO(3) \times SO(3))$ can be reduced to a Lie-Poisson system on the fifteen-dimensional space $\mathfrak{s}_-^*$, the dual of the Lie algebra $\mathfrak{s}$ of S. In fact, $\mathfrak{s} = (\mathfrak{so}(3) \times \mathfrak{so}(3)) \times_{\rho'} \mathbb{R}^{3\times 3}$ where ρ' is the induced Lie algebra representation, given by

$$\rho'(\alpha, \beta)\, v \;=\; \alpha v \;-\; v\beta. \tag{18}$$

Here $(\alpha,\beta)\in\mathfrak{so}(3)\times\mathfrak{so}(3)$ and $v\in\mathbb{R}^{3\times 3}$. This implies that the Lie bracket of two elements of the Lie algebra $\mathfrak{s}$ is given by

$$\begin{aligned}[((\alpha,\beta,v),((\alpha',\beta'),v')] &= ([(\alpha,\beta),(\alpha',\beta')],\ \rho'(\alpha,\beta)\,v'-\rho'(\alpha',\beta')\,v)\\ &= (\alpha\alpha'-\alpha'\alpha,\beta\beta'-\beta'\beta,\alpha v'-v'\beta-\alpha' v+v\beta').\end{aligned}$$

Again using the definition for the Kronecker product, elements in the Lie algebra $\mathfrak{s}$ can be represented by 16×16 matrices of the form

$$\begin{bmatrix} \alpha & 0 & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \alpha\otimes I+I\otimes\beta & \tilde{v}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

where, as before, $\widetilde{v}=(e_1^Tv,e_2^Tv,e_3^Tv)^T$. The Lie bracket of two Lie algebra elements in this form is given simply by the matrix commutator.

In body coordinates, which describe the isomorphism $SO(3)\times\mathfrak{so}(3)^*_-\cong T^*SO(3)$, the reduction mapping is given by

$$\begin{array}{ccc} SO(3)\times\mathfrak{so}(3)^*_-\times SO(3)\times\mathfrak{so}(3)^*_- & \longrightarrow & \mathfrak{s}^*_- = \mathfrak{so}(3)^*_-\times\mathfrak{so}(3)^*_-\times V^*_-\\ (R_A,\pi_A,R_B,\pi_B) & \mapsto & (\pi_A,\pi_B,\Delta,\Sigma,\Gamma),\end{array}$$

where we denote the three columns of $(R_A^TKR_B)^T$ by Δ,Σ and Γ, respectively. In the particular case $K=I_3$ the embedded reduced phase space $\mathfrak{t}^*_-$ of (3) is defined by the equations

$$\begin{array}{lll} \Delta^2=1, & \Sigma^2=1, & \Gamma^2=1,\\ \Sigma\cdot\Gamma=0, & \Gamma\cdot\Delta=0, & \Delta\cdot\Sigma=0,\\ \det(\Delta,\Sigma,\Gamma)=1 & & \end{array}\tag{19}$$

and $(\Delta,\Sigma,\Gamma)\in SO(3)$ measures the relative orientation $R_B^TR_A$ of the two bodies.

Since we have performed reduction by means of the left action, the reduced phase space is $\mathfrak{s}^*_-$ (with lower signs in (12)). Defining the pairing between matrices in $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 3^*}$ to be $\langle Y,Z\rangle=\mathrm{Tr}(Y^TZ)$ yields the structure matrix

$$\Lambda(\pi_A,\pi_B,\Delta,\Sigma,\Gamma) = \begin{bmatrix} \widehat{\pi}_A & 0 & a & b & c\\ 0 & \widehat{\pi}_B & \widehat{\Delta} & \widehat{\Sigma} & \widehat{\Gamma}\\ -a^T & \widehat{\Delta} & 0 & 0 & 0\\ -b^T & \widehat{\Sigma} & 0 & 0 & 0\\ -c^T & \widehat{\Gamma} & 0 & 0 & 0\end{bmatrix}\tag{20}$$

of the Poisson bracket (12). Here $\Delta\mapsto\widehat{\Delta}$ is the usual isomorphism between vectors in $\mathbb{R}^3$ and skew-symmetric 3×3 matrices, while the 3×3 matrices a,b and c are given by

$$a=\begin{bmatrix}0\\ \Gamma^T\\ -\Sigma^T\end{bmatrix},\quad b=\begin{bmatrix}-\Gamma^T\\ 0\\ \Delta^T\end{bmatrix},\quad c=\begin{bmatrix}\Sigma^T\\ -\Delta^T\\ 0\end{bmatrix}.$$

For a given Hamiltonian, $H(\pi_A, \pi_B, \Delta, \Sigma, \Gamma)$, expressed in terms of the reduced variables, we may now write the system dynamics as

$$(\dot{\pi}_A, \dot{\pi}_B, \dot{\Delta}, \dot{\Sigma}, \dot{\Gamma})^T \;=\; \Lambda \nabla H(\pi_A, \pi_B, \Delta, \Sigma, \Gamma). \tag{21}$$

3.2 Case B

Here $F = SE(3) \times SE(3)$ and the symmetry group G, the semi-direct product of $SO(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$, is embedded as

$$F_K \;=\; \left\{ \, (R_A, b_A, R_B, b_B) \in F \;\middle|\; R_A^T K R_B = K \, \right\}$$

with the same dummy variable $K \in \mathbb{R}^{3\times 3}$ as in Case A. Analogous to Case A, we recover the action (8) for the fixed value $K = I_3$ of interest. The representation $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{3\times 3})$ is given by $\rho(R_A, b_A, R_B, b_B)K = R_A K (R_B)^T$ and thus is essentially the same as in Case A. Hence, the reduction proceeds via Theorem 3.1 as before, with the reduction mapping now reading

$$\begin{array}{ccc} SE(3) \times \mathfrak{se}(3)^*_- \times SE(3) \times \mathfrak{se}(3)^*_- & \longrightarrow & \mathfrak{s}^*_- \;=\; \mathfrak{se}(3)^*_- \times \mathfrak{se}(3)^*_- \times V^*_- \\ (R_A, b_A, \pi_A, p_A, R_B, b_B, \pi_B, p_B) & \mapsto & (\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) \end{array}$$

with the same convention for Δ, Σ, Γ as in Case A. The variables p_A, p_B are the linear momenta of the two bodies (measured in their respective body frames).

The elements $(R, b) \in SE(3)$ are in one-to-one correspondence with 4×4 matrices $\begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix}$ and computations similar to those of Case A yield the structure matrix

$$\Lambda(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) \;=\; \begin{bmatrix} \widehat{\pi}_A & \widehat{p}_A & 0 & 0 & a & b & c \\ \widehat{p}_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\pi}_B & \widehat{p}_B & \widehat{\Delta} & \widehat{\Sigma} & \widehat{\Gamma} \\ 0 & 0 & \widehat{p}_B & 0 & 0 & 0 & 0 \\ -a^T & 0 & \widehat{\Delta} & 0 & 0 & 0 & 0 \\ -b^T & 0 & \widehat{\Sigma} & 0 & 0 & 0 & 0 \\ -c^T & 0 & \widehat{\Gamma} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

of the Poisson bracket (12).

