SHAPE CONTROL OF A MULTI-AGENT SYSTEM USING TENSEGRITY STRUCTURES

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Abstract: We present a new coordinated control law for a group of vehicles in the plane that stabilizes an arbitrary desired group shape. The control law is derived for an arbitrary shape using models of tensegrity structures which are spatial networks of interconnected struts and cables. The symmetries in the coupled system and the energy-momentum method are used to investigate stability of relative equilibria corresponding to steady translations of the prescribed rigid shape.

Keywords: Co-operative control, Geometric approaches, Nonlinear control

1. INTRODUCTION

We address shape control of a group of mobile agents or vehicles in the plane. Shape refers to the geometry, configuration or formation of the group and is invariant under translation and rotation of the group as a whole. In (Zhang *et al.*, 2003), shape coordinates are defined based on Jacobi coordinates and a law to control small formations on Jacobi shape space is derived using a control Lyapunov function. In this paper, we use models of *tensegrity structures* to synthesize and analyze the shape dynamics of a group.

Shape control and, more generally, collective control of multi-agent systems have applications in a variety of engineering problems. One specific application motivating this research is the control of a fleet of autonomous underwater vehicles (AUVs) recently used for an adaptive sampling experiment in Monterey Bay, CA (AOSN), see e.g. (Leonard *et al.*, 2006; Fiorelli *et al.*, 2004). For the design of mobile sensors carrying out sampling or searching tasks, the configuration of the group can be critical. Depending on the field that is being surveyed, smaller or larger formations might be more efficient, and certain shapes of the group might be preferable for estimating field parameters such as gradients or higher-order derivatives from noisy measurements made by the mobile sensors, see e.g., the problem of generating a contour plot with a mobile sensor network (Zhang and Leonard, 2005).

We present a constructive method to stabilize an arbitrary planar shape for a group of n vehicles using virtual tensegrity structures. We model each of the n vehicles as a particle, moving in the plane, under the influence of a control force. The control forces are designed as if the particle group forms a tensegrity structure in which particles are treated like nodes and connections between particles simulate struts or cables. Stabilization of relative equilibria corresponding to the desired shape in steady translation is investigated using symmetries in the multi-agent system together with the energy-momentum method.

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Tensegrity structures (Skelton *et al.*, 2001) are geometric structures formed by a combination of *struts* (in compression) and *cables* (in tension) which we classify together more generally as *edges*. The edges of a tensegrity structure meet at *nodes*. A generic combination of cables and struts will not be in equilibrium; if the corresponding structure were physically built, it would collapse. We define tensegrity structures as only those structures that are in equilibrium. The artist Kenneth Snelson (Snelson, 1965) built the first tensegrity structure, and Buckminster Fuller (Fuller, 1962) coined the term tensegrity by combining the words tension and integrity.

Tensegrity structures have been widely studied with different motivations and approaches. For instance, there is growing interest in tensegrities in the context of designing structures whose shape can be adjusted and controlled. Models of the forces and formalization of the notion of stability for tensegrity structures were proposed by Connelly through an energy approach in (Connelly, 1982; Connelly, 2005; Connelly, 1999; Connelly and Whiteley, 1996). Tensegrity structures have also been used to model biological systems such as proteins, (Zanotti and Guerra, 2003) or cellular structure, (Ingber, 1993). It is known (Skelton *et al.*, 2001) that the shape of a tensegrity structure can be changed substantially with little change in the potential energy of the structure. This motivates us in part to use tensegrity structures as a model for shape control of a group of vehicles.

In this paper, we define a tensegrity structure that realizes any arbitrary desired shape. Each vehicle, modelled as a particle, is identified with one node of the tensegrity structure. The edges of the tensegrity structure correspond to communications and direction of forces between the vehicles. If an edge is a cable, the force is attractive; if the edge is a strut then the force is repulsive. The magnitude of the forces depends on the tensegrity structure parameters and the relative distance between the vehicles associated with the edge. In this setting, it is possible to see a tensegrity structure as an undirected graph with the interconnection between nodes weighted by the magnitude of the force. This allows us to use the formalism and results from algebraic graph theory. We note that because we use *virtual* tensegrity structures our model cannot impose the constraints that physical struts only increase in length and cables only decrease in length; an important consequence is the need for a nonlinear model that isolates the desired equilibrium shape. In Sections 2 and 3 of this paper we discuss different models for the forces. In Section 4 we present a systematic method to generate any shape. In Section 5 we investigate the stability of the generated shapes.

