# One More Step Towards Reality: Cooperative Bandits with Imperfect Communication 

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#### Abstract

The cooperative bandit problem is increasingly becoming relevant due to its applications in large-scale decision-making. However, most research for this problem focuses exclusively on the setting with perfect communication, whereas in most real-world distributed settings, communication is often over stochastic networks, with arbitrary corruptions and delays. In this paper, we study cooperative bandit learning under three typical real-world communication scenarios, namely, (a) message-passing over stochastic time-varying networks, (b) instantaneous reward-sharing over a network with random delays, and (c) message-passing with adversarially corrupted rewards, including byzantine communication. For each of these environments, we propose decentralized algorithms that achieve competitive performance, along with near-optimal guarantees on the incurred group regret as well. Furthermore, in the setting with perfect communication, we present an improved delayed-update algorithm that outperforms the existing state-of-the-art on various network topologies. Finally, we present tight network-dependent minimax lower bounds on the group regret. Our proposed algorithms are straightforward to implement and obtain competitive empirical performance.


## 1 Introduction

The cooperative multi-armed bandit problem involves a group of $N$ agents collectively solving a multi-armed bandit while communicating with one another. This problem is relevant for a variety of applications that involve decentralized decision-making, for example, in distributed controls and robotics (Srivastava, Reverdy and Leonard, 2014) and communication (Lai, Jiang and Poor, 2008). In the typical formulation of this problem, a group of agents are arranged in a network $G=(\mathcal{V}, \mathcal{E})$, wherein each agent interacts with the bandit, and communicates with its neighbors in $G$, to maximize the cumulative reward.

A large body of recent work on this problem assumes the communication network $G$ to be fixed (Kolla, Jagannathan and Gopalan, 2018; Landgren, Srivastava and Leonard, 2021). Furthermore, these algorithms inherently require precise communication, as they construct careful confidence intervals for cumulative arm statistics across agents, e.g., for stochastic bandits, it has been shown that the standard UCB1 algorithm (Auer, Cesa-Bianchi and Fischer, 2002) with a neighborhood confidence interval is close to optimal (Dubey and Pentland, $2020 a$; Kolla, Jagannathan and Gopalan, 2018; Madhushani and Leonard, $2020 a, b$ ), and correspondingly, for adversarial bandits, a

[^0]neighborhood-weighted loss estimator can be utilized with the EXP3 algorithm to provide competitive regret (Cesa-Bianchi, Gentile and Mansour, 2019). Such approaches are indeed feasible when communication is perfect, e.g., the network $G$ is fixed, and messages are not lost or corrupted. In real-world environments, however, this is rarely true: messages can be lost, agents can be byzantine, and communication networks are rarely static (Leskovec, 2008). This aspect has hence received much attention in the distributed optimization literature (Yang et al., 2019). However, contrary to network optimization where dynamics in communication can behave synergistically (Hosseini, Chapman and Mesbahi, 2016), bandit problems additionally bring a decision-making component requiring an explore-exploit trade-off. As a result, external randomness and corruption are incompatible with the default optimal approaches, and require careful consideration (Vernade, Cappé and Perchet, 2017; Lykouris, Mirrokni and Paes Leme, 2018). This motivates us to study the multi-agent bandit problem under real-world communication, which regularly exhibits external randomness, delays and corruptions. Our key contributions include the following.

Contributions. We provide a set of algorithms titled Robust Communication Learning (RCL) for the cooperative stochastic bandit under three real-world communication scenarios.

First, we study stochastic communication, where the communication network $G$ is time-varying, with each edge being present in $G$ with an unknown probability $p$. For this setting, we present a UCBlike algorithm, RCL-LF (Link Failures), that directs agent $i$ to discard messages with an additional probability of $1-p_{i}$ in order to control the bias in the (stochastic) reward estimates. RCL-LF obtains a group regret of $\mathcal{O}\left(\left(\sum_{i=1}^{N}\left(1-p \cdot p_{i}\right)+\sum_{\mathcal{C} \in \mathfrak{C}}\left(\max _{i \leq \mathcal{C}} p_{i}\right) \cdot p\right)\left(\sum_{k=1}^{K} \frac{\log T}{\Delta_{k}}\right)\right)$, where $\mathcal{C}$ is a non overlapping clique covering of $G, T$ is time horizon, and $\Delta_{k}$ is the difference in reward mean between the optimal and $k$ th arm. The regret exhibits a smooth interpolation between known rates for no communication $(p=0)$ and perfect communication $(p=1)$.

Second, we study the case where messages from any agent can be delayed by a random (but bounded) number of trials $\tau$ with expectation $\mathbb{E}[\tau]$. For this setting, simple reward-sharing with a natural extension of the UCB algorithm (RCL-SD (Stochastic Delays)) obtains a regret of

$$
\mathcal{O}\left(\bar{\chi}(G) \cdot\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)+(N \cdot \mathbb{E}[\tau]+\log (T)+\sqrt{N \cdot \mathbb{E}[\tau] \log (T)}) \cdot \sum_{k>1} \Delta_{k}\right)
$$

, which is reminiscent of that of single-agent bandits with delays (Joulani, Gyorgy and Szepesvári, 2013) (Remark 4). Here $\bar{\chi}(G)$ is the clique covering number of $G$.

Third, we study the corrupted setting, where any message can be (perhaps in a byzantine manner) corrupted by an unknown (but bounded) amount $\epsilon$. This setting presents the two-fold challenge of receiving feedback after (variable) delays as well as adversarial corruptions, making existing arm elimination (Lykouris, Mirrokni and Paes Leme, 2018; Chawla et al., 2020; Gupta et al., 2021) or cooperative estimation (Dubey and Pentland, 2020a) methods inapplicable. We present algorithm RCL-AC (Adversarial Corruptions) that overcomes this issue by limiting exploration only to wellpositioned agents in $G$, who explore using a hybrid robust arm elimination and local confidence bound approach. RCL-AC obtains a regret of $\mathcal{O}\left(\psi\left(G_{\gamma}\right) \cdot \sum_{k=1}^{K} \frac{\log T}{\Delta_{k}}+N \sum_{k=1}^{K} \frac{\log \log T}{\Delta_{k}}+N T K \gamma \epsilon\right)$, where $\psi\left(G_{\gamma}\right)$ denotes the domination number of the $\gamma$ graph power of $G$, which matches the rates obtained for corrupted single-agent bandits without knowledge of $\epsilon$.

Finally, for perfect communication, we present a simple modification of cooperative UCB1 that provides significant empirical improvements, and also provides minimax lower bounds on the group regret of algorithms based on message-passing.

Related Work. A variant of the networked adversarial bandit problem without communication constraints (e.g., delay, corruption) was studied first in the work of Awerbuch and Kleinberg (2008), who demonstrated an average regret bound of order $\sqrt{(1+K / N) T}$. This line of inquiry was generalized to networked communication with at most $\gamma$ rounds of delays in the work of (Cesa-Bianchi, Gentile and Mansour, 2019), that demonstrate an average regret of order $\sqrt{\left(\gamma+\alpha\left(G_{\gamma}\right) / N\right) K T}$ where $\alpha\left(G_{\gamma}\right)$ denotes the independence number of $G_{\gamma}$, the $\gamma$-power of network graph $G$. This line of inquiry has been complemented for the stochastic setting with problem-dependent analyses in the work of Kolla, Jagannathan and Gopalan (2018) and Dubey and Pentland (2020a). The former presents a UCB1-style algorithm with instantaneous reward-sharing that obtains a regret bound of $\mathcal{O}\left(\alpha(G) \cdot \sum_{k=1}^{K} \frac{\log T}{\Delta_{k}}\right)$ that was generalized to message-passing communication with delays in the latter.

Alternatively, Landgren, Srivastava and Leonard (2021) consider the multi-agent bandit where communication is done instead using a running consensus protocol, where neighboring agents average their reward estimates using the DeGroot consensus model (DeGroot, 1974). This algorithm was refined in the work of Martínez-Rubio, Kanade and Rebeschini (2019) by a delayed mixing scheme that reduces the bias in the consensus reward estimates. A specific setting of Huber contaminated communication was explored in the work of Dubey and Pentland (2020b); however, in contrast to our algorithms, that work assumes that the total contamination likelihood is known $a$ priori. Additionally, multi-agent networked bandits with stochastic communication was considered in Madhushani and Leonard (2019, 2021a,b), however, only for regular networks and multi-star networks.