For a given Hamiltonian, $H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma)$, expressed in terms of the reduced variables, we may now write the system dynamics as

$$(\dot{\pi}_A, \dot{p}_A, \dot{\pi}_B, \dot{p}_B, \dot{\Delta}, \dot{\Sigma}, \dot{\Gamma})^T \;=\; \Lambda \nabla H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma). \tag{23}$$

3.3 Case C

Again $F = SE(3) \times SE(3)$, while the symmetry group $G = SE(3)$ is now embedded as

$$F_{(K,k)} \;=\; \left\{ \, (R_A, b_A, R_B, b_B) \in F \;\middle|\; R_A^T K R_B = K \, , \; R_A k + b_A - K b_B = k \, \right\}, \tag{24}$$

where the dummy variable has become $(K,k) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^3$. For the fixed value $(K,k) = (I_3, 0)$ we recover the diagonal action (10). One may easily check that $F_{(K,k)}$ may also be written

$$F_{(K,k)} = \left\{ (R_A, b_A, R_B, b_B) \in F \;\middle|\; \begin{bmatrix} R_A & 0 \\ -b_A^T R_A & 1 \end{bmatrix}^T \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_B & b_B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \right\}. \tag{25}$$

We now define the representation $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{3\times 3} \times \mathbb{R}^3)$ by

$$\rho(R_A, b_A, R_B, b_B) \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_A & 0 \\ -b_A^T R_A & 1 \end{bmatrix}^T \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_B & b_B \\ 0 & 1 \end{bmatrix}.$$

One may use the definition (15) to demonstrate that there is a direct correspondence between the elements of

$$\begin{bmatrix} R_A & 0 \\ -b_A^T R_A & 1 \end{bmatrix}^T \begin{bmatrix} K & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_B & b_B \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

and those of

$$\begin{bmatrix} R_A^T & -R_A^T b_A \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} R_B^T & 0 \\ b_B^T & 1 \end{bmatrix} \widetilde{K} \in \mathbb{R}^{16}$$

where $\widetilde{K} = (e_1^T K, k_1, e_2^T K, k_2, e_3^T K, k_3, 0, 0, 0, 1)^T$. We can thus equivalently define $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(\mathbb{R}^{16})$ by

$$\rho(R_A, b_A, R_B, b_B)\widetilde{K} = \begin{bmatrix} R_A^T & -R_A^T b_A \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} R_B^T & 0 \\ b_B^T & 1 \end{bmatrix} \widetilde{K}.$$

Note that $\rho(R_A, b_A, R_B, b_B)$ fixes the 12-dimensional affine subspace V of $\mathbb{R}^{16}$ that consists of vectors of the above form $\widetilde{K}$, i.e. vectors that have as last four components $0,0,0,1$. We can give V the structure of a vector space by (re)defining addition to only operate on the first 12 (or 15) entries of $\widetilde{K}$ and leaving the 16th component fixed at 1. In this way we are led to the desired $\rho : SE(3) \times SE(3) \longrightarrow \mathrm{GL}(V)$. Given this parameter space $V \cong \mathbb{R}^{12}$, we may write the entire 24-dimensional space $S = F \times_\rho V = (SE(3) \times SE(3)) \times_\rho V$ as a semi-direct product and proceed as in Case A. In particular, G may be identified with the isotropy subgroup

$$F_{\widetilde{K}} = \left\{ (R_A, b_A, R_B, b_B) \in F \;\middle|\; \rho^*(R_A, b_A, R_B, b_B)\widetilde{K} = \widetilde{K} \right\},$$

where the associated right representation of the symmetry group G is

$$\rho^*(R_A, b_A, R_B, b_B) = \begin{bmatrix} R_A^T & -R_A^T b_A \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} R_B^T & 0 \\ b_B^T & 1 \end{bmatrix}.$$

By Theorem 3.1 the Hamiltonian dynamics on $T^*(SE(3) \times SE(3))$ can be projected to a Lie-Poisson system on the 24-dimensional space $\mathfrak{s}^*_-$. This performs the reduction from 12 to 9 degrees of freedom and provides at the same time a realization of the reduced phase space (1) as a *Poisson submanifold* of the flat Poisson space $\mathfrak{s}^*_-$, the dual of the Lie algebra $\mathfrak{s}$ of S. In fact, $\mathfrak{s} = (\mathfrak{se}(3) \times \mathfrak{se}(3)) \times_{\rho'} V$ where ρ' is the induced Lie algebra representation

$$\rho'(\alpha, v_A, \beta, v_B) \;=\; \begin{bmatrix} \alpha & 0 \\ -v_A^T & 0 \end{bmatrix} \otimes I_4 + I_4 \otimes \begin{bmatrix} \beta & v_B \\ 0 & 0 \end{bmatrix}.$$

The reduction mapping is given by

$$\begin{array}{ccc} SE(3) \times \mathfrak{se}(3)^*_- \times SE(3) \times \mathfrak{se}(3)^*_- & \longrightarrow & \mathfrak{s}^*_- \;=\; \mathfrak{se}(3)^*_- \times \mathfrak{se}(3)^*_- \times V^*_- \\ (R_A, b_A, \pi_A, p_A, R_B, b_B, \pi_B, p_B) & \mapsto & (\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta) \end{array}$$

where in addition to the three columns Δ, Σ, Γ of $(R_A^T K R_B)^T$ we now have $\beta = R_A^T(k - b_A + K b_B)$. In the particular case $(K, k) = (I_3, 0)$ the embedded reduced phase space is still defined by (19) and the new variable β measures the relative position of the two bodies (in the body frame of underwater vehicle A). The Poisson bracket (12) has the structure matrix

$$\Lambda \;=\; \begin{bmatrix} \widehat{\pi}_A & \widehat{p}_A & 0 & 0 & a & b & c & \widehat{\beta} \\ \widehat{p}_A & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & \widehat{\pi}_B & \widehat{p}_B & \widehat{\Delta} & \widehat{\Sigma} & \widehat{\Gamma} & 0 \\ 0 & 0 & \widehat{p}_B & 0 & 0 & 0 & 0 & -(\Delta, \Sigma, \Gamma) \\ -a^T & 0 & \widehat{\Delta} & 0 & 0 & 0 & 0 & 0 \\ -b^T & 0 & \widehat{\Sigma} & 0 & 0 & 0 & 0 & 0 \\ -c^T & 0 & \widehat{\Gamma} & 0 & 0 & 0 & 0 & 0 \\ \widehat{\beta} & -I_3 & 0 & (\Delta, \Sigma, \Gamma)^T & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

For a given Hamiltonian, $H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta)$, expressed in terms of the reduced variables, we may now write the system dynamics as

$$(\dot{\pi}_A, \dot{p}_A, \dot{\pi}_B, \dot{p}_B, \dot{\Delta}, \dot{\Sigma}, \dot{\Gamma}, \dot{\beta})^T \;=\; \Lambda \nabla H(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta). \tag{27}$$

4 Casimirs in flat spaces

In the previous section we were able to realize the abstract reduced spaces of §2 as submanifolds of (concrete) flat Poisson spaces. We now study these Poisson spaces in their own right, determining the (maximal) rank of the bracket and looking for the Casimir functions. These Casimirs play a central role in proving stability for the coordinated rigid body networks. While we do not lose sight of our particular case(s) $K = I_3$ (and $k = 0$) which lead to the submanifolds $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ defined by (19), letting K (and k) vary allows to decompose the whole flat space(s) into various reduced manifolds. One may think of $\mathfrak{s}^*_-$ as collecting all possible reductions of F_K-actions for all possible matrices $K \in \mathbb{R}^{3\times 3}$ (and $k \in \mathbb{R}^3$).