2. LINEAR FORCE MODEL

In this section we describe the simplest way of modelling the forces induced by the two types of edges of a tensegrity structure. We then find the relationship between the choice of cables, struts and parameters for the corresponding model and the equilibria. We model cables as springs with zero rest length and struts as springs with zero rest length and with a negative spring constant (Connelly, 1982; Connelly, 2005). Hence if we consider two nodes i, j we have

$$\overrightarrow{f}_{i\to j} = \omega_{ij}(\overrightarrow{q_i} - \overrightarrow{q_j}) = -\overrightarrow{f}_{j\to i}, \tag{1}$$

where $\overrightarrow{f}_{i\to j} \in \mathbb{R}^2$ is the force applied to node j as a result of the presence of node i. Here, $\overrightarrow{q_i} = (x_i, y_i) \in \mathbb{R}^2$ is the position vector of node i and ω_{ij} is the spring constant of the edge ij. The spring constant ω_{ij} is positive if ij is a strut, negative if ij is a cable and zero if there is no connection between the nodes i and j. We call ω_{ij} the *stress* of the edge ij. The absolute position of the structure in the plane is given by a vector $\mathbf{q} \in \mathbb{R}^{2n}$ which we call a *placement*. Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$, then $\mathbf{q} = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$. The potential energy of a tensegrity structure is

$$E_\omega(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} \omega_{ij}\|\overrightarrow{q_j} - \overrightarrow{q_i}\|^2. \tag{2}$$

We write $\sum\limits_{i<j}$ to represent $\sum\limits_{i=1}^{n}\sum\limits_{j=i+1}^{n}$. We note that this potential increases as we stretch the cables or shrink the struts. Using cartesian coordinates in the plane, equation (2) becomes

$$E_\omega(\mathbf{q}) = \frac{1}{2}\Big(\sum_{i<j}\omega_{ij}(x_j - x_i)^2 + \sum_{i<j}\omega_{ij}(y_j - y_i)^2\Big). \tag{3}$$

The equilibria of the system are the critical points of the potential (3). We rewrite (3) to more easily calculate the critical points.

Using notations from algebraic graph theory, we consider the undirected graph $G = (V, E)$, where V is the set of nodes and E the set of edges. Let d_j be the degree of node j, then the Laplacian L of the graph G is the $n \times n$ matrix defined by

$$L_{ij} = \begin{cases} d_j & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In our setting communications are not identical from one edge to the other, but rather are weighted by the spring constants ω_{ij}.

Our goal is to solve for and stabilize a tensegrity structure. To do this we solve for the weights ω_{ij}. A tensegrity structure can then be viewed as an undirected graph for which we define the weighted Laplacian Ω by

$$\Omega_{ij} = \begin{cases} \sum_{j=1}^{n} \omega_{ij} & \text{if } i = j \\ -\omega_{ij} & \text{if } i \neq j. \end{cases}$$

This matrix Ω introduced by Connelly (but derived in a different way) in (Connelly, 1982) is called the *stress matrix*. The stress matrix is an $n \times n$ symmetric matrix and the n-dimensional vector $\mathbf{1} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T$ is in the kernel of Ω (the last property is true for all Laplacians). Using this matrix, we can rewrite the potential (3) as

$$E_\omega(\mathbf{q}) = \frac{1}{2}\mathbf{q}^T(\Omega \otimes I_2)\mathbf{q},$$

where I_2 is the 2×2 identity matrix and $\Omega \otimes I_2$ is the $2n \times 2n$ block diagonal matrix $\begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}$. It is now easy to see that the critical points, and hence the equilibria, are given by

$$\mathbf{q}^T(\Omega \otimes I_2) = 0. \tag{5}$$

Using the fact that the stress matrix is symmetric, we see that a placement $\mathbf{q}^e$ is an equilibrium if and only if $\mathbf{x}^e$ and $\mathbf{y}^e$ are in the kernel of Ω. Recall that $\mathbf{1}$ is in the kernel of Ω. Assuming that the nodes are not all in a line, $\mathbf{x}^e, \mathbf{y}^e$ and $\mathbf{1}$ are linearly independent. We can conclude that with this model, a combination of cables and struts will have an equilibrium if and only if $\text{rank}(\Omega) \leq n-3$. We assume from now on that $n \geq 4$. By choosing the stresses of the edges of the structure so that $\text{rank}(\Omega) = n-3$, the kernel of the stress matrix Ω is exactly three dimensional, and we can prescribe the shape of the equilibrium.