Our work also relates to aspects of stochastic delayed feedback and corruptions in the context of single-agent multi-armed bandits. There has been considerable research in these areas, beginning from the early work of Weinberger and Ordentlich (2002) that proposes running multiple bandit algorithms in parallel to account for (fixed) delayed feedback. Vernade, Cappé and Perchet (2017) discuss the multi-armed bandit with stochastic delays, and provide algorithms using optimism indices based on the UCB1 (Auer, Cesa-Bianchi and Fischer, 2002) and KL-UCB (Garivier and Cappé, 2011) approaches. Stochastic bandits with adversarial corruptions have also received significant attention recently. Lykouris, Mirrokni and Paes Leme (2018) present an arm elimination algorithm that provides a regret that scales linearly with the total amount of corruption, and present lower bounds demonstrating that the linear dependence is inevitable. This was followed up by Gupta, Koren and Talwar (2019) who introduce the algorithm BARBAR that improves the dependence on the corruption level by a better sampling of worse arms. Alternatively, Altschuler, Brunel and Malek (2019) discuss best-arm identification under contamination, which is a weaker adversary compared to the one discussed in this paper. The corrupted setting discussed in our paper combines both issues of (variable) delayed feedback along with adversarial corruptions, and hence requires a novel approach.

In another line of related work, Chawla et al.Chawla et al. (2020) discuss gossip-based communication protocols for cooperative multi-armed bandits. While the paper provides similar results, there are several differences in the setup considered in Chawla et al compared to our setup. First, we can see that Chawla et al.do not provide a uniform $\mathcal{O}\left(\frac{1}{N}\right)$ speedup, but in fact, their regret depends on the difficulty of the first $\frac{K}{N}$ arms, which is a $\mathcal{O}\left(\frac{1}{N}\right)$ speed up only when all arms are "uniformly" suboptimal, i.e., $\Delta_{i} \approx \Delta_{j} \forall i, j \in[K]$. In contrast, our algorithm will always provide a speed up of order $\frac{\alpha\left(G_{\gamma}\right)}{N}$ regardless of the arms themselves, and when we run our algorithm by setting the delay parameter $\gamma=d_{\star}(G)$ (diameter of the graph $G$ ), we obtain an $\mathcal{O}\left(\frac{1}{N}\right)$ speedup

Table 1: Quantity (with notation) for any graph $G$.

| Average degree $(d)$ | Maximum degree $\left(d_{\max }\right)$ | Degree of $i\left(d_{i}\right)$ | Independence number $(\alpha)$ |
| :--- | :--- | :--- | :--- |
| Message life $(\gamma)$ | Minimum degree $\left(d_{\min }\right)$ | Neighborhood of $i\left(\mathcal{N}_{i}\right)$ | Domination number $(\psi)$ |
| $k$-power of $G\left(G_{k}\right)$ | Diameter $\left(d_{\star}\right)$ | $\mathcal{N}_{i} \cup\{i\}\left(\mathcal{N}_{i}^{+}\right)$ | Clique covering number $(\bar{\chi})$ |

regardless of the sparsity of $G$. Additionally, our constants (per-agent) scale as $\mathcal{O}(K)$ in the worst case, whereas Chawla et al obtain a constant between $\mathcal{O}\left(K+(\log N)^{\beta}\right)$ and $\mathcal{O}\left(K+N^{\beta}\right)$ for some $\beta \gg 1$, based on the graph structure, which can dominate the $\log T$ term when we have a large number of agents present.

## 2 Preliminaries

Notation (Table 1). We denote the set $a, \ldots, b$ as $[a, b]$, and as $[b]$ when $a=1$. We define the indicator of a Boolean predicate $x$ as $\mathbf{1}\{x\}$. For any graph $G$ with diameter $d_{\star}(G)$, and any $1 \leq \gamma \leq d_{\star}(G)$, we define $G_{\gamma}$ as the $\gamma$-power of $G$, i.e., the graph with edge $(i, j)$ if $i, j$ are at most a distance $\gamma$.
Problem Setting. We consider the cooperative stochastic multi-armed bandit problem with $K$ arms and a group $\mathcal{V}$ of $N$ agents. In each round $t \in[T]$, each agent $i \in \mathcal{V}$ pulls an arm $A_{i}(t) \in[K]$ and receives a random reward $X_{i}(t)$ (realized as $r_{i}(t)$ ) drawn i.i.d. from the corresponding arm's distribution. We assume that each reward distribution is sub-Gaussian with an unknown mean $\mu_{k}$ and unknown variance proxy $\sigma_{k}^{2}$ upper bounded by a known constant $\sigma^{2}$. Without loss of generality we assume that $\mu_{1} \geq \mu_{2} \ldots \geq \mu_{K}$ and define $\Delta_{k}:=\mu_{1}-\mu_{k}, \forall k>1$, to be the reward gap (in expectation) of arm $k$. Let $\bar{\Delta}:=\min _{k>1} \Delta_{k}$ be the minimum expected reward gap. For brevity in our theoretical results, we define $g(\xi, \sigma):=8(\xi+1) \sigma^{2}=o(1)$ and $f(M, G):=M \sum_{k>1} \Delta_{k}+$ $4 \sum_{i=1}^{N}\left(3 \log \left(3\left(d_{i}(G)+1\right)\right)+\left(\log \left(d_{i}(G)+1\right)\right)\right) \cdot \sum_{k>1} \Delta_{k}=o\left((M+N \log N) \cdot \sum_{k>1} \Delta_{k}\right)$.
Networked Communication (Figure 1). Let $G=(\mathcal{V}, \mathcal{E})$ be a connected, undirected graph encoding the communication network, where $\mathcal{E}$ contains an edge $(i, j)$ if agents $i$ and $j$ can communicate directly via messages with each other. After each round $t$, each agent $j$ broadcasts a message $\boldsymbol{m}_{j}(t)$ to all their neighbors. Each message is forwarded at most $\gamma$ times through $G$, after which it is discarded. For any value of $\gamma>1$, the protocol is called message-passing (Linial, 1992), but for $\gamma=1$ it is called instantaneous reward sharing, as this setting has no delays in communication.
Exploration Strategy (Figure 2). For Sections 3 and 4 we use a natural extension of the UCB1 algorithm for exploration. Thus we modify UCB1 (Auer, Cesa-Bianchi and Fischer, 2002) such that at each time step $t$ for each arm $k$ each agent $i$ constructs an upper confidence bound, i.e., the sum of its estimated expected reward $\widehat{\mu}_{k}^{i}(t-1)$ (empirical average of all the observed rewards) and the uncertainty associated with the estimate $C_{k}^{i}(t-1):=\sigma \sqrt{\frac{2(\xi+1) \log t}{N_{k}^{i}(t-1)}}$ where $\xi>1$, and pulls the arm with the highest bound.
Regret. The performance measure we consider, group regret, is a straightforward extension of pseudo regret for a single agent. Group regret is the regret (in expectation) incurred by the group $\mathcal{V}$ by pulling suboptimal arms. The group regret is given by $\operatorname{Reg}_{G}(T)=\sum_{i=1}^{N} \sum_{k>1} \Delta_{k} \cdot \mathbb{E}\left[n_{k}^{i}(t)\right]$, where $n_{k}^{i}(t)$ is the number of times agent $i$ pulls the suboptimal arm $k$ up to (and including) round $t$.

Before presenting our algorithms and regret upper bounds we present some graph terminology.

For $t=1,2, \ldots$ each agent $i \in \mathcal{V}$

1. Plays arm $A_{i}(t)$, gets reward $r_{i}(t)$, computes $\boldsymbol{m}_{i}(t)=\left\langle A_{i}(t), r_{i}(t), i, t\right\rangle$.
2. Adds $\boldsymbol{m}_{i}(t)$ to the set of messages $\mathbf{M}_{i}(t)$, broadcasts all messages in $\mathbf{M}_{i}(t)$ to its neighbors and receives messages $\mathbf{M}_{i}^{\prime}(t)$ from its neighbors.
3. Computes $\mathbf{M}_{i}(t+1)$ from $\mathbf{M}_{i}^{\prime}(t)$ by discarding all messages sent prior to round $t-\gamma$.

This is called instantaneous reward sharing for $\gamma=1$ (no delays), and message-passing for $\gamma>1$.
Figure 1: The cooperative bandit protocol with delay parameter $\gamma$.
For $t=1,2, \ldots$, each agent $i \in \mathcal{V}$

1. Calculates, for each arm $k \in[K], Q_{k}^{i}(t-1)=\widehat{\mu}_{k}^{i}(t-1)+\sigma \sqrt{\frac{2(\xi+1) \log (t-1)}{N_{k}^{i}(t-1)}}$, where $N_{k}^{i}(t-1)$ is the number of reward samples available for arm $k$ at time $t$.
2. Plays $\operatorname{arm} A_{i}(t)=\arg \max _{k} Q_{k}^{i}(t-1)$

Figure 2: Cooperative UCB1 which uses additional arm pulls from messages.

Definition 1 (Clique covering number). A clique cover $\mathcal{C}$ of any graph $G=(\mathcal{V}, \mathcal{E})$ is a partition of $\mathcal{V}$ into subgraphs $C \in \mathcal{C}$ such that each subgraph $C$ is fully connected, i.e., a clique. The size of the smallest possible covering $\mathcal{C}^{\star}$ is known as the clique covering number $\bar{\chi}(G)$.