4.1 Case A

On the whole 15-dimensional space $\mathfrak{s}^*_-$ the rank of the Poisson structure (20) cannot be more than 12; this follows because of the identically vanishing lower right 9×9-block. For points in $\mathfrak{t}^*_-$ we already know this rank to be equal to 8 since the symplectic leaves of $\mathfrak{t}^*_-$ are precisely the quotient manifolds $P_\mu = \mathbf{J}^{-1}(\mu)/_{G_\mu}$ of §2.1. However, regular reduction of the F_K-action with

$$K \ = \ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{28}$$

does indeed lead to 12-dimensional quotient manifolds. Correspondingly, on the whole space $\mathfrak{s}^*_-$ there are only three Casimir functions

$$\begin{aligned} C_1 &= \Delta^2 + \Sigma^2 + \Gamma^2 \\ C_2 &= \|\Sigma\times\Gamma\|^2 + \|\Gamma\times\Delta\|^2 + \|\Delta\times\Sigma\|^2 \\ C_3 &= (\Delta\times\Sigma)\cdot\Gamma \ = \ \det(\Delta,\Sigma,\Gamma). \end{aligned} \tag{29}$$

The functions $c_1 = \Delta^2, c_2 = \Sigma^2, c_3 = \Gamma^2, c_4 = \Sigma\cdot\Gamma, c_5 = \Gamma\cdot\Delta, c_6 = \Delta\cdot\Sigma$ of (19) are not themselves Casimirs. We may, however, use these functions to define a 10-dimensional Poisson submanifold,

$$\mathfrak{r}^*_- \ = \ \left\{ \Delta^2 = \Sigma^2 = \Gamma^2,\ \Sigma\cdot\Gamma = \Gamma\cdot\Delta = \Delta\cdot\Sigma = 0 \right\}, \tag{30}$$

on which the three Casimirs (29) are constrained by the two relations

$$3C_2 = C_1^2 \quad \text{and} \quad 27C_3^2 = C_1^3.$$

Thus on $\mathfrak{r}^*_-$ we have, effectively, only one Casimir remaining from (29). Since the rank of the Poisson bracket on $\mathfrak{r}^*_-$ is 8, there must be a further Casimir on this subspace; it is given by

$$C_0 \ = \ ((\Delta\times\Sigma)\cdot\Gamma)\left((\pi_A)^2 + (\pi_B)^2\right) + \frac{2(\Delta^2+\Sigma^2+\Gamma^2)}{3}\,\pi_B^T(\Delta,\Sigma,\Gamma)\pi_A. \tag{31}$$

Thus, in the particular case $K = \gamma R$ with nonzero scalar γ and $R \in SO(3)$, regular reduction of the F_K-action leads to 8-dimensional quotients. We may go on to identify the hypersurface $\mathfrak{t}^*_- \subseteq \mathfrak{r}^*_-$ as given by $C_3 \equiv 1$ (whence $C_1 \equiv 3$ and $C_2 \equiv 3$), confirming that $\mathfrak{t}^*_-$ is a Poisson submanifold of $\mathfrak{s}^*_-$ as well. On $\mathfrak{t}^*_-$ the (first) factors in the two terms of (31) become 1 and 2, respectively, and we see that the Casimir C_0 equals the total angular momentum (7) of the system. One easily computes that the rank of the Poisson structure on $\mathfrak{r}^*_-$ drops from 8 to 6 when $\pi_A = \pi_B$ and that there is no further drop of the rank when both vanish.

The Poisson submanifold $\mathfrak{r}^*_-$ of $\mathfrak{s}^*_-$ also contains the non-generic points $C_3 \equiv 0$ (whence $C_1 \equiv 0$ and $C_2 \equiv 0$, note that these points are not contained in $\mathfrak{t}^*_-$) where the rank of the Poisson structure drops from 8 to 4 for π_A and π_B in general position. Here $F_K = F_0 = SO(3)\times SO(3)$ and there are two sub-Casimirs π_A^2 and π_B^2; reducing by this large symmetry group abstracts from all inter-body information and leads to two uncoupled free rigid bodies.

4.2 Case B

On the whole 21-dimensional space $\mathfrak{s}^*_-$, the rank of the Poisson structure (22) cannot be more than 12 since the rows associated with the momentum variables can be linearly combined from the last 9 rows for a proper choice of Δ, Σ, Γ; this is possible if e.g. K is given by (28). We also know from §2.2 that the (maximal) symplectic leaves of the reduced phase space $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ defined by (19) are 12-dimensional as well. Thus, there is no drop in the rank of the Poisson bracket on $\mathfrak{s}^*_-$ when $K = I_3$. The parametrisation by invertible $K \in \mathbb{R}^{3\times 3}$ provides a foliation into symplectic leaves that all have the same dimension. However, the extreme choice $K = 0$ still leads to $F_0 = F$. In addition to the three Casimirs in (29) we found the Casimirs

$$\begin{aligned} C_4 &= (p_A)^2 \\ C_5 &= (p_B)^2 \\ C_6 &= p_B^T(\Delta, \Sigma, \Gamma) p_A \end{aligned} \tag{32}$$

which equal (9) on $\mathfrak{t}^*_-$. We conjecture that there are three further Casimirs C_7, C_8, C_9, but these have not yet been found.

By (30) we again define a submanifold $\mathfrak{r}^*_- \subseteq \mathfrak{s}^*_-$ which is now 16-dimensional and on which the Poisson structure has maximal rank 12. The 12-dimensional symplectic leaves of $\mathfrak{r}^*_-$ are the level sets of the remaining Casimirs C_3, C_4, C_5, C_6. Since $\mathfrak{t}^*_- \subseteq \mathfrak{r}^*_-$ is once more defined by $C_3 \equiv 1$, those level sets with $C_3 = 1$ yield the symplectic leaves of $\mathfrak{t}^*_-$. There is a further drop of the rank for certain special configurations of the four vectors π_A, p_A, π_B and p_B or when $K = 0$.

4.3 Case C

Here the (maximal) symplectic leaves of the 18-dimensional reduced phase space $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ (defined by (19)) have dimension 16, while the maximal rank of the Poisson structure (26) on the whole 24-dimensional Poisson space $\mathfrak{s}^*_-$ is 18. Indeed, the extra rows in (26) yield, e.g. for K given by (28), 6 additional linear independent rows in comparison with (22). Correspondingly, we have the Casimir functions C_1, C_2, C_3 of (29) and possibly the three others C_7, C_8, C_9 conjectured to exist for Case B.

On the Poisson submanifold $\mathfrak{t}^*_- \subseteq \mathfrak{s}^*_-$ defined by (19) we have $C_1 = C_2 = 3C_3 = 3$, whence the symplectic leaves are given by the two Casimirs

$$\begin{aligned} C_{10} &= (p_A)^2 + (p_B)^2 + 2p_B^T(\Delta, \Sigma, \Gamma) p_A \\ C_{11} &= \pi_A \cdot p_A + p_B^T(\Delta, \Sigma, \Gamma)\pi_A + \pi_B^T(\Delta, \Sigma, \Gamma) p_A \\ &\quad + \pi_B \cdot p_B + ((\Delta, \Sigma, \Gamma)\beta) \cdot (p_B \times (\Delta, \Sigma, \Gamma) p_A). \end{aligned}$$

We stress that C_{10} and C_{11} are Casimirs only on the Poisson manifold $\mathfrak{t}^*_-$, but not on the whole space $\mathfrak{s}^*_-$. They simplify to (11) when $(K, k) = (I_3, 0)$.