However, we cannot prescribe the size of the equilibrium configuration. Indeed if $\ker(\Omega) = \text{span}\{\mathbf{x}^e, \mathbf{y}^e, \mathbf{1}\}$ then the structure described by $\mathbf{q}^e = (\alpha\mathbf{x}^e, \beta\mathbf{y}^e)$ is also an equilibrium $\forall \alpha, \beta \in \mathbb{R}$. For real tensegrities this is not a problem because the cable and strut constraints preclude the existence of any but the original equilibrium. In the virtual setting, however, where we cannot impose the constraints, we get a continuum of equilibria which is not desirable. For example, if we prescribe the tensegrity to be a square, it will be the case that not only all squares but also all rectangles will be equilibria. In the next section we exploit the simple equation (5), derived using the linear model (1), that determines the tensegrity shape as a function of the parameters ω_{ij}. We propose a nonlinear model for the forces along edges that isolates a tensegrity, fixing both shape and size.

3. NONLINEAR FORCE MODEL

In the previous section we chose to model the forces between a pair of nodes with a linear function of the relative distance between the nodes. We now model the forces along the edges as nonlinear springs with finite, nonzero rest length. Cables will always be longer than their rest length and struts will always be shorter than their rest length. We consider two nodes i, j and we define

$$\overrightarrow{f}_{i \to j} = \alpha_{ij}|\omega_{ij}|\frac{r_{ij} - L_{ij}}{r_{ij}}(\overrightarrow{q}_i - \overrightarrow{q}_j). \tag{6}$$

Here $r_{ij} = \|\overrightarrow{q}_i - \overrightarrow{q}_j\|$ is the relative distance between nodes i and j, L_{ij} is the rest length of the spring that models the edge ij, ω_{ij} is the spring constant from model (1) and α_{ij} is a scalar parameter that fixes the spring constant of model (6) for the edge ij.

The corresponding potential energy is

$$E_\omega(\mathbf{q}) = \frac{1}{2}\sum_{i<j} \alpha_{ij}|\omega_{ij}|(r_{ij} - L_{ij})^2. \tag{7}$$

The equilibria of this system can be found by solving for the critical points of the potential (7). After some manipulation, we find that the critical points of (7) are given by

$$\sum_{j=1}^{n} \alpha_{ij}|\omega_{ij}|(\overrightarrow{q_j} - \overrightarrow{q_i})(1 - \frac{L_{ij}}{r_{ij}}) = 0, \; i = 1, \ldots, n.$$

From this set of equations, we can define an analogue of the stress matrix Ω of (2). The new stress matrix is not a constant matrix. Rather it depends on the relative distances between pairs of nodes. We define

$$\widetilde{\omega}_{ij}(\mathbf{x}, \mathbf{y}) = \alpha_{ij}|\omega_{ij}|(1 - \frac{L_{ij}}{r_{ij}}) \tag{8}$$

to be the stress of the edge ij. The entries of the new stress matrix are given by

$$\widetilde{\Omega}_{ij}(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_{k=1}^{n} \widetilde{\omega}_{ik}(\mathbf{x}, \mathbf{y}) & \text{if } i = j \\ -\widetilde{\omega}_{ij}(\mathbf{x}, \mathbf{y}) & \text{if } i \neq j. \end{cases}$$

We note that the vector $\mathbf{1}$ is also in the kernel of $\widetilde{\Omega}$, $\forall\, (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$. Now if we wish the placement $\mathbf{q}^e = (\mathbf{x}^e, \mathbf{y}^e)$ to be the tensegrity structure (i.e the stable equilibrium of the system), we need to pick (if possible) the parameters $\alpha_{ij}, \omega_{ij}, L_{ij}$ so that

$$\widetilde{\Omega}(\mathbf{x}^e, \mathbf{y}^e)\mathbf{x}^e = 0$$
$$\widetilde{\Omega}(\mathbf{x}^e, \mathbf{y}^e)\mathbf{y}^e = 0.$$