Definition 2 (Independence number). The independence number $\alpha(G)$ of $G=(\mathcal{V}, \mathcal{E})$ is the size of the largest subset of $\mathcal{V}_{\alpha} \subseteq \mathcal{V}$ such that no two vertices in $\mathcal{V}_{\alpha}$ are connected.

Definition 3 (Domination number). The domination number $\psi(G)$ of $G=(\mathcal{V}, \mathcal{E})$ is the size of the smallest subset $\mathcal{V}_{\psi} \subseteq \mathcal{V}$ such that each vertex not in $\mathcal{V}_{\psi}$ is adjacent to at least one agent in $\mathcal{V}_{\psi}$.

Organization. In this paper, we study three specific forms of communication errors. Section 3 discusses the case when, for both message-passing and instantaneous reward-sharing, any message forwarding fails independently with probability $p$, resulting in stochastic communication failures. Section 4 discusses the case when instantaneous reward-sharing incurs a random (but bounded) delay. Section 5 discusses the case when the outgoing reward from any message may be corrupted by an adversarial amount at most $\epsilon$. Finally, in Section 6, we discuss an improved algorithm for the case with perfect communication and present minimax lower bounds on the problem. We present all proofs in the Appendix and present proof-sketches highlighting the central ideas in the main paper.

## 3 Probabilistic Message Selection for Random Communication Failures

The fundamental advantage of cooperative estimation is the ability to leverage observations about suboptimal arms from neighboring agents to reduce exploration. However, when agents are communicating over an arbitrary graph, the amount of information an agent receives varies according to its connectivity in $G$. For example, agents with a large number of neighbors receive more information, leading them to begin exploitation earlier than agents with fewer neighbors. This means
that well-connected agents exhibit better performance early on, but because they quickly do only exploiting, agents that are poorly connected typically only observe exploitative arm pulls, which requires them to explore for longer in order to obtain similarly good estimates for suboptimal arms, increasing their regret. The disparity between performance in well-connected versus poorly connected agents is exacerbated in the presence of random link failures, where any message sent by an agent can fail to reach its recipient with a failure probability $1-p$ (drawn i.i.d. for each message).

Indeed, it is natural to expect the group regret to decrease with decreasing link failure probability, i.e., increasing communication probability $p$. However, what we observe experimentally (Section 7) is that this holds only for graphs $G$ that are regular (i.e., each agent has the same degree), or close to regular. When $G$ is irregular, as we increase $p$ from 0 to 1 , the group performance oscillates. While, in some cases, the improved performance in the well-connected agents can outweigh the degradation observed in the weakly-connected agents (leading to lower group regret), it is prudent to consider an approach that mitigates this disparity by regulating information flow in the network.

Information Regulation in Cooperative Bandits. Our approach to regulate information is straightforward: we direct each agent $i$ to discard any incoming message with an agent-specific probability $1-p_{i}$, while always utilizing its own observations. For specific values of $p_{i}$, we can obtain various weighted combinations of internal versus group observations. Our first algorithm RCL-LF (Link Failures) is built on this regulation strategy, coupled with UCB1 exploration using all selected observations for each arm. Essentially, each agent runs UCB1 using the cumulative set of observations it has received from its network. After pulling an arm, it broadcasts its pulled arm and reward through the network, but incorporates each incoming message only with a probability $p_{i}$. Pseudo code for the algorithm is given in the appendix. We first present a regret bound for RCL-LF when run with the instantaneous reward-sharing protocol.

Theorem 1 (RCL-LF Regret with instantaneous reward-sharing). RCL-LF running with the instantaneous reward-sharing protocol (Figure 1, $\gamma=1$ ) obtains cumulative group regret of

$$
\operatorname{Reg}_{G}(T) \leq g(\xi, \sigma)\left(\sum_{i=1}^{N}\left(1-p_{i} \cdot p\right)+\sum_{\mathcal{C} \in \mathcal{C}}\left(\max _{i \leq \mathcal{C}} p_{i}\right) \cdot p\right)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)+f(5 N, G)
$$

where $\mathcal{C}$ is a non-overlapping clique covering of $G$.
Proof sketch. We follow an approach similar to the analysis of UCB1 by Auer, Cesa-Bianchi and Fischer (2002) with several key modifications. First, we partition the communication graph $G$ into a set of non-overlapping cliques and then analyze the regret of each clique. The group regret can be obtained by taking the summation of the regret over each clique. Two major technical challenges in proving the regret bound for RCL-LF are (a) deriving a tail probability bound for probabilistic communication, and (b) bounding the regret accumulated by agents by losing information due to communication failures and message discarding. We overcome the first challenge by noticing that communication is independent of the decision making process thus $\mathbb{E}\left(\exp \left(\lambda \sum_{\tau=1}^{t} X_{\tau}^{i} \mathbf{1}\left\{A_{\tau}^{i}=k\right\}-\mu_{k} N_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t)\right)\right) \leq 1$ holds under probabilistic communication. We obtain the tail bound by combining this result with the Markov inequality and optimizing over $\lambda$ using a peeling type argument. We address the second challenge by proving that the number of times agents do not share information about any suboptimal arm $k$ can be bounded by a term that increases logarithmically with time and scales with number of agents, $G$, and communication probabilities, as $\sum_{i=1}^{N}\left(1-p_{i} \cdot p\right)+\sum_{\mathcal{C} \in \mathbb{C}}\left(\max _{i \leq \mathcal{C}} p_{i}\right) \cdot p$.

Remark 1 (Regret bound optimality). Under perfect communication ( $p=1$ ) and no message discarding, i.e., $p_{i}=p=1, \forall i \in[N]$ the dominant term in our regret bound scales with $\bar{\chi}(G)$, obtaining identical performance to deterministic communication over $G$ (Dubey and Pentland, $2020 a$ ). Alternatively, when $p_{i}=p=0$, there is no communication, and hence, the regret bound is $\mathcal{O}(N \log T)$. Theorem 1 quantifies the benefit of communication in reducing the group regret under probabilistic link failure and when agents incorporate observations with an agent-specific probability. Note that $\sum_{i=1}^{N}\left(1-p_{i} \cdot p\right)+\sum_{\mathcal{C} \in \mathcal{C}}\left(\max _{i \leq \mathcal{C}} p_{i}\right) \cdot p=N-p \cdot\left(\sum_{i=1}^{N} p_{i}-\sum_{\mathcal{C} \in \mathfrak{e}}\left(\max _{i \leq \mathcal{C}} p_{i}\right)\right)$. Since the clique covering is non-overlapping, the results show that agents obtain improved group performance for any communication probability $p>0$ for any nontrivial graph as compared to the case with no communication in which each agent learns on its own.

Remark 2 (Controlling information disparity). In order to regulate the information disparity across the network we set $p_{i}=\frac{d_{\min }(G)}{d_{i}(G)}$. Thus, the agent(s) with minimum degree $d_{\text {min }}$ incorporate each message they receive with probability 1 and we have that the expected number of messages for each agent is the same, i.e., $T \cdot d_{\min }(G)$. Therefore, every agent receives the same amount of information (in expectation), providing a large performance improvement for irregular graphs (see Section 7).

Message-Passing. Under this communication protocol each agent $i$ communicates with neighbors at distance at most $\gamma$, where each hop adds a 1 -step delay. Our algorithm RCL-CF obtains a similar regret bound in this setting as well, when all agents use the same UCB1 exploration strategy (Figure 2).

Theorem 2 (RCL-LF Regret with message-passing). Let $\mathcal{C}$ be a minimal clique covering of $G_{\gamma}$. For any $\mathcal{C} \in \mathcal{C}$ and $i, j \in \mathcal{C}$ let $\gamma_{i}=\max _{j \in \mathcal{C}} d(i, j)$ be the maximum distance (in graph $G$ ) between agents $i$ and $j$. RCL-LF running with the message-passing protocol (Figure 1) with delay parameter $\gamma$ obtains cumulative group regret of

$$
\operatorname{Reg}_{G}(T) \leq g(\xi, \sigma)\left(\sum_{i=1}^{N}\left(1-p_{i} \cdot p^{\gamma_{i}}\right)+\bar{\chi}\left(G_{\gamma}\right) \cdot\left(\max _{i \leq N} p_{i} \cdot p^{\gamma_{i}}\right)\right)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)+f\left((\gamma+4) N, G_{\gamma}\right) .
$$

Proof sketch. We partition the graph $G_{\gamma}$ into non-overlapping cliques, analyze the regret of each clique and take the summation of regrets over cliques to obtain group regret. In addition to the challenges encountered in Theorem 1 here we are required to account for having different probabilities of failures for messages due to having multiple paths of different length between agents and to account for the delay incurred by each hop when passing messages. We overcome the first challenge by noting that agent $i$ receives each message with at least probability $p^{\gamma_{i}}$. We overcome the second challenge by identifying that regret incurred by delays can be upper bounded using $\left(\sum_{i=1}^{N} \gamma_{i}-N\right) \sum_{k>1} \Delta_{k}$.