5 Controlled equations of motion

The controlled equations of motion are obtained taking the Poisson bracket with the Hamiltonian, where the Hamiltonian is the sum of the kinetic and the potential energies. In our examples, the kinetic energy is completely specified by the problem description; it is simply the rotational kinetic energy of two satellites in $SO(3)$ (Case A) or the translational plus rotational kinetic energies of two underwater vehicles in $SE(3)$ (Cases B and C). In contrast, the potential energy for the uncontrolled system is assumed to be zero. In the controlled system we design the potential energy to achieve the desired control, assuming that the control design is able to realize the ensuing requirements (this particular kind of control design was previously discussed in [6]). Specifically, for orientation alignment (Cases A, B and C), it is required that each body can provide three independent torque inputs. For relative position control (Case C only), it is also required that each body can provide three independent force inputs. In Cases A and C the vehicles are therefore required to be fully actuated.

We require this designed potential energy to be invariant under the action of a subgroup G of $SO(3)\times SO(3)$ or $SE(3)\times SE(3)$, respectively, since we only want the relative orientation (and relative position in Case C) to enter. This subgroup G is of the form $F_{\widetilde{K}}$ as in (13) of Theorem 3.1 with parameter $\widetilde{K}$. This can be ensured by defining the potential energy on the reduced phase space $\mathfrak{k}^*_-$, as the restriction of some formal potential energy on $\mathfrak{s}^*_-$. In future work we will incorporate both the interaction with the environment and the mission one wants the satellites/underwater vehicles to achieve by means of potential energy terms that *do* break the G-symmetry (and thus have to be formulated on the original phase space).

5.1 Case A

Here we consider the problem of aligning two satellites that have purely rotational dynamics. We model each satellite as a free rigid body with moment of inertia matrix J, and write Ω_k for the angular velocity vector of the body k with respect to its body-fixed frame.

In this case, the kinetic energy of the system is merely rotational. In the unreduced space, $T^*SO(3)\times T^*SO(3)$, we write it as

$$T \;=\; \frac{1}{2}\left(\Omega_A^T J\Omega_A + \Omega_B^T J\Omega_B\right) \tag{33}$$

whence the expression in the reduced variables reads

$$T \;=\; \frac{1}{2}\left[\pi_A^T J^{-1}\pi_A \;+\; \pi_B^T J^{-1}\pi_B\right].$$

To enforce orientation matching we formulate on $\mathfrak{s}^*_-$ the control with potential energy

$$U \;=\; \sigma\left(\Delta\cdot e_1 + \Sigma\cdot e_2 + \Gamma\cdot e_3\right) \tag{34}$$

where σ is a scalar control gain. For $K=I_3$ and negative σ, an aligned orientation $R_A=R_B$ then minimizes the potential energy (recall that Δ,Σ and Γ denote the three columns of $(R_A^T K R_B)^T$ and simplify for $K=I_3$ and $R_A=R_B$ to the canonical basis $\{e_1,e_2,e_3\}$ of $\mathbb{R}^3$). Writing $H=$

$T+U$ and substituting into (21), one finds that the reduced equations of motion are

$$\begin{aligned}
\dot{\pi}_A &= \pi_A \times J^{-1}\pi_A \,-\, \sigma\,(\Delta\times e_1 + \Sigma\times e_2 + \Gamma\times e_3) \\
\dot{\pi}_B &= \pi_B \times J^{-1}\pi_B \,+\, \sigma\,(\Delta\times e_1 + \Sigma\times e_2 + \Gamma\times e_3) \\
\dot{\Delta} &= \frac{\pi_A^3}{J_3}\Sigma - \frac{\pi_A^2}{J_2}\Gamma \,+\, \Delta\times J^{-1}\pi_B \\
\dot{\Sigma} &= \frac{\pi_A^1}{J_1}\Gamma - \frac{\pi_A^3}{J_3}\Delta \,+\, \Sigma\times J^{-1}\pi_B \\
\dot{\Gamma} &= \frac{\pi_A^2}{J_2}\Delta - \frac{\pi_A^1}{J_1}\Sigma \,+\, \Gamma\times J^{-1}\pi_B.
\end{aligned} \tag{35}$$

The control torque applied to body A is thus computed to be

$$u_A \;=\; -\sigma\,(\Delta\times e_1 + \Sigma\times e_2 + \Gamma\times e_3)$$

and the control torque applied to body B is $u_B = -u_A$. As discussed in §4.1 the Casimir functions (29) and (31) are conserved under the flow of this dynamical system. In the unreduced space $T^*SO(3)\times T^*SO(3)$ the potential (34) takes the (rather intuitive) form

$$U \;=\; \sigma\sum_{i=1}^{3} e_i^T R_A^T K R_B e_i\,. \tag{36}$$

5.2 Case B

Here we consider the case of two underwater vehicles as an example of a pair of bodies with translational as well as rotational dynamics. We assume they are identical and model each as an ellipsoidal body of mass m, as in [5]. For each vehicle, the matrix J will now denote the sum of the body inertia and the *added inertia* from the potential flow model of the fluid. Similarly, let M denote the sum of the body mass m multiplied by the identity matrix and the *added mass matrix*. We assume that m is also the mass of the displaced fluid so that each vehicle is neutrally buoyant. If each ellipsoid has uniformly distributed mass, the center of buoyancy is coincident with the center of gravity and both J and M are diagonal in a coordinate system defined by the ellipsoid's principal axes. We shall denote the translational velocity of each vehicle in body coordinates as v_k, and write $M = \mathrm{diag}(m_1, m_2, m_3)$, assuming $m_3 > m_2 > m_1$. In this case, the kinetic energy is now written as

$$T \;=\; \frac{1}{2}\left(\Omega_A^T J\Omega_A + \Omega_B^T J\Omega_B \,+\, v_A^T M v_A + v_B^T M v_B\right) \tag{37}$$

and turns into

$$T \;=\; \frac{1}{2}\left[\pi_A^T J^{-1}\pi_A + \pi_B^T J^{-1}\pi_B \,+\, p_A^T M^{-1} p_A + p_B^T M^{-1} p_B\right] \tag{38}$$

in terms of the reduced variables. We continue to write the control potential energy as (34) which still takes the form (36) in the unreduced space $T^*SE(3)\times T^*SE(3)$. Writing $H = T+U$

and substituting into (23), one finds that the reduced equations of motion become

$$
\begin{aligned}
\dot{\pi}_A &= \pi_A \times J^{-1}\pi_A \,+\, p_A \times M^{-1} p_A \,-\, \sigma\,(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3) \\
\dot{p}_A &= p_A \times J^{-1}\pi_A \\
\dot{\pi}_B &= \pi_B \times J^{-1}\pi_B \,+\, p_B \times M^{-1} p_B \,+\, \sigma\,(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3) \\
\dot{p}_B &= p_B \times J^{-1}\pi_B \\
\dot{\Delta} &= \frac{\pi_A^3}{J_3}\Sigma - \frac{\pi_A^2}{J_2}\Gamma \,+\, \Delta \times J^{-1}\pi_B \\
\dot{\Sigma} &= \frac{\pi_A^1}{J_1}\Gamma - \frac{\pi_A^3}{J_3}\Delta \,+\, \Sigma \times J^{-1}\pi_B \\
\dot{\Gamma} &= \frac{\pi_A^2}{J_2}\Delta - \frac{\pi_A^1}{J_1}\Sigma \,+\, \Gamma \times J^{-1}\pi_B.
\end{aligned}
\tag{39}
$$

The control inputs are the same torques as in Case A.