As a first step we choose parameters α_{ij} and L_{ij} for all i, j so that $\widetilde{\Omega}(\mathbf{x}^e, \mathbf{y}^e) = \Omega$. If edge ij is a cable, then $\omega_{ij} > 0$, and by (8) we have $\widetilde{\omega}_{ij} = \alpha_{ij}\omega_{ij}(1 - \frac{L_{ij}}{r_{ij}})$. To make $\widetilde{\omega}_{ij}(\mathbf{x}^e, \mathbf{y}^e) = \omega_{ij}$, we make $\alpha_{ij}(1 - \frac{L_{ij}}{r_{ij}^e}) = 1$ where r_{ij}^e is the relative distance between nodes i and j for the desired placement. This last equation is solved by picking $\alpha_{ij} = 2$ and $L_{ij} = \frac{1}{2}r_{ij}^e$. If edge ij is a strut, then

$\omega_{ij} < 0$, and by (8) we have $\widetilde{\omega}_{ij} = -\alpha_{ij}\omega_{ij}(1 - \frac{L_{ij}}{r_{ij}})$. We make $\alpha_{ij}(1 - \frac{L_{ij}}{r^e_{ij}}) = -1$ by picking $\alpha_{ij} = 1$ and $L_{ij} = 2r^e_{ij}$. The choice of L_{ij} and α_{ij} is not unique (in case of a strut or a cable); the effect of picking other values for parameters α_{ij} and L_{ij} is to be determined.

We show in the next section, that we can also find parameters ω_{ij} independent of parameters α_{ij} and L_{ij}, so that $\ker(\Omega) = \operatorname{span}\{\mathbf{x}^e, \mathbf{y}^e, \mathbf{1}\}$, and such that the nonzero eigenvalues of Ω are all positive. This makes the equilibrium $\mathbf{q}^e = (\mathbf{x}^e, \mathbf{y}^e)$ an isolated minimum of the potential (modulo rigid transformations), i.e., our choices ensure that we have the right combination of struts and cables to make $\mathbf{q}^e = (\mathbf{x}^e, \mathbf{y}^e)$ a tensegrity structure.

4. CONSTRUCTION OF THE CONSTANT STRESS MATRIX

In this section we solve the following problem: given a desired placement $\mathbf{q}^e = (\mathbf{x}^e, \mathbf{y}^e)$, find stresses ω_{ij} such that $\ker(\Omega) = \operatorname{span}\{\mathbf{x}^e, \mathbf{y}^e, \mathbf{1}\}$ and the nonzero eigenvalues of Ω are positive.

We know that Ω is symmetric, hence it has only real eigenvalues and can be diagonalized using an orthonormal basis. As mentioned previously, if we do not consider the case when all the nodes are in a line, then $\mathbf{x}^e, \mathbf{y}^e$ and $\mathbf{1}$ are linearly independent. We can complete these three vectors with $n-3$ others so that we have a basis of $\mathbb{R}^n$. Then if we apply the Gram-Schmidt procedure to those vectors, we get an orthonormal basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $\mathbb{R}^n$ that satisfies

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{x}^e, \mathbf{y}^e, \mathbf{1}\}.$$

Now we define the $n \times n$ diagonal matrix D with diagonal elements $(0, 0, 0, 1, \ldots, 1)$ and the orthonormal $n \times n$ matrix $\Lambda = \begin{pmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{pmatrix}$. If we compute $\Lambda D \Lambda^T$ we have a symmetric positive semi-definite matrix with its kernel equal to $\operatorname{span}\{\mathbf{x}^e, \mathbf{y}^e, \mathbf{1}\}$. Setting $\Omega = \Lambda D \Lambda^T$ determines the values of stresses ω_{ij} that make the placement $\mathbf{q}^e = (\mathbf{x}^e, \mathbf{y}^e)$ a tensegrity structure. The effect of choosing smaller or larger positive eigenvalues for Ω (set here to 1) is to be determined.