Remark 3. Finding an optimal observation probability $\left\{p_{i}\right\}_{1=1}^{N}$ for RCL-LF with message-passing is difficult due to the delays added by each hop when forwarding messages. If messages are forwarded without a delay, optimal performance can be obtained by using $p_{i}=\frac{d_{\min }\left(G_{\gamma}\right)}{d_{i}\left(G_{\gamma}\right)}$. For dense $G_{\gamma}$, the above choice of observation probability provides near-optimal performance. When $\gamma=d_{\star}(G)$ we have that $G_{\gamma}$ is a complete graph, $p_{i}=\frac{d_{\min }\left(G_{\gamma}\right)}{d_{i}\left(G_{\gamma}\right)}=1$, and agents do not discard any message. However, when $\gamma<d_{\star}(G)$, the graph $G_{\gamma}$ is not complete. Therefore agents receive different
amounts of information which are approximately proportional to the degree distribution of $G_{\gamma}$. As explained earlier this information disparity leads to a performance disparity among agents. As a result group performance decreases. In this case we design the algorithm such that each agent $i$ discards messages with $1-p_{i}$ where $p_{i}=\frac{d_{\min }\left(G_{\gamma}\right)}{d_{i}\left(G_{\gamma}\right)}$. This regulates the information flow mitigating the bias introduced by information disparity. As a result the group obtains near-optimal performance.

## 4 Instantaneous Reward-sharing Under Stochastic Delays

Next, we consider a communication protocol, where any message is received after an arbitrary (but bounded) stochastic delay. We assume for simplicity that each message is sent only once in the network (and not forwarded multiple times as in message-passing), and leave the message-passing setting as future work. We assume, furthermore that the delays are identically and independently drawn from a bounded distribution with expectation $\mathbb{E}[\tau]$ (similar to prior work, e.g., Joulani, Gyorgy and Szepesvári (2013); Vernade, Cappé and Perchet (2017)). For this setting, we demonstrate that cooperative UCB1, along with incorporating all messages as soon as they are available, provides efficient performance, both empirically and theoretically. We denote this algorithm as RCL-SD (Stochastic Delays), and demonstrate that this approach incurs only an extra $\mathcal{O}(\sqrt{N \log T}+\log T)$ overhead compared to perfect communication.

Theorem 3 (RCL-SD Regret). Let $D_{\text {total }}=N \cdot \mathbb{E}[\tau]+2 \log T+2 \sqrt{N \cdot \mathbb{E}[\tau] \log T}$ denote an upper bound on the total number of outstanding messages. RCL-SD obtains, with probability at least $1-\frac{1}{T}$, cumulative group regret of

$$
\operatorname{Reg}_{G}(T) \leq g(\xi, \sigma) \cdot \bar{\chi}(G) \cdot\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)+D_{\text {total }} \cdot\left(\sum_{k>1} \Delta_{k}\right)+f(5 N, G)
$$

Proof sketch. We first demonstrate that the additional group regret due to stochastic delays can be bounded by the maximum number of cumulative outstanding messages over all agents at any given time step. Then we apply a result similar to Lemma 2 of Joulani, Gyorgy and Szepesvári (2013) to bound the total number of outstanding messages using the cumulative expected delay $N \cdot \mathbb{E}[\tau]$, giving the result.

Remark 4. The $D_{\text {total }}$ term is a succinct upper bound on the maximum number of cumulative outstanding messages over all agents, and when the expected delay $\mathbb{E}[\tau]=o(1)$, we see that the contribution of $D_{\text {total }}$ is $\mathcal{O}(\sqrt{N \log T}+\log T)$. We conjecture that this cannot be improved without restricting communication, as each agent will send $T$ messages in total. The result obtained by Joulani, Gyorgy and Szepesvári (2013) has a similar dependence for a single agent.

## 5 Hybrid Arm Elimination for Adversarial Reward Corruptions

In this section, we assume that any reward when transmitted can be corrupted by a maximum value of $\epsilon$, i.e., $\max _{t, n}\left|r_{n}(t)-\tilde{r}_{n}(t)\right| \leq \epsilon$ where $\tilde{r}_{n}(t)$ denotes the transmitted reward. Furthermore, we assume that the corruptions can be adaptive, i.e., can depend on the prior actions and rewards of each agent. This model includes natural settings, where messages can be corrupted during transmission, as well as byzantine communication (Dubey and Pentland, 2020b). If $\epsilon$ were known, we could then extend algorithms for misspecified bandits (Ghosh, Chowdhury and Gopalan,
2017) to create a robust estimator and a subsequent UCB1-like algorithm that obtains a regret of $\mathcal{O}\left(\bar{\chi}\left(G_{\gamma}\right) K\left(\frac{\log T}{\Delta}\right)+T N K \epsilon\right)$. However, this approach has two issues. First, $\epsilon$ is typically not known, and the dependence on $G_{\gamma}$ can be improved as well. We present an arm-elimination algorithm called RCL-AC (Adversarial Corruptions) that provides better guarantees on regret, without knowledge of $\epsilon$ in Algorithm 1.