5.3 Case C

Here we focus attention on the case of the two underwater vehicles of Case B where we now seek to add a control to stabilise the configuration where both vehicles are not only aligned, but also separated by a prescribed relative position vector d_{AB}, directed from body A to body B. In the reduced variables the kinetic energy still reads (38) and we choose

$$
U \;=\; \frac{1}{2}\,(\beta + d_{AB})^T\, C\,(\beta + d_{AB}) \,+\, \sigma\,(\Delta \cdot e_1 + \Sigma \cdot e_2 + \Gamma \cdot e_3) \tag{40}
$$

as potential energy where C is a constant 3×3 matrix. Using (27) the reduced equations of motion defined by the Hamiltonian $H = T + U$ are written as

$$
\begin{aligned}
\dot{\pi}_A &= \pi_A \times J^{-1}\pi_A \,+\, p_A \times M^{-1} p_A \,+\, \beta \times C(\beta + d_{AB}) \\
&\quad -\, \sigma\,(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3) \\
\dot{p}_A &= p_A \times J^{-1}\pi_A \,+\, C(\beta + d_{AB}) \\
\dot{\pi}_B &= \pi_B \times J^{-1}\pi_B \,+\, p_B \times M^{-1} p_B \,+\, \sigma\,(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3) \\
\dot{p}_B &= p_B \times J^{-1}\pi_B \,-\, (\Delta, \Sigma, \Gamma)\, C\,(\beta + d_{AB}) \\
\dot{\Delta} &= \frac{\pi_A^3}{J_3}\Sigma - \frac{\pi_A^2}{J_2}\Gamma \,+\, \Delta \times J^{-1}\pi_B \\
\dot{\Sigma} &= \frac{\pi_A^1}{J_1}\Gamma - \frac{\pi_A^3}{J_3}\Delta \,+\, \Sigma \times J^{-1}\pi_B \\
\dot{\Gamma} &= \frac{\pi_A^2}{J_2}\Delta - \frac{\pi_A^1}{J_1}\Sigma \,+\, \Gamma \times J^{-1}\pi_B \\
\dot{\beta} &= \beta \times J^{-1}\pi_A \,-\, M^{-1} p_A \,+\, (\Delta, \Sigma, \Gamma)^T M^{-1} p_B.
\end{aligned}
\tag{41}
$$

Here the control torque applied to body A is thus computed to be

$$
u_A \;=\; \beta \times C(\beta + d_{AB}) \,-\, u_B,
$$

where the control torque applied to body B is, as in Cases A and B,

$$u_B \;=\; \sigma\,(\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3)\,.$$

Additionally, we find there are control forces

$$f_A \;=\; C\,(\beta + d_{AB}) \quad \text{and} \quad f_B \;=\; -(\Delta,\Sigma,\Gamma)\,C\,(\beta + d_{AB})$$

acting on bodies A and B, respectively.

In the unreduced space $T^*SE(3) \times T^*SE(3)$, the kinetic energy is still given by (37) and we now use the control with potential energy

$$\begin{aligned} U \;&=\; \frac{1}{2}\left(R_A^T(R_A k + b_A - K b_B) + d_{AB}\right)^T C\left(R_A^T(R_A k + b_A - K b_B) + d_{AB}\right) \\ &\quad + \;\sigma \sum_{i=1}^{3} e_i^T R_A^T K R_B e_i. \end{aligned}$$

Recall that we made in §3.3 the arbitrary choice to express the inter-vehicle distance in terms of the body frame of vehicle A. We observe that for $(K,k) = (I_3, 0)$, positive definite matrix C and $\sigma < 0$, the aligned configuration $R_A = R_B$ with the vehicles having the desired relative position $d_{AB} = -R_A^T(b_A - K b_B)$ minimizes the potential energy.

6 Equilibria of the reduced system

The first step in analysing the closed-loop dynamical systems defined by (35), (39) and (41) consists in identifying the simplest possible motions: the (relative) equilibria. In the interest of avoiding a great deal of unwieldy algebra, we shall only exhaustively discuss the equilibria for Case A.

6.1 Case A

Careful consideration of the last three equations of (35) indicates that, as might be expected, all equilibria in this case are non-generic, i.e. they are contained in the lower-dimensional symplectic leaf $\pi_A = \pi_B$. Let us set $\pi_0 = \pi_A = \pi_B$.

Since the control torques of bodies A and B are equal and opposite, it is clear that for real equilibria to exist we require

$$\pi_0 \times J^{-1}\pi_0 \;=\; 0. \tag{42}$$

For bodies with neither spherical nor axial symmetry, elementary considerations reveal that no real solutions exist for this equation with all three components of π_0 different from zero. While real solutions to (42) exist with merely one component of π_0 equal to zero, it is not possible to simultaneously satisfy the equations $\dot{\Delta} = 0\,, \dot{\Sigma} = 0\,, \dot{\Gamma} = 0$ (since this requires two of Δ, Σ

and Γ to be parallel when, by definition, they must be orthogonal). We must therefore have two components of π_0 equal to zero. Additionally, an equilibrium in this system requires that

$$\Delta \times e_1 + \Sigma \times e_2 + \Gamma \times e_3 \;=\; \begin{pmatrix} \Gamma_2 - \Sigma_3 \\ \Delta_3 - \Gamma_1 \\ \Sigma_1 - \Delta_2 \end{pmatrix} \;=\; 0 \tag{43}$$

and, consequently, the matrix (Δ, Σ, Γ) must be symmetric.

We use the unit quaternion representation (see [11]) for the matrix $(\Delta, \Sigma, \Gamma) \in SO(3)$ to rewrite its columns as

$$\begin{aligned}
\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ 2(q_1 q_2 - q_0 q_3) \\ 2(q_1 q_3 + q_0 q_2) \end{bmatrix} \\
\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{bmatrix} &= \begin{bmatrix} 2(q_1 q_2 + q_0 q_3) \\ q_0^2 - q_1^2 + q_2^2 - q_3^2 \\ 2(q_2 q_3 - q_0 q_1) \end{bmatrix} \\
\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} &= \begin{bmatrix} 2(q_1 q_3 - q_0 q_2) \\ 2(q_2 q_3 + q_0 q_1) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}
\end{aligned}$$

where the q_j are real numbers such that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

The condition that (Δ, Σ, Γ) is symmetric is thus equivalent to stating that either $q_1 = q_2 = q_3 = 0$, in which case (Δ, Σ, Γ) is the identity matrix and the bodies A and B are aligned or, $q_0 = 0$, in which case the bodies need not be aligned.