5. RELATIVE EQUILIBRIUM STABILITY

In this section we use the energy-momentum method to look at the stability of the relative equilibrium corresponding to the tensegrity structure modelled by (6) in steady translation. The *energy-momentum method* is a technique for proving stability of relative equilibria (Marsden, 2004). For simple mechanical systems, we have the following setting: a configuration space Q, a symplectic manifold $P = T^*Q$ with a symplectic action of a Lie group G on P, an equivariant momentum map $\mathbf{J} : P \mapsto \mathfrak{g}^*$ and a G-invariant Hamiltonian $H : P \mapsto \mathbb{R}$. Here $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. If the Hamiltonian vector at the point $\mathbf{z}^e \in P$ points in the direction of the group orbit through $\mathbf{z}^e$, then the point is called a *relative equilibrium*. Let $\mu = \mathbf{J}(\mathbf{z}^e)$.

Theorem 1. **Relative Equilibrium Theorem (Marsden, 2004)** $\mathbf{z}^e$ is a relative equilibrium if and only if there is a $\xi \in \mathfrak{g}$ such that $\mathbf{z}^e$ is a critical point of the augmented Hamiltonian $H_\xi(\mathbf{z}) := H(\mathbf{z}) - \langle \mathbf{J} - \mu, \xi \rangle$.

Definition 1. ***(Marsden, 2004).*** Let S be a subspace of $T_{\mathbf{z}^e}P$ such that $S \subset \ker \mathbf{DJ}(\mathbf{z}^e)$ and S is transverse to the G_μ-orbit within $\ker \mathbf{DJ}(\mathbf{z}^e)$, where $G_\mu = \{g \in G \mid g \cdot \mu = \mu\}$, $\mu \in \mathfrak{g}^*$ and $g \cdot \mu$ is the coadjoint action of G on $\mathfrak{g}^*$.

Theorem 2. ***Energy Momentum Theorem (Marsden, 2004).*** If $\delta^2 H_\xi(\mathbf{z}^e)$ is definite on the subspace S, then $\mathbf{z}^e$ is G_μ-orbitally stable in $\mathbf{J}^{-1}(\mu)$ and G-orbitally stable in P.

For our system, the configuration space is $Q = (\mathbb{R}^2)^n$, $\mathbf{q} = (\mathbf{x}, \mathbf{y}) \in Q$ and an element in the cotangent bundle $\mathbf{z} \in T^*Q$ can be written as $\mathbf{z} = (\mathbf{q}, \mathbf{p}) = (\mathbf{x}, \mathbf{y}, \mathbf{p}_x, \mathbf{p}_y)$. Assuming unit mass nodes, the system kinetic energy is $1/2(\|\dot{\mathbf{x}}\|^2 + \|\dot{\mathbf{y}}\|^2)$ and $\mathbf{p}_x = \dot{\mathbf{x}}$ and $\mathbf{p}_y = \dot{\mathbf{y}}$. The Hamiltonian of the system is given by

$$H(\mathbf{z}) = \frac{1}{2}(\|\mathbf{p}_x\|^2 + \|\mathbf{p}_y\|^2) + \frac{1}{2}\sum_{i<j} \alpha_{ij}|\omega_{ij}|(r_{ij} - L_{ij})^2.$$

Noting the fact that the potential energy of the system only depends on the relative distances between nodes, we have that the Hamiltonian of the system is invariant under the following action of the Lie group $SE(2)$ on Q:

$$g.\mathbf{q} = (\cos\theta\mathbf{x} - \sin\theta\mathbf{y} + T_x\mathbf{1}, \sin\theta\mathbf{x} + \cos\theta\mathbf{y} + T_y\mathbf{1}), \tag{9}$$

where $g = (\theta, T_x, T_y) \in SE(2)$. Let $g(t) \in SE(2)$ such that $g(0) = (0, 0, 0)$ and $\dot{g}(0) = \xi = (\omega, V_x, V_y) \in \mathfrak{se}(2) = \mathfrak{g}$. The infinitesimal generator corresponding to the action (9) is

$$\begin{aligned}\xi_Q(q) &= \left.\frac{d}{dt}\right|_{t=0} g(t) \cdot (\mathbf{x}, \mathbf{y}) \\ &= (-\omega\mathbf{y} + V_x\mathbf{1}, \omega\mathbf{x} + V_y\mathbf{1}).\end{aligned}$$