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Algorithm 1: RCL-RC: Cooperative Hybrid Arm Elimination
    Parameters: Confidence \(\delta \in(0,1)\), horizon \(T\), graph \(G\) with exploration set \(\mathcal{I} \subseteq \mathcal{V}\).
    Initialize \(T_{i}(0)=K, \forall i \in \mathcal{I} \lambda=1024 \log \left(\frac{8 K \psi\left(G_{\gamma}\right)}{\delta} \log _{2} T\right)\) and \(\Delta_{k}^{i}(0)=1, \forall k \in[K]\) and
    \(i \in \mathcal{I}\).
    for each subgraph \(\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\) where \(i \in \mathcal{I}\) do
        for \(t=1, \ldots, K\), each agent \(j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\) do
            Play arm \(K\) and get reward \(r_{j}(t)\).
        end
        for epoch \(m_{i}=1,2, \ldots\), do
            Set \(n_{k}^{i}\left(m_{i}\right)=\lambda\left(\Delta_{k}^{i}\left(m_{i}-1\right)\right)^{-2} \forall k \in[K]\).
            \(N_{i}\left(m_{i}\right)=\sum_{k} n_{k}^{i}\left(m_{i}\right)\) and \(T_{i}\left(m_{i}\right)=T_{i}\left(m_{i}\right)+N_{i}\left(m_{i}\right)+2 \gamma\).
            for agent \(j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\) do
            for \(t=T_{i}\left(m_{i}-1\right)\) to \(s=T_{i}\left(m_{i}-1\right)+2 \gamma\) do
                if \(j \neq i\) then
                            if \(t \leq K+d(i, j)\) then
                Pull random arm.
                    end
                        else
                            Pull \(A_{j}(t)=A_{i}(t-d(i, j))\) and get reward \(r_{j}(t)\).
                            end
                end
                else
                            Pull \(A_{j}(t)=\mathrm{UCB} 1(t)\)
                end
            end
            for \(t=T_{i}\left(m_{i}-1\right)+2 \gamma\) to \(T_{i}\left(m_{i}\right)\) do
                if \(j \neq i\) then
                    Pull \(A_{j}(t)=A_{i}(t-d(i, j))\) and get reward \(r_{j}(t)\).
                end
                else
                    Pull an arm \(A_{i}(t)=k \in[K]\) with probability \(n_{k}^{i}\left(m_{i}\right) / N_{k}\left(m_{i}\right)\).
                end
            end
            end
        end
    end
```

The central motif in RCL-AC's design is to eliminate bad arms by an epoch-based exploration, an idea that has been successful in the past for adversarially-corrupted stochastic bandits (Lyk-
ouris, Mirrokni and Paes Leme, 2018; Gupta, Koren and Talwar, 2019). The challenge, however, in a message-passing decentralized setting is two-fold. First, agents have different amounts of information based on their position in the network, and hence badly positioned agents in $G$ may be exploring for much larger periods. Secondly, communication between agents is delayed, and hence any agent naively incorporating stale observations may incur a heavy bias from delays. To ameliorate the first issue, we partition the group of agents into two sets - exploring agents $(\mathcal{I})$ and imitating agents $(\mathcal{V} \backslash \mathcal{I})$. The idea is to only allow well-positioned agents in $\mathcal{I}$ to direct the exploration strategy for their neighboring agents, and the rest simply imitate their exploration strategy. We select $\mathcal{I}$ as a minimal dominating set of $G_{\gamma}$, hence $|\mathcal{I}|=\psi\left(G_{\gamma}\right)$. Furthermore, since $\mathcal{V} \backslash \mathcal{I}$ is a vertex cover, this ensures that each imitating agent is connected (at distance at most $\gamma$ ) to at least one agent in $\mathcal{I}$. Next, observe that there are two sources of delay: first, any imitating agent must wait at most $\gamma$ trials to observe the latest action from its corresponding exploring agent, and second, each exploring agent must wait an additional $\gamma$ trials for the feedback from all of its imitating agents. We propose that each exploring agent run UCB1 for $2 \gamma$ rounds after each epoch of arm elimination, using only local pulls. This prevents a large bias due to these delays, at a small cost of $\mathcal{O}(\log \log T)$ suboptimal pulls.

Theorem 4 (RCL-RC Regret). RCL-RC obtains, with probability at least $1-\delta$, group regret of

$$
\operatorname{Reg}_{G}(T)=\mathcal{O}\left(K T N \gamma \epsilon+\psi\left(G_{\gamma}\right) \cdot \sum_{k>1} \frac{\log T}{\Delta_{k}} \log \left(\frac{K \psi\left(G_{\gamma}\right) \log T}{\delta}\right)+N \sum_{k>1} \Delta_{k}+\sum_{k>1} \frac{N \log (\gamma \log T)}{\Delta_{k}}\right) .
$$

Proof sketch. Since the dominating set covers $\mathcal{V}$, we can decompose the group regret into the cumulative regret of the subgraphs corresponding to each agent in $\psi\left(G_{\gamma}\right)$. For each subgraph, we can consider the cumulative regret incurred when the exploring agent follows UCB1 versus arm elimination. We have that arm elimination occurs for $\log T$ epochs, and since UCB1 runs for $2 \gamma$ rounds between succesive epochs, we have that in any subgraph of size $n$, the cumulative regret from UCB1 rounds is of $\mathcal{O}(n K \log (\gamma \log T))$. For arm elimination, we can bound the subgraph regret using a modification of the approach in Gupta, Koren and Talwar (2019): the difference in our approach is to construct a multi-agent filtration for arbitrary (reward-dependent) corruptions from message-passing, and then applying Freedman's bound on the resulting martingale sequence. Subsequently, the regret in each epoch is bounded in a manner similar to Gupta, Koren and Talwar (2019), and finally applying a union bound.

Remark 5 (Regret Optimality). Theorem 4 demonstrates a trade-off between communication density and the adversarial error, as seen by the first two terms in the regret bound. The first term $(K T N \gamma \epsilon)$ is a bound on the cumulative error introduced due to message-passing, which is increasing in $\gamma$, whereas the second term denotes the logarithmic regret due to exploration, where $\psi\left(G_{\gamma}\right)$ decreases as $\gamma$ increases: for $\gamma=d_{\star}(G), \psi\left(G_{\gamma}\right)=1$, matching the lower bound in Dubey and Pentland (2020a). This too is expected, as fewer exploring agents are needed with a higher communication budget. Furthermore, we conjecture that the first term is optimal (in terms of $T$, up to graphical constants): a linear lower bound has been demonstrated for the single-agent setting in Lykouris, Mirrokni and Paes Leme (2018).

Remark 6 (Computational complexity). While the dominating set problem is known to be NPcomplete (Karp, 1972), the problem admits a polynomial-time approximation scheme (PTAS) (Crescenzi,

Kann and Halldórsson, 1995) for certain graphs, for which our bounds hold exactly. However, RCL-RC can work on any dominating set of size $n$, and suffer regret of $\widetilde{\mathcal{O}}\left(K T N \gamma \epsilon+n \sum_{k>1} \frac{\log T}{\Delta_{k}}\right)^{1}$.

## 6 An Algorithm for Perfect Communication and Lower Bounds

For perfect communication, we present Delayed MP-UCB, a simple improvement to UCB1 with message-passing where each agent $i$ only incorporates messages originated prior to $\bar{\gamma} \leq \gamma$ time steps, reducing disparity in information across agents.

Theorem 5 (Delayed MP-UCB Regret). Delayed(MP)-UCB obtains cumulative group regret of

$$
\begin{gathered}
\operatorname{Reg}_{G}(T) \leq g(\xi, \sigma) \bar{\chi}\left(G_{\gamma}\right)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)+\left(N-\bar{\chi}\left(G_{\gamma}\right)(\gamma-1) \sum_{k>1} \Delta_{k}+f\left(5 N, G_{\gamma}\right)+h\left(G_{\gamma}, \bar{\gamma}\right)\right. \\
\text { where } h\left(G_{\gamma}, \bar{\gamma}\right)=\left(\left(N-\bar{\chi}\left(G_{\gamma}\right) \bar{\gamma}+\sum_{t>\bar{\gamma}}^{T} \frac{\log \left(1-\frac{d_{i}\left(G_{\gamma}\right) \bar{\gamma}}{\left.\left(d_{i}\left(G_{\gamma}\right)+1\right) t\right)}\right)}{\log _{1.3}} \frac{1}{t^{(\xi+1)\left(1-\frac{0.09}{16}\right)}}\right) \sum_{k>1} \Delta_{k} .\right.
\end{gathered}
$$

Proof sketch. Following a similar approach to the proof of Theorem 2 we partition the graph $G_{\gamma}$ into a set of non-overlapping cliques, analyze the regret of each clique via a UCB1 type analysis and take the summation of regret over cliques. However, using less information (due to delayed information usage) in estimates leads to a large confidence bound $C_{k}^{i}(t)$ and this reduces the contribution to the regret from tail probabilities. Note that $\log \left(1-\frac{d_{i}\left(G_{\gamma}\right) \bar{\gamma}}{\left.\left(d_{i}\left(G_{\gamma}\right)+1\right) t\right)}\right)$ is negative $\forall t>\bar{\gamma}$, and hence lower regret achieved due to low tail probabilities is given by the second term of $h\left(G_{\gamma}, \bar{\gamma}\right)$.