For the case where the bodies are aligned we may immediately write down the form of two equilibria. This simplest possible case consists of the two bodies being stationary and aligned,

$$(\pi_A, \pi_B, \Delta, \Sigma, \Gamma) \;=\; (0, 0, e_1, e_2, e_3), \tag{44}$$

while the next simplest consists of the two rigid bodies spinning synchronously about their ith axis of inertia, i.e.

$$(\pi_A, \pi_B, \Delta, \Sigma, \Gamma) \;=\; (\pi_0 e_i, \pi_0 e_i, e_1, e_2, e_3). \tag{45}$$

For the non-aligned configurations we introduce spherical coordinates

$$\begin{aligned}
q_1 &= \sin\varphi \cos\psi \\
q_2 &= \sin\varphi \sin\psi \\
q_3 &= \cos\varphi
\end{aligned}$$

to write these equilibria as

$$\begin{aligned}
\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} &= \begin{bmatrix} \sin^2\varphi\cos(2\psi) - \cos^2\varphi \\ \sin^2\varphi\sin(2\psi) \\ \sin(2\varphi)\cos\psi \end{bmatrix} \\
\begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{bmatrix} &= \begin{bmatrix} \sin^2\varphi\sin(2\psi) \\ -\sin^2\varphi\cos(2\psi) - \cos^2\varphi \\ \sin(2\varphi)\sin\psi \end{bmatrix} \\
\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} &= \begin{bmatrix} \sin(2\varphi)\cos\psi \\ \sin(2\varphi)\sin\psi \\ \cos(2\varphi) \end{bmatrix}.
\end{aligned} \tag{46}$$

For the case of two stationary rigid bodies in a non-aligned configuration, there are no additional restrictions on the columns of the relative rotation matrix given in (46). In contrast, for two bodies rotating and non-aligned with, say, $\pi_A = \pi_B = \pi_0 e_i$, we see from (35) that $(\Delta,\Sigma,\Gamma)^T e_i$ must be parallel to e_i and, hence, from (46), the relative rotation matrix is of the form:

$$(\Delta,\Sigma,\Gamma) = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \tag{47}$$

where one of the ε_i is positive and the other two are negative. This configuration corresponds to one axis of the two bodies A and B being in alignment, while the other two axes are out of alignment by 180 degrees. This equilibrium therefore corresponds to the two bodies ‘pointed’ in opposite directions.

6.2 Cases B & C

Here we merely give the principal equilibria of interest, ones where the two underwater vehicles are not spinning and have aligned orientation. In Case B the bodies may still move in different directions, whence the equilibria are of the form

$$(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma) = (0, C_A e_i, 0, C_B e_j, e_1, e_2, e_3). \tag{48}$$

In Case C the two bodies have to move in the same direction and maintain their relative position, yielding

$$(\pi_A, p_A, \pi_B, p_B, \Delta, \Sigma, \Gamma, \beta) = (0, Ce_i, 0, Ce_i, e_1, e_2, e_3, -d_{AB}).$$

We shall prove the stability of the equilibrium (48) for motion along the shortest axis in §7.2.

7 Stability of Equilibria

We now obtain results on the stability of the equilibria of the controlled network dynamics found in the previous section. Where possible, we use the energy-Casimir method [8] to prove stability for the aligned equilibria of Cases A and B. We note that the application of the energy-Casimir method does not require any knowledge of the Poisson structure other than (some of) the Casimirs. In particular, only the partial derivatives of these (and of the Hamiltonian) within the flat space $\mathfrak{s}^*_-$ are needed, while the precise way in which the reduced manifold $\mathfrak{t}^*_-$ lies within $\mathfrak{s}^*_-$ is relevant only to the extent that this determines the Casimirs.

7.1 Case A

We claim that the equilibria of the form (44) are stable in the sense of Lyapunov. Recall that these equilibria correspond to two aligned and stationary rigid bodies. To find a Lyapunov function, we make the ansatz

$$H_\Phi \;=\; H \,+\, \Phi(C_1, C_2, C_3)$$

where we have included as arguments of Φ only those Casimirs (29) that are valid on all of $\mathfrak{s}^*_-$. Let

$$\Phi^{(l)} \;:=\; \frac{\partial \Phi}{\partial C_l}(3,3,1)$$

denote the partial derivative of Φ with respect to C_l ($l = 1,2$ or 3) at the equilibrium (44). One readily checks [18] that the first variation of H_Φ vanishes at the equilibrium provided that $\Phi^{(1)} = -\sigma/2$ and $\Phi^{(2)} = \Phi^{(3)} = 0$. We may thus choose Φ to have the simple form

$$\Phi \;=\; -\frac{\sigma}{2}\left(\Delta^2 + \Sigma^2 + \Gamma^2\right).$$

For the use of the energy-Casimir method we now check for definiteness of the second variation at the equilibrium point. The Hessian of H_Φ has an associated block-diagonal matrix given by

$$\begin{bmatrix} J^{-1} & & & & \\ & J^{-1} & & & \\ & & -\sigma I_3 & & \\ & & & -\sigma I_3 & \\ & & & & -\sigma I_3 \end{bmatrix}.$$

The Hessian is thus positive definite provided $\sigma < 0$. Stability follows from the statement of the energy-Casimir method ([8], §1.7).

For the other equilibria in Case A, it is less straightforward to find a function Φ for which it is easy to show the Hessian to be definite. Stability results are proved for these cases using alternative energy methods in [13].

7.2 Case B

We claim that the equilibria of the form (48) with $i = j = 3$ are stable in the sense of Lyapunov. Since we have assumed $m_3 > m_2 > m_1$, these equilibria correspond to two aligned underwater vehicles moving in the same direction each with their shortest principal axis aligned with the direction of motion. Here we make the ansatz

$$H_\Phi \;=\; H + \Phi(C_1, C_2, C_3, C_4, C_5, C_6), \tag{49}$$

where we now include as arguments of Φ also the Casimirs (32), which are valid on all $\mathfrak{s}_-^*$. Recall from §4.2 that there may be three further Casimirs, so it is not a priori clear that our ansatz will indeed yield the desired result. The first variation of H_Φ vanishes at the equilibrium provided that

$$\begin{aligned}
\frac{C_A}{m_3} + 2C_A\Phi^{(4)} + C_B\Phi^{(6)} &= 0\\
\frac{C_B}{m_3} + 2C_B\Phi^{(5)} + C_A\Phi^{(6)} &= 0\\
\sigma + 2\Phi^{(1)} + 4\Phi^{(2)} + \Phi^{(3)} &= 0\\
\sigma + C_AC_B\Phi^{(6)} + 2\Phi^{(1)} + 4\Phi^{(2)} + \Phi^{(3)} &= 0.
\end{aligned}$$

This implies $\Phi^{(6)} = 0$, and taking $\Phi^{(2)} = \Phi^{(3)} = 0$ we may choose Φ to have the simple form

$$\Phi \;=\; ((p_A)^2 - C_A^2)^2 - \frac{(p_A)^2}{2m_3} + ((p_B)^2 - C_B^2)^2 - \frac{(p_B)^2}{2m_3} - \frac{\sigma}{2}(\Delta^2 + \Sigma^2 + \Gamma^2). \tag{50}$$

Here the Hessian of H_Φ has an associated matrix given by

$$\begin{bmatrix}
J^{-1} &&&&&& \\
& \widetilde{M}_A &&&&& \\
&& J^{-1} &&&& \\
&&& \widetilde{M}_B &&& \\
&&&& -\sigma I_3 && \\
&&&&& -\sigma I_3 & \\
&&&&&& -\sigma I_3
\end{bmatrix} \tag{51}$$

where

$$\widetilde{M}_k \;=\; \begin{bmatrix}
1/m_1 - 1/m_3 && \\
& 1/m_2 - 1/m_3 & \\
&& 8C_k^2
\end{bmatrix}$$

for $k = A, B$, and is thus again positive definite provided $\sigma < 0$. As before, stability follows from §1.7 in [8].