The momentum map $\mathbf{J} : T^*Q \mapsto \mathfrak{g}^*$ is given by the formula (Marsden and Ratiu, 1999)

$$\langle \mathbf{J}(\mathbf{x}, \mathbf{y}, \mathbf{p}_x, \mathbf{p}_y), \xi \rangle = \langle (\mathbf{p}_x, \mathbf{p}_y), \xi_Q(q) \rangle. \tag{10}$$

From equation (10), we get

$$\mathbf{J}(\mathbf{z}) = \begin{pmatrix} \langle \mathbf{x}, \mathbf{p}_y \rangle - \langle \mathbf{y}, \mathbf{p}_x \rangle \\ \sum p_{x_i} \\ \sum p_{y_i} \end{pmatrix}. \tag{11}$$

The components of the momentum map are the total angular momentum of the tensegrity about the origin and the total linear momenta in the x and y directions.

Let $\mathbf{q}^e = (\mathbf{x}^e, \mathbf{y}^e)$ correspond to a tensegrity as designed in the previous sections. Let $\xi = (\omega, V_x, V_y) \in \mathfrak{se}(2)$. By Theorem 1, the relative equilibria $\mathbf{z}^e = (\mathbf{x}^e, \mathbf{y}^e, \mathbf{p}_x^e, \mathbf{p}_y^e) \in \mathbb{R}^{4n}$ satisfy

$$\frac{\partial H_\xi}{\partial \mathbf{x}}(\mathbf{z}^e) = -\omega \mathbf{p}_y^e + \Omega \mathbf{x}^e = \mathbf{0} \tag{12}$$

$$\frac{\partial H_\xi}{\partial \mathbf{y}}(\mathbf{z}^e) = \omega \mathbf{p}_x^e + \Omega \mathbf{y}^e = \mathbf{0} \tag{13}$$

$$\frac{\partial H_\xi}{\partial \mathbf{p}_x}(\mathbf{z}^e) = \mathbf{p}_x^e + \omega \mathbf{y}^e - V_x \mathbf{1} = \mathbf{0} \tag{14}$$

$$\frac{\partial H_\xi}{\partial \mathbf{p}_y}(\mathbf{z}^e) = \mathbf{p}_y^e - \omega \mathbf{x}^e - V_y \mathbf{1} = \mathbf{0}\,. \tag{15}$$

By design $\Omega \mathbf{x}^e = \Omega \mathbf{y}^e = \mathbf{0}$. Choosing $\xi = (0, V_x, V_y)$ and $(\mathbf{p}_x^e, \mathbf{p}_y^e) = (V_x \mathbf{1}, V_y \mathbf{1})$, we satisfy equations (12)-(15). Therefore $\mathbf{z}^e = (\mathbf{x}^e, \mathbf{y}^e, V_x\mathbf{1}, V_y\mathbf{1})$ is a relative equilibria of the system.

Next we compute $\delta^2 H_\xi(\mathbf{z}^e)$, the second variation of H_ξ evaluated at the relative equilibrium:

$$\delta^2 H_\xi(\mathbf{z}^e) = \begin{pmatrix} \Omega + L_{\omega x}(\mathbf{q}^e) & L_{\omega xy}(\mathbf{q}^e) & 0_n & 0_n \\ L_{\omega xy}(\mathbf{q}^e) & \Omega + L_{\omega y}(\mathbf{q}^e) & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \end{pmatrix} \tag{16}$$

where the ijth element of each matrix is

$$L_{\omega x}(i,j) = \begin{cases} -\alpha_{ij}|\omega_{ij}| \dfrac{(x_i - x_j)^2 L_{ij}}{r_{ij}^3} & \text{if } i \neq j \\ \displaystyle\sum_{j=1 j\neq i}^{n} \alpha_{ij}|\omega_{ij}| \dfrac{(x_i - x_j)^2 L_{ij}}{r_{ij}^3} & \text{if } i = j \end{cases}$$

$$L_{\omega y}(i,j) = \begin{cases} -\alpha_{ij}|\omega_{ij}| \dfrac{(y_i - y_j)^2 L_{ij}}{r_{ij}^3} & \text{if } i \neq j \\ \displaystyle\sum_{j=1 j\neq i}^{n} \alpha_{ij}|\omega_{ij}| \dfrac{(y_i - y_j)^2 L_{ij}}{r_{ij}^3} & \text{if } i = j \end{cases}$$

and

$$L_{\omega xy}(i,j) = \begin{cases} -\alpha_{ij}|\omega_{ij}| \dfrac{(x_i - x_j)(y_i - y_j) L_{ij}}{r_{ij}^3} & \text{if } i \neq j \\ \displaystyle\sum_{j=1 j\neq i}^{n} \alpha_{ij}|\omega_{ij}| \dfrac{(x_i - x_j)(y_i - y_j) L_{ij}}{r_{ij}^3} & \text{if } i = j. \end{cases}$$