Remark 7. Incorporating only the messages originated before $\bar{\gamma}$ time steps is similar to communicating over $G_{\bar{\gamma}}$ after a delay of $\bar{\gamma}$ time steps. When $G$ is connected and $\bar{\gamma}=\gamma=d_{*}$ this is similar to communicating over a complete graph with a delay of $d_{*}$. Thus Delayed MP-UCB mitigates the disparity in information used by each agent, leading to improved group performance.

Lower Bounds. Without strict assumptions, a lower bound of $\mathcal{O}\left(\sum_{k>1} \log T / \Delta_{k}\right)$ has been demonstrated both for $\gamma=1$ (instantaneous reward-sharing, Kolla, Jagannathan and Gopalan (2018)) and $\gamma>1$ (message-passing, Dubey and Pentland (2020a)), which both suggest that a speedup of $\frac{1}{N}$ is potentially achievable. For a more restrictive class of individually consistent and non-altruistic policies (i.e., that do not contradict their local feedback), a tighter lower bound of $\mathcal{O}\left(\alpha\left(G_{2}\right) \sum_{k>1} \log T / \Delta_{k}\right)$ can be demonstrated for reward-sharing (Kolla, Jagannathan and Gopalan, 2018), and consequently $\mathcal{O}\left(\alpha\left(G_{\gamma+1}\right) \sum_{k>1} \log T / \Delta_{k}\right)$ for message-passing. To supplement these results, we present a lower bound to characterize the minimax optimal rates for the problem. We present first an assumption on multi-agent policies.

Assumption 1 (Agnostic decentralized policies). $A$ set of $N$ policies $\pi_{1}, \ldots, \pi_{N}$ are termed agnostic decentralized policies, if for every pair $(i, j)$ of agents that communicate in $G$ and each $t \in[T], \pi_{i}(t)$ is independent of $\left\{\pi_{j}(\tau)\right\}_{\tau=1}^{t-d(i, j)}$ conditioned on the rewards $\left\{\left(A_{j}(\tau), X_{j}(\tau)\right)\right\}_{\tau=1}^{t-d(i, j)}$.

[^1]Theorem 6 (Minimax Rate). For any policy $\mathcal{A}$, there exists a $K$-armed environment over $N$ agents with $\Delta_{k} \leq 1$ for any connected graph $G$ and $\gamma \geq 1$ such that, for some absolute constant $c$,

$$
\operatorname{Reg}_{G}(\mathcal{A}, T) \geqslant c \sqrt{K N(T+\widetilde{d}(G))}
$$

Furthermore, if $\mathcal{A}$ is an agnostic decentralized policy, there exists a $K$-armed environment over $N$ agents with $\Delta_{k} \leq 1$ for any connected graph $G$ and $\gamma \geq 1$ such that, for some absolute constant $c^{\prime}$,

$$
\operatorname{Reg}_{G}(\mathcal{A}, T) \geqslant c^{\prime} \sqrt{\alpha^{\star}\left(G_{\gamma}\right) K N T}
$$

Here $\tilde{d}(G)=\sum_{i=1}^{d^{\star}(G)} \bar{d}_{=i} \cdot i$ denotes the average delay incurred by message-passing across the network $G$, and $\alpha^{\star}\left(G_{\gamma}\right)=\frac{N}{1+\bar{d}_{\gamma}}$ is Turan's lower bound (Turán, 1941) on $\alpha\left(G_{\gamma}\right)$.
Remark 8 (Tightness of lower bound). The first minimax bound does not make any assumptions on the policy $\mathcal{A}$, and hence we only see an additive dependence of the average delay incurred by communication over $G$. This dependence generalizes the minimax rate for delayed multi-armed bandits (Neu et al., 2010) to graphical feedback. For the latter bound, observe that a variety of cooperative extensions of single-agent bandit algorithms (Kolla, Jagannathan and Gopalan, 2018; Dubey and Pentland, 2020a; Cesa-Bianchi, Gentile and Mansour, 2019) obey this assumption, where the decision-making for any agent is independent of any other agent, conditioned on the observed rewards. In this setting, agents merely treat messages as additional pulls to construct stronger estimators, and do not strategize collectively. This bound is exact (up to constants) for a variety of communication graphs $G$. For instance, for linear and circular graphs, $\frac{\alpha^{\star}\left(G_{\gamma}\right)}{\alpha\left(G_{\gamma}\right)}=o(1)$, and for $d$-regular graphs, $\alpha^{\star}\left(G_{\gamma}\right)=\alpha\left(G_{\gamma}\right)$ (Turán, 1941).

## 7 Experimental Results

We consider the 10 -armed bandit with rewards drawn from Gaussian distributions with $\sigma_{k}=1$ for each arm, such that $\mu_{1}=1$ and $\mu_{k}=0.5$ for $k \neq 1$, and the number of agents $N=50$, where we repeat each experiment 100 times with $G$ selected randomly from different families of random graphs. The bottom row of Figure 3 corresponds to Erdos-Renyi graphs with $p=0.7$. The top row of Figure 3 (a), (c) and (d) corresponds to multi-star graphs and (b) and (e) to random tree graphs. We set $\xi=1.1$ and $\gamma=\max \left\{3, d_{\star}(G) / 2\right\}$.

Stochastic Link Failure. Figure 3(a) and Figure 3(b) summarize performance of RCL (RS) -LF and RCL (MP) -LF, comparing it with the corresponding reward-sharing and message-passing UCB-like algorithms in which $p_{i}=1, \forall i \in[N]$, for different $p$ values. The group regret is given at $T=500$. The results validate our claim that probabilistic message discarding improves performance for irregular graphs and provides competitive performance for near-regular graphs.

Stochastic Delays. We compare performance of RCL-SD with UCB1. We draw delays from a bounded distribution with $\mathbb{E}[\tau]=10$ and $\tau_{\max }=50$. The results are summarized in Figure 3(c).

Adversarial Communication. We compute the (approximate) dominating set using the algorithm provided in networkx for each connected component in $G_{\gamma}$. We draw corruptions uniformly from the range $[0, \epsilon]$ for each message, where $\epsilon$ is increased from $10^{-3}$ to $10^{-2}$. The group regret at $T=500$ as a function of $\epsilon$ is shown in Figure 3(d) and compared against individual UCB1 and cooperative UCB with message-passing (MP-UCB), which incur larger regret increasing linearly with $\epsilon$.


Figure 3: Experimental results for various imperfect communication settings.
Perfect Communication. We compare the regret curve for $T=1000$ for our Delayed(MP) -UCB against regular MP-UCB in Figure 3(e). We use $\bar{\gamma}=2$. It is evident that delayed incorporation of messages markedly improves performance across both networks.

## 8 Conclusions

In this paper, we studied the cooperative bandit problem in three different imperfect communication settings. For each setting, we proposed algorithms with competitive empirical performance and provided theoretical guarantees on the incurred regret. Further, we provided an algorithm for perfect communication that comfortably outperforms existing baseline approaches. We additionally provided a tighter network-dependent minimax lower bound for the cooperative bandit problem. We believe that our contributions can be of immediate utility in applications. Moreover, future inquiry can be pursued in several different directions, including multi-agent reinforcement learning and contextual bandit learning.

Ethical Considerations. Our work is primarily theoretical, and we do not foresee any negative societal consequences arising specifically from our contributions in this paper.

## Acknowledgement

This research has been supported in part by ONR grants N00014-18-1-2873 and N00014-19-1-2556, ARO grant W911NF-18-1-0325, and the MIT Trust::Data Consortium.

## References

Altschuler, Jason, Victor-Emmanuel Brunel and Alan Malek. 2019. "Best arm identification for contaminated bandits." Journal of Machine Learning Research 20(91):1-39.

Angluin, Dana and Leslie G Valiant. 1979. "Fast probabilistic algorithms for Hamiltonian circuits and matchings." Journal of Computer and System Sciences 18(2):155-193.

Auer, Peter, Nicolo Cesa-Bianchi and Paul Fischer. 2002. "Finite-time analysis of the multiarmed bandit problem." Machine Learning 47(2-3):235-256.

Awerbuch, Baruch and Robert Kleinberg. 2008. "Online linear optimization and adaptive routing." Journal of Computer and System Sciences 74(1):97-114.

Bubeck, Sébastien. 2010. Bandits games and clustering foundations PhD thesis.
Cesa-Bianchi, Nicolo, Claudio Gentile and Yishay Mansour. 2019. "Delay and cooperation in nonstochastic bandits." The Journal of Machine Learning Research 20(1):613-650.

Cesa-Bianchi, Nicolo and Gábor Lugosi. 2006. Prediction, learning, and games. Cambridge University Press.

Chawla, Ronshee, Abishek Sankararaman, Ayalvadi Ganesh and Sanjay Shakkottai. 2020. The gossiping insert-eliminate algorithm for multi-agent bandits. In International Conference on Artificial Intelligence and Statistics. PMLR pp. 3471-3481.

Crescenzi, Pierluigi, Viggo Kann and M Halldórsson. 1995. "A compendium of NP optimization problems.".
URL: http://www.nada.kth.se/viggo/problemlist/compendium.html
DeGroot, Morris H. 1974. "Reaching a consensus." Journal of the American Statistical Association 69(345):118-121.

Dubey, Abhimanyu and Alex Pentland. 2020a. Cooperative multi-agent bandits with heavy tails. In International Conference on Machine Learning. PMLR pp. 2730-2739.

Dubey, Abhimanyu and Alex Pentland. 2020b. Private and byzantine-proof cooperative decisionmaking. In Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems. pp. 357-365.