8 Simulation Example

As an example, we consider Case B of two ellipsoidal underwater vehicles each with $M = J = \mathrm{diag}(1,2,3)$. We set these vehicles to have initial conditions

$$\pi_A \;=\; p_A \;=\; \pi_B \;=\; p_B \;=\; (1,1,1)^T \tag{52a}$$

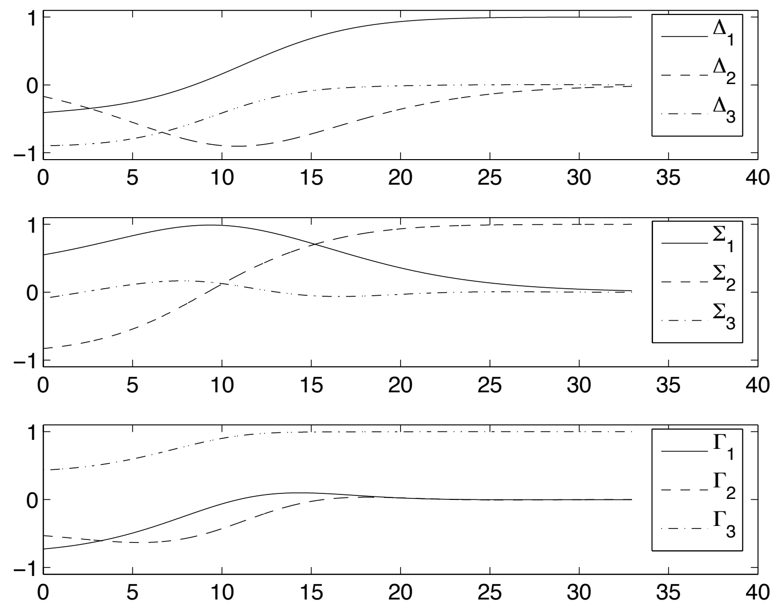

Figure 2: Dynamics of the Δ, Σ, and Γ variables for the example given in (52).

which means that each body (initially) moves in a direction that is not a principal axis and (initially) spins around that same axis. For the relative orientation we randomly choose the matrix

$$(\Delta,\Sigma,\Gamma) \;=\; \begin{pmatrix} -0.7079 & -0.7013 & -0.0843 \\ -0.6254 & 0.5667 & 0.5364 \\ -0.3284 & 0.4324 & -0.8397 \end{pmatrix} \tag{52b}$$

as an initial condition. Clearly, for this non-identiy choice of the relative orientation matrix the vector $(1,1,1)^T$ in body coordinates of vehicle A and $(1,1,1)^T$ in body coordinates of vehicle B do not lead to the same vector measured in an inertial frame.

We formulated our model in the Hamiltonian context, where energy is conserved and friction excluded from the outset. When applying such models to real underwater vehicles, there is always dissipation. Alternatively, one may also add control terms that introduce dissipation (cf. [12]). In these ways the equilibria (48) with $e_i = e_j = e_3$ turn from merely being stable in the sense of Lyapunov into asymptotically stable equilibria. We therefore integrate (39) with a modest amount of linear dissipation added to the $\dot{\pi}_A$, $\dot{p}_A$, $\dot{\pi}_B$ and $\dot{p}_B$ equations, with damping factor set to 0.5. The resulting dynamics of the Δ, Σ and Γ variables are shown in Figure 2 where we observe that Δ_1, Σ_2 and Γ_3 all tend to 1 while the remaining variables tend to zero, i.e. $(\Delta,\Sigma,\Gamma) \to I_3$ as desired. Snapshots of two schematic vehicles rotating in space for the example of (52) are shown in Figure 3, with the translational motion of the vehicles neglected from the figure for clarity. Because of the suitable damping, the two bodies converge on the equilibrium with orientations in alignment, while the translational motion converges to the (now common) direction of the shortest principal axis and the rotational motion is damped out completely.

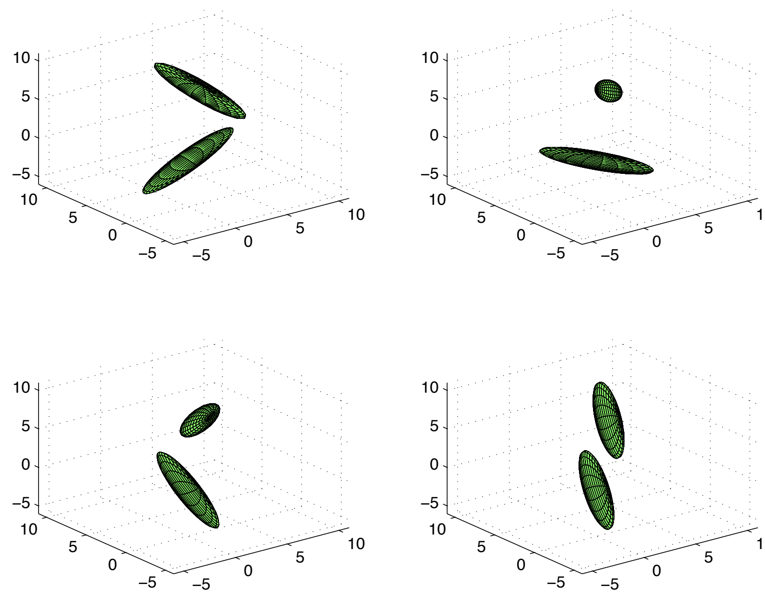

Figure 3: The two underwater vehicles of (52) converging towards alignment when suitable damping is added.

9 Systems of N vehicles

The case of N underwater vehicles, when considered as $N(N-1)/2$ possible pairwise interactions, can be treated as a natural extension of the two-vehicle problem. We concentrate on Case B and label each of our N vehicles with an index (i) for $i = 1, \ldots N$, and now seek an appropriate artificial potential to stabilise the equilibrium $R_{(j)} = K_{(jk)} R_{(k)}$ for all pairs (j,k) in the set

$$\mathfrak{I} \; = \; \left\{ \, (j,k) \in \mathbb{N}^2 \;\middle|\; 1 \leq j < k \leq N \, \right\}$$

where $K_{(jk)} \in \mathbb{R}^{3\times 3}$ is a (matrix) parameter relating the orientation of body (j) to that of body (k). When re-interpreting equilibria singled out by a choice of K as relative equilibria of the dynamics of our group of vehicles, it is important that we satisfy the two sets (53) of nonlinear equations

$$K_{(jk)} K_{(jk)}^T \;=\; I_3 \quad \text{for all } i,j,k \text{ with } j < i < k \tag{53a}$$

$$\det K_{(jk)} \;=\; 1 \quad \text{for all } i,j,k \text{ with } j < i < k \tag{53b}$$

to ensure $K_{(jk)} \in \mathrm{SO}(3)$ for all $(j,k) \in \mathfrak{I}$ and, for consistency concerning subgroups of three bodies,

$$K_{(ji)} K_{(ik)} \;=\; K_{(jk)} \quad \text{for all } i,j,k \text{ with } j < i < k\,. \tag{53c}$$

These three requirements are trivially satisfied in the case

$$K_{(jk)} \;=\; I_3 \quad \text{for all } (j,k) \in \mathfrak{I}. \tag{54}$$

As in §3, the choice (54) corresponds to our desire to stabilize the equilibrium consisting of all N vehicles having the same orientation. We note that it is not necessary to include potential functions for all possible $N(N-1)/2$ pairwise interactions in order to stabilise this equilibrium. In the language of Graph Theory, all possible pairwise would correspond to a complete graph, where each of the vehicles is represented by a node on the graph, and the presence of a potential function coupling the orientations of any two vehicles may be described as an *undirected* link in the graph. Rather than this complete graph of all $N(N-1)/2$ pairwise interactions, we merely need at least $N-1$ of them to ensure that the graph is *connected*. Indeed, the proper generalization of the symmetry group $SO(3)\times_\delta(\mathbb{R}^3\times\mathbb{R}^3)$ for two bodies in Case B is the semi-direct product

$$G \;=\; SO(3)\;\times_\delta\;(\mathbb{R}^3\times\ldots\times\mathbb{R}^3)$$

of $SO(3)$ with $\mathbb{R}^{3N}$, where the group of rotations acts diagonally on each of the N three-dimensional spaces. This leads straightforwardly to the reduced phase space

$$T^*SE(3)^N\big/G \;\cong\; (\mathfrak{se}(3)^*)^N\times(SO(3))^{N-1} \tag{55}$$

where the $(N-1)$-fold identity in the second factor stands for alignment of all bodies.