We first show that this matrix is positive semi-definite. This is equivalent to proving that the top left $2n$ by $2n$ block in (16) given by

$$K = \begin{pmatrix} \Omega + L_{\omega x}(\mathbf{q}^e) & L_{\omega xy}(\mathbf{q}^e) \\ L_{\omega xy}(\mathbf{q}^e) & \Omega + L_{\omega y}(\mathbf{q}^e) \end{pmatrix} \tag{17}$$

is positive semi-definite. Recall that we have designed Ω to be positive semi-definite. The matrices $L_{\omega x}$ and $L_{\omega y}$ are symmetric and have all their off diagonal terms negative. The diagonal terms are positive so that each row sums to 0. In other words, $L_{\omega x}$ and $L_{\omega y}$ are diagonally dominant symmetric matrices. From the Gersgorin theorem (Horn and Johnson, 1985) those two matrices are positive semi-definite. We assert (and will prove in a later publication) that $\begin{pmatrix} L_{\omega x} & L_{\omega xy} \\ L_{\omega xy} & L_{\omega y} \end{pmatrix} \geq 0$. This implies that $K \geq 0$ and $\delta^2 H_\xi(\mathbf{z}^e) \geq 0$.

Lemma 1. The kernel of $\delta^2 H_\xi(\mathbf{z}^e)$ is equal to

$$\text{span} \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} -\mathbf{y}^e \\ \mathbf{x}^e \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right\}.$$

Proof: See the appendix.

We now investigate if the vectors in the kernel of $\delta^2 H_\xi(\mathbf{z}^e)$ are in $\mathcal{S}$ (given by Definition 1). First we compute $\mathbf{DJ}(\mathbf{z}^e)$ to be

$$\mathbf{DJ}(\mathbf{z}^e) = \begin{pmatrix} V_y \mathbf{1}^T & -V_x \mathbf{1}^T & -\mathbf{y}^{eT} & \mathbf{x}^{eT} \end{pmatrix}.$$

Next we compute G_μ. In our case, $G = SE(2)$ and $\mu \in \mathfrak{se}(2)^*$. By (11), we have $\mu = \mathbf{J}(\mathbf{z}^e) = (\mu_1, nV_x, nV_y)$. Let $g = (\theta, T_x, T_y) \in SE(2)$ then the coadjoint action $g \cdot \mu$ is

$$\begin{aligned} &Ad^*_{(\theta,T_x,T_y)^{-1}}(\mu_1, nV_x, nV_y) \\ &= (\mu_1 - R(\theta)\mathbf{V} \cdot \mathbb{J}\mathbf{T}, R(\theta)\mathbf{V}), \end{aligned}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} T_x \\ T_y \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} nV_x \\ nV_y \end{pmatrix}.$$

If $\mathbf{V} \neq 0$ then

$$G_\mu = \{g \in SE(2) \mid R(\theta) = I \text{ and } V_x T_y = V_y T_x\}.$$

Having computed $\mathbf{DJ}(\mathbf{z}^e)$ and G_μ, we can now compute S as follows (Marsden, 2004). Let

$$\mathcal{V} = \{\delta\mathbf{q} \in T_{\mathbf{q}^e} Q \mid \langle \delta\mathbf{q}, \xi_Q(\mathbf{q}^e) \rangle = 0 \ \forall \xi \in \mathfrak{g}_\mu\}. \tag{18}$$