Garivier, Aurélien and Olivier Cappé. 2011. The KL-UCB algorithm for bounded stochastic bandits and beyond. In Proceedings of the 24th Annual Conference on Learning Theory. JMLR Workshop and Conference Proceedings pp. 359-376.

Ghosh, Avishek, Sayak Ray Chowdhury and Aditya Gopalan. 2017. Misspecified linear bandits. In Proceedings of the AAAI Conference on Artificial Intelligence. Vol. 31.

Gupta, Anupam, Tomer Koren and Kunal Talwar. 2019. Better algorithms for stochastic bandits with adversarial corruptions. In Conference on Learning Theory. PMLR pp. 1562-1578.

Gupta, Samarth, Shreyas Chaudhari, Gauri Joshi and Osman Yağan. 2021. "Multi-armed bandits with correlated arms." IEEE Transactions on Information Theory .

Hosseini, Saghar, Airlie Chapman and Mehran Mesbahi. 2016. "Online distributed convex optimization on dynamic networks." IEEE Transactions on Automatic Control 61(11):3545-3550.

Joulani, Pooria, Andras Gyorgy and Csaba Szepesvári. 2013. Online learning under delayed feedback. In International Conference on Machine Learning. PMLR pp. 1453-1461.

Karp, Richard M. 1972. Reducibility among combinatorial problems. In Complexity of Computer Computations. Springer pp. 85-103.

Kolla, Ravi Kumar, Krishna Jagannathan and Aditya Gopalan. 2018. "Collaborative learning of stochastic bandits over a social network." IEEE/ACM Transactions on Networking 26(4):17821795.

Lai, Lifeng, Hai Jiang and H Vincent Poor. 2008. Medium access in cognitive radio networks: A competitive multi-armed bandit framework. In $42 n d$ Asilomar Conference on Signals, Systems and Computers. IEEE pp. 98-102.

Landgren, Peter, Vaibhav Srivastava and Naomi Ehrich Leonard. 2021. "Distributed cooperative decision making in multi-agent multi-armed bandits." Automatica 125:109445.

Leskovec, Jurij. 2008. Dynamics of large networks PhD thesis Carnegie Mellon University, School of Computer Science, Machine Learning.

Linial, Nathan. 1992. "Locality in distributed graph algorithms." SIAM Journal on Computing 21(1):193-201.

Lykouris, Thodoris, Vahab Mirrokni and Renato Paes Leme. 2018. Stochastic bandits robust to adversarial corruptions. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing. pp. 114-122.

Madhushani, Udari and Naomi Ehrich Leonard. 2019. Heterogeneous stochastic interactions for multiple agents in a multi-armed bandit problem. In European Control Conference. pp. 35023507.

Madhushani, Udari and Naomi Ehrich Leonard. 2020a. Distributed learning: Sequential decision making in resource-constrained environments. In "Practicle Machine Learning for Developing Countries"ICLR 2020 workshop.

Madhushani, Udari and Naomi Ehrich Leonard. 2020b. A Dynamic Observation Strategy for Multiagent Multi-armed Bandit Problem. In European Control Conference.

Madhushani, Udari and Naomi Ehrich Leonard. 2021a. Distributed Bandits: Probabilistic Communication on $d$-regular Graphs. In European Control Conference.

Madhushani, Udari and Naomi Ehrich Leonard. 2021b. "Heterogeneous explore-exploit strategies on multi-star networks." IEEE Control Systems Letters 5(5):1603-1608.

Martínez-Rubio, David, Varun Kanade and Patrick Rebeschini. 2019. Decentralized cooperative stochastic bandits. In Advances in Neural Information Processing Systems. pp. 4531-4542.

Neu, Gergely, Andras Antos, András György and Csaba Szepesvári. 2010. Online Markov decision processes under bandit feedback. In Advances in Neural Information Processing Systems. pp. 1804-1812.

Srivastava, Vaibhav, Paul Reverdy and Naomi Ehrich Leonard. 2014. Surveillance in an abruptly changing world via multi-armed bandits. In Conference on Decision and Control. IEEE pp. 692697.

Turán, Paul. 1941. "On an external problem in graph theory." Mat. Fiz. Lapok 48:436-452.
Vernade, Claire, Olivier Cappé and Vianney Perchet. 2017. Stochastic bandit models for delayed conversions. In Conference on Uncertainty in Artificial Intelligence.

Weinberger, Marcelo J and Erik Ordentlich. 2002. "On delayed prediction of individual sequences." IEEE Transactions on Information Theory 48(7):1959-1976.

Yang, Tao, Xinlei Yi, Junfeng Wu, Ye Yuan, Di Wu, Ziyang Meng, Yiguang Hong, Hong Wang, Zongli Lin and Karl H Johansson. 2019. "A survey of distributed optimization." Annual Reviews in Control 47:278-305.

## A Proof of Theorem 1

We consider the case where each message fails with probability $1-p$ and each agent $i$ uses the messages it receives from its neighbors with probability $p_{i}$. This is equivalent to each agent $i$ receiving messages from its neighbors with probability $p_{i} p$. Let $\mathbf{1}\left\{(i, j) \in E_{t}\right\}$ be the indicator random variable that takes value 1 if agent $i$ receives reward value and arm id from agent $j$ at time $t$ and 0 otherwise.

We start by proving some useful lemmas.
Lemma 1. (Restatement of results from (Auer, Cesa-Bianchi and Fischer, 2002)) Let $\eta_{k}=\left(\frac{8(\xi+1) \sigma^{2}}{\Delta_{k}^{2}}\right) \log T$. For any suboptimal arm $k$ and $\forall i, t$ we have

$$
\mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) \leq \mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)
$$

Proof. Let $Q_{k}^{i}(t)=\widehat{\mu}_{k}^{i}(t)+C_{k}^{i}(t)$. Note that for any $k>1$ we have

$$
\begin{aligned}
\left\{A_{i}(t+1)=k\right\} & \subset\left\{Q_{k}^{i}(t) \geq Q_{1}^{i}(t)\right\} \\
& \subset\left\{\left\{\mu_{1}<\mu_{k}+2 C_{k}^{i}(t)\right\} \cup\left\{\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right\} \cup\left\{\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right\}\right\}
\end{aligned}
$$

Let $\eta_{k}=\left(\frac{8(\xi+1) \sigma^{2}}{\Delta_{k}^{2}}\right) \log T$. Since $N_{k}^{i}(t)>\eta_{k}$ the event $\left\{\mu_{1}<\mu_{k}+2 C_{k}^{i}(t)\right\}$ does not occur. Thus we have

$$
\mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) \leq \mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)
$$

This concludes the proof of Lemma 1.
Lemma 2. Let $\bar{\chi}(G)$ is the clique covering number of graph $G$. Let $\eta_{k}=\left(\frac{8(\xi+1) \sigma_{k}^{2}}{\Delta_{k}^{2}}\right) \log T$. Then we have

$$
\begin{align*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right] & \leq\left(\sum_{i=1}^{N}\left(1-p_{i} p\right)+\bar{\chi}(G) p_{\max } p\right) \eta_{k}+2 N  \tag{1}\\
& +\sum_{i=1}^{N} \sum_{t=1}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right] \tag{2}
\end{align*}
$$

Proof. Let $\mathcal{C}$ be a non overlapping clique covering of $G$. Note that for each suboptimal arm $k>1$ we have

$$
\begin{equation*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right]=\sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)=\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) . \tag{3}
\end{equation*}
$$

Let $\tau_{k, \mathcal{C}}$ denote the maximum time step when the total number of times arm $k$ has been played by all the agents in clique $\mathcal{C}$ is at most $\eta_{k}+|\mathcal{C}|$ times. This can be stated as $\tau_{k, \mathcal{C}}:=\max \{t \in[T]:$ $\left.\sum_{i \in \mathcal{C}} n_{k}^{i}(t) \leq \eta_{k}+|\mathcal{C}|\right\}$. Then, we have that $\eta_{k}<\sum_{i \in \mathcal{C}} n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right) \leq \eta_{k}+|\mathcal{C}|$.

For each agent $i \in \mathcal{C}$ let

$$
\bar{N}_{k}^{i}(t):=\sum_{j \in \mathcal{C}} \sum_{\tau=1}^{t} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}
$$

denote the sum of the total number of times agent $i$ pulled arm $k$ and the total number of observations it received from agents in its clique about arm $k$ until time $t$. Define $\bar{\tau}_{k, \mathcal{C}}^{i}:=\max \{t \in[T]:$ $\left.\bar{N}_{k}^{i}(t) \leq \eta_{k}\right\}$. Then we have that $\eta_{k}-|\mathcal{C}|<\bar{N}_{k}^{i}\left(\bar{\tau}_{k, \mathcal{C}}^{i}\right) \leq \eta_{k}$.

Note that $N_{k}^{i}(t) \geq \bar{N}_{k}^{i}(t), \forall t$, hence for all $i \in \mathcal{C}$ we have $N_{k}^{i}(t)>\eta_{k}, \forall t>\bar{\tau}_{k, \mathcal{C}}^{i}$. Here we consider that $\bar{\tau}_{k, \mathcal{C}}^{(i)} \geq \tau_{k, \mathcal{C}}, \forall i$. From regret results it follows that regret for this case is greater than the regret for the case where $\bar{\tau}_{k, \mathcal{C}}^{i}<\tau_{k, \mathcal{C}}$ for some (or all) $i$.

We analyse the expected number of times agents pull suboptimal arm $k$ as follows,

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{4}\\
& =\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\}+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\bar{\tau}_{k, \mathcal{C}}^{i}}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{5}\\
& \leq \sum_{\mathcal{C} \in \mathcal{C}}\left(\eta_{k}+|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\}+|\mathcal{C}|  \tag{6}\\
& +\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\bar{\tau}_{k, \mathcal{C}}^{i}}^{T-1} \mathbf{1}\left\{A_{i}(t+1)=k\right\} \mathbf{1}\left\{N_{k}^{i}(t)>\eta_{k}\right\} . \tag{7}