When designing our control inputs we aim for the flexibility and (structural) stability of e.g. fish in a school that try to align with all of their nearest neighbours (and not only one of them). We therefore use the generality of the semi-direct product reduction to let our dummy variable K vary in a vector space V that is possibly larger than $(\mathbb{R}^{3\times3})^{N-1}$. Let therefore $\mathfrak{J}\subseteq\mathfrak{I}$ be a set of $\mathfrak{n}$ ordered pairs (j,k) with $N-1\le\mathfrak{n}\le N(N-1)/2$, such that the associated graph is connected and consider

$$V \;=\; (\mathbb{R}^{3\times3})^{\mathfrak{J}} \;\cong\; \mathbb{R}^{9\mathfrak{n}}.$$

For this latter isomorphism we define for each $K_{(jk)}$ an associated vector, $\widetilde{K}_{(jk)}=(e_1^TK_{(jk)},e_2^TK_{(jk)},e_3^TK_{(jk)})^T\in\mathbb{R}^9$ and then $\widetilde{K}\in\mathbb{R}^{9\mathfrak{n}}$ by

$$\widetilde{K} \;=\; \left(\widetilde{K}_{(jk)}\right)_{(j,k)\in\mathfrak{J}}.$$

Similar to (16) the associated right representation $\rho^*:\mathrm{SE}(3)^N\to\mathrm{GL}(\mathbb{R}^{9\mathfrak{n}})$ is given by

$$\rho(R_{(1)},b_{(1)},\ldots,R_{(N)},b_{(N)})\widetilde{K} \;=\; \left((R_{(j)}^T\otimes R_{(k)}^T)\widetilde{K}_{(jk)}\right)_{(j,k)\in\mathfrak{J}}$$

whence the isotropy subgroup reads

$$F_{\widetilde{K}} \;=\; \left\{\,(R_{(j)},b_{(j)})_j\in SE(3)^N \;\middle|\; \rho^*(R_{(j)},b_{(j)})_j)\widetilde{K}=\widetilde{K}\,\right\}.$$

Note that for each $K\in V$ satisfying (53) we have $G\cong F_{\widetilde{K}}$ and (55) is embedded in

$$\mathfrak{s}_-^* \;=\; (\mathfrak{se}(3)^*)^N\times V_-^*.$$

With these definitions, the semi-direct product reduction follows through in much the same way as in §3. In particular, the resulting structure matrix will be block diagonal with each block of the form (22). Most importantly, one can use the linear coordinates

$$\big((\pi_{(j)}, p_{(j)})_j\,, (\Delta_{jk}, \Sigma_{jk}, \Gamma_{jk})_{(j,k)\in\mathfrak{J}}\big) \;\in\; \mathfrak{s}_-^*$$

on the reduced phase space to define the artificial potential

$$U \;=\; \sum_{(j,k)\in\mathfrak{J}} \sigma_{jk}\left(\Delta_{jk}\cdot e_1 + \Sigma_{jk}\cdot e_2 + \Gamma_{jk}\cdot e_3\right)$$

where $(\sigma_{jk})_{(j,k)\in\mathfrak{J}} \in \mathbb{R}^{\mathfrak{n}\times\mathfrak{n}}$ is a matrix of control gains. In the unreduced space $T^*SE(3)^N$ this potential takes the form

$$U \;=\; \sum_{(j,k)\in\mathfrak{J}} \sigma_{jk} \sum_{i=1}^{3} e_i^T R_{(j)}^T K_{(jk)} R_{(k)} e_i$$

and by construction we have $F_{\widetilde{K}}$-symmetry. The kinetic energy is the sum of terms (37) which turns into the corresponding sum of terms (38) on the reduced phase space. Using the energy-Casimir method we can now show that the equilibrium consisting of all bodies aligned ($K_{(jk)} = I_3$ for all $(j,k)\in\mathfrak{J}$) is stable provided all the control gains σ_{jk} are negative. This proof is straightforward since the Hessian of the augmented Hamiltonian, $H_\Phi = H + \Phi$, where Φ is a sum of terms of the form (50), is again block diagonal where each block is of the form (51). This inherent block diagonality ensures that our control law, at least for fixed and connected coupling topology, is *scalable* (with number of bodies N), i.e. the control inputs, the dynamics and the stability analysis do not get any more complex for very large N. Further, the control inputs should still provide stabilisation of alignment even if the number of bodies in the group drops due to malfunction. For the latter it is necessary to choose the number $\mathfrak{n}$ of pairs in $\mathfrak{J}$ sufficiently generous. Such a choice will be almost automatic in Case C where $\mathfrak{J}$ should contain the pairs of nearest neighbours.

10 Final Remarks

We have presented reduction for two rigid bodies coupled by a control law that depends only on their relative configuration. The semi-direct product reduction provides flat (Euclidean) spaces from which we can efficiently formulate control laws. We have applied the reduction results to several cases of interest including orientation alignment of a pair of rigid satellites and orientation and position alignment of a pair of underwater vehicles moving at constant speed.

We note that the control law presented in §5 for orientation matching is related to those of [1, 15] constructed for the asymptotic tracking of a desired attitude for satellites and helicopters. The control laws in these works consist of state feedback laws which required cancellation of the natural dynamics of those vehicles to achieve asymptotic stabilisation. In contrast, our control law is derived from the addition of a potential function designed to preserve the Hamiltonian nature of the system (and therefore the natural dynamics of the system); this Hamiltonian structure was used to prove stabilitiy of the aligned configuration in §7.

We also point out that the stabilization proof of §7.2 applies to two aligned underwater vehicles with each moving along its short (stable) axis. An additional control term is necessary to stably coordinate two (or more) aligned underwater vehicles with each moving along its long (unstable) axis. The problem of stable coordination of a network of mechanical systems of a particular class, each with otherwise unstable dynamics, is addressed in [12, 13]. In [12] stable coordination is proved for a controlled network of carts, each moving steadily while balancing an inverted pendulum. The addition of a control term that introduces dissipation is included and asymptotic stability analyzed. In [13] stabilization is proved for a group of satellites with orientations aligned and each spinning stably about its otherwise unstable (middle) axis. The role of an additional dissipation term provided by control and the analysis of asymptotic stability in the context of the problems discussed in [13] and the present paper will be presented in a future publication.

At the end of the present paper, we discussed extensions to groups of coupled rigid bodies with $N > 2$. We considered fixed, connected interconnection topologies. The reduction and this preliminary extension provide a foundation for further study of control inputs for alignment of groups that may include a varying number of vehicles and a limited communication radius.

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