Then,

$$\mathcal{S} = \{\delta\mathbf{z} \in \ker \mathbf{DJ}(\mathbf{z}^e) \mid T\pi_Q \cdot \delta\mathbf{z} \in \mathcal{V}\} \tag{19}$$

where $\pi_Q : T^*Q \to Q$ is the projection. This leads to $\delta\mathbf{z} = (\delta\mathbf{q}, \mathbf{0}, \mathbf{0}) \in \mathcal{S}$ if and only if

$$\begin{pmatrix} V_x \mathbf{1}^T & V_y \mathbf{1}^T \end{pmatrix} \begin{pmatrix} \delta\mathbf{x} \\ \delta\mathbf{y} \end{pmatrix} = 0 \tag{20}$$

$$\begin{pmatrix} V_y \mathbf{1}^T & -V_x \mathbf{1}^T \end{pmatrix} \begin{pmatrix} \delta\mathbf{x} \\ \delta\mathbf{y} \end{pmatrix} = 0. \tag{21}$$

Since there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\delta\mathbf{q} = \begin{pmatrix} \delta\mathbf{x} \\ \delta\mathbf{y} \end{pmatrix} = \alpha \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \beta \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} + \begin{pmatrix} -\mathbf{y}^e \\ \mathbf{x}^e \end{pmatrix} \tag{22}$$

satisfies both (20)-(21), then $\delta\mathbf{z} = (\delta\mathbf{q}, \mathbf{0}, \mathbf{0}) \in \mathcal{S}$ is also in the kernel of $\delta^2 H_\xi(\mathbf{z}^e)$.

Thus, the Energy Momentum Theorem does not provide a conclusive result on the stability of the relative equilibrium $\mathbf{z}^e$. However, we note that $\delta\mathbf{z} = (\delta\mathbf{q}, \mathbf{0}, \mathbf{0})$ with $\delta\mathbf{q}$ given by (22) corresponds to a rotation of the tensegrity about its center of mass. Any rotated tensegrity moving with constant velocity is just another relative equilibrium. Indeed there is a continuum of relative equilibria for a given tensegrity shape moving at constant velocity, parameterized by the orientation of the tensegrity. Simulations restricted to a constant momentum surface with $\mu = (0, nV_x, nV_y)$ reveal no rotational drift. In the full space, simulations exhibit drift only in the SE(2) directions, i.e., translational and rotational drift. Of particular note, there are solutions in the full space, near the relative equilibrium studied, that correspond to a rotating and translating formation.

6. FINAL REMARKS

We present a new coordinated control law for a group of vehicles in the plane that creates an arbitrary desired group shape. The control law is derived for an arbitrary shape using tensegrity structures modelled by (6). The symmetries in the coupled system and the energy-momentum method are used to investigate stability of relative equilibria corresponding to steady translations of the prescribed rigid shape. The energy momentum method alone does not provide conclusive results; however, the relative equilibrium does appear to be stable when the dynamics are restricted to the constant momentum surface. The analysis led to the discovery of solutions in the full space corresponding to rotating and translating formations; these solutions are under further investigation.

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APPENDIX: Proof of Lemma 1

Proving Lemma 1 is equivalent to proving that the kernel of K is exactly equal to the span of

$$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} -\mathbf{y}^e \\ \mathbf{x}^e \end{pmatrix} \right\}.$$

We can rewrite K as

$$K = K_1 + K_2 = \begin{pmatrix} \Omega & 0_n \\ 0_n & \Omega \end{pmatrix} + \begin{pmatrix} L_{\omega x} & L_{\omega xy} \\ L_{\omega xy} & L_{\omega y} \end{pmatrix}.$$

Since K_1, K_2 are symmetric positive semi-definite,

$$\mathbf{q} \in \ker(K) \Longleftrightarrow \mathbf{q} \in \ker(K_1) \text{ and } \mathbf{q} \in \ker(K_2).$$

By design $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are in the kernel of K_1. By direct computation, they are also in the kernel of K_2. We now show that the kernel of K_2 is exactly equal to the span of $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Since we exclude the case of all nodes in a line, the span of $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is three dimensional. We complete $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ with $2n - 3$ vectors $\{\mathbf{w}_4, \ldots, \mathbf{w}_{2n}\}$ from the canonical basis of $\mathbb{R}^{2n}$ making sure that we have $2n$ linearly independent vectors. It is then easy to check that

$$K_2 \mathbf{v}_i \neq 0 \quad \forall i \geq 4$$

and so the kernel of K_2 and therefore the kernel of K is exactly spanned by $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. $\square$