\end{align*}
$$

Taking expectation we have

$$
\begin{align*}
\sum_{\mathcal{C} \in \mathcal{C}} & \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) \leq \sum_{\mathcal{C} \in \mathcal{C}}\left(\eta_{k}+2|\mathcal{C}|\right)  \tag{8}\\
& +\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathrm{P}\left(A_{i}(t)=k\right)+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}^{i}}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) . \tag{9}
\end{align*}
$$

Note that we have

$$
\begin{align*}
& \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{10}\\
& =\sum_{i \in \mathcal{C}} \bar{N}_{k}^{i}\left(\bar{\tau}_{k, \mathcal{C}}^{i}\right)-\sum_{i \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}-\sum_{i \in \mathcal{C}} \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{j}(t)=k\right\} \mathbf{1}\left\{(i, j) \in E_{t}\right\}  \tag{11}\\
& =\sum_{i \in \mathcal{C}} \bar{N}_{k}^{i}\left(\bar{\tau}_{k, \mathcal{C}}^{i}\right)-\sum_{i \in \mathcal{C}} n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right)-\sum_{i \in \mathcal{C}} \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{j}(t)=k\right\} \mathbf{1}\left\{(i, j) \in E_{t}\right\}  \tag{12}\\
& \leq|\mathcal{C}| \eta_{k}-\eta_{k}-\sum_{i \in \mathcal{C}} \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{j}(t)=k\right\} \mathbf{1}\left\{(i, j) \in E_{t}\right\} \tag{13}
\end{align*}
$$

$$
\begin{equation*}
\leq|\mathcal{C}| \eta_{k}-\eta_{k}-\sum_{i \in \mathcal{C}} \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} 1\left\{A_{j}(t)=k\right\} 1\left\{(i, j) \in E_{t}\right\} \tag{14}
\end{equation*}
$$

Taking the expectation

$$
\begin{align*}
\sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathrm{P}\left(A_{i}(t)=k\right) & \leq|\mathcal{C}| \eta_{k}-\eta_{k}-\sum_{i \in \mathcal{C}} p_{i} p \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathrm{P}\left(A_{j}(t)=k\right)  \tag{15}\\
& =|\mathcal{C}| \eta_{k}-\eta_{k}-\sum_{i \in \mathcal{C}} p_{i} p \sum_{j \neq i, j \in \mathcal{C}} \mathbb{E}\left(n_{k}^{j}\left(\tau_{k, \mathcal{C}}\right)\right)  \tag{16}\\
& =|\mathcal{C}| \eta_{k}-\eta_{k}-\left(\sum_{i \in \mathcal{C}} p_{i} p\right)\left(\sum_{i \in \mathcal{C}} \mathbb{E}\left(n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right)\right)\right)+\sum_{i \in \mathcal{C}} p_{i} p \mathbb{E}\left(n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right)\right)  \tag{17}\\
& \leq|\mathcal{C}| \eta_{k}-\eta_{k}-p\left(\sum_{j \in \mathcal{C}} p_{j}-p_{\max }\right) \mathbb{E}\left(\sum_{i \in \mathcal{C}} n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right)\right)  \tag{18}\\
& \leq|\mathcal{C}| \eta_{k}-\eta_{k}-p\left(\sum_{j \in \mathcal{C}} p_{j}-p_{\max }\right) \eta_{k}  \tag{19}\\
& =\left(|\mathcal{C}|-1-p\left(\sum_{j \in \mathcal{C}} p_{j}-p_{\max }\right)\right) \eta_{k} \tag{20}
\end{align*}
$$

Substituting this results to (9) we get

$$
\begin{align*}
\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) \leq & \sum_{\mathcal{C} \in \mathcal{C}}\left(\eta_{k}+2|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}}\left(|\mathcal{C}|-1-p\left(\sum_{j \in \mathcal{C}} p_{j}-p_{\max }\right)\right) \eta_{k}  \tag{21}\\
& +\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\bar{\tau}_{k, \mathcal{C}}^{i}}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) \tag{22}
\end{align*}
$$

Thus from Lemma 1 and (22) we have

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)  \tag{23}\\
& \leq \sum_{\mathcal{C} \in \mathcal{C}} \eta_{k}+2 N+\sum_{\mathcal{C} \in \mathcal{C}}\left(|\mathcal{C}|-1-p\left(\sum_{j \in \mathcal{C}} p_{j}-p_{\max }\right)\right) \eta_{k}  \tag{24}\\
& \quad+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right]  \tag{25}\\
& \stackrel{(a)}{=} \bar{\chi}(G) \eta_{k}+\left(N-\sum_{i=1}^{N} p_{i} p-\mathcal{X}(G)\left(1-p_{\max } p\right)\right) \eta_{k}+2 N \tag{26}
\end{align*}
$$

$$
\begin{align*}
& +\sum_{i=1}^{N} \sum_{t>\tau_{k, \mathcal{C}}}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right]  \tag{27}\\
\leq & \left(\sum_{i=1}^{N}\left(1-p_{i} p\right)+\bar{\chi}(G) p_{\max } p\right) \eta_{k}+2 N  \tag{28}\\
& +\sum_{i=1}^{N} \sum_{t=1}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right] \tag{29}
\end{align*}
$$

where (a) follows from the fact that clique covering is non overlapping. This concludes the proof of Lemma 2.

Lemma 3. Let $d_{i}(G)$ be the degree of agent $i$ in graph $G$. For any $\sigma_{k}>0$ some constant $\zeta>1$

$$
\begin{equation*}
\mathrm{P}\left(\left|\widehat{\mu}_{k}^{i}(t)-\mu_{k}\right|>\sigma_{k} \sqrt{\frac{2(\xi+1) \log t}{N_{k}^{i}(t)}}\right) \leq \frac{\log \left(\left(d_{i}(G)+1\right) t\right)}{\log \zeta} \frac{1}{t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)}} . \tag{30}
\end{equation*}
$$

Proof. For all $k$ let $X_{k}^{i}(t)$ for all $i, t$ be iid copies of $X_{k}$. Then we have $X_{t}^{i} \mathbf{1}\left\{A_{i}(t)=k\right\}=X_{k}^{i}(t) \mathbf{1}\left\{A_{i}(t)=k\right\}$. Recall that reward distribution of arm $k$ has mean $\mu_{k}$ and variance proxy $\sigma_{k}$. Thus $\forall i, t$ we have

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda\left(X_{k}^{i}(t)-\mu_{k}\right)\right)\right) \leq \exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2}\right) \tag{31}
\end{equation*}
$$

Define local history at every agent $i$ as follows

$$
\begin{equation*}
\mathcal{H}_{t}^{i}:=\sigma\left(X_{\tau}^{i}, A_{i}(\tau), X_{\tau}^{j} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}, A_{j}(\tau) \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}, \forall \tau \in[t], j \in \mathcal{N}_{i}(G)\right) \tag{32}
\end{equation*}
$$

Since $\mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}$ for $j \in \mathcal{N}_{i}(G)$ is a $\mathcal{H}_{\tau-1}^{i}$ measurable random variable, we have

$$
\begin{align*}
& \mathbb{E}\left(\exp \left(\lambda\left(X_{\tau}^{j}-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{33}\\
& =\mathbb{E}\left(\exp \left(\lambda\left(X_{k}^{j}(\tau)-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{34}\\
& \leq \exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) \tag{35}
\end{align*}
$$

Define a new random variable such that $\forall \tau>0$.

$$
\begin{align*}
Y_{k}^{i}(\tau) & =\sum_{j=1}^{N}\left(X_{k}^{j}(\tau) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}-\mathbb{E}\left[X_{k}^{j}(\tau) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\} \mid \mathcal{H}_{\tau-1}^{i}\right]\right)  \tag{36}\\
& =\sum_{j=1}^{N}\left(X_{k}^{j}(\tau)-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\} \tag{37}
\end{align*}
$$

Note that $\mathbb{E}\left(Y_{k}^{i}(\tau)\right)=\mathbb{E}\left(Y_{k}^{i}(\tau) \mid \mathcal{H}_{\tau-1}^{i}\right)=0$. Let $Z_{k}^{i}(t)=\sum_{\tau=1}^{t} Y_{k}^{i}(\tau)$. For any $\lambda>0$

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda Y_{k}^{i}(\tau)\right) \mid \mathcal{H}_{\tau-1}^{i}\right) \tag{38}
\end{equation*}
$$

$$
\begin{align*}
& =\mathbb{E}\left(\exp \left(\lambda \sum_{j=1}^{N}\left(X_{k}^{j}(\tau)-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{39}\\
& =\mathbb{E}\left(\prod_{j=1}^{N} \exp \left(\lambda\left(X_{k}^{j}(\tau)-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{40}\\
& \stackrel{(a)}{=} \prod_{j=1}^{N} \mathbb{E}\left(\exp \left(\lambda\left(X_{k}^{j}(\tau)-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{41}\\
& \leq \prod_{j=1}^{N} \exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right)  \tag{42}\\
& =\exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2} \sum_{j=1}^{N} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right) . \tag{43}
\end{align*}
$$

Equality (a) follows from the fact that random variables $\left\{\exp \left(\lambda\left(X_{k}^{j}(\tau)-\mu_{k}\right) \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right)\right\}_{j=1}^{N}$ are conditionally independent with respect to $\mathcal{H}_{\tau-1}^{i}$. Since $\mathbf{1}\left\{A_{j}(\tau)=k\right\}, \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}$ are $\mathcal{H}_{\tau-1}^{i}$ measurable, and so

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda Y_{k}^{i}(\tau)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} \sum_{j=1}^{N} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}\right)\left|\mid \mathcal{H}_{\tau-1}^{i}\right) \leq 1\right. \tag{44}
\end{equation*}
$$

Let $N_{k}^{i}(t)=\sum_{\tau=1}^{t} \sum_{j=1}^{N} \mathbf{1}\left\{A_{i}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right\}$. Further, using the tower property of conditional expectation we have

$$
\begin{equation*}
\mathbb{E}\left(\left.\exp \left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t)\right) \right\rvert\, \mathcal{H}_{t-1}^{i}\right) \leq \exp \left(\lambda Z_{k}^{i}(t-1)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t-1)\right) \tag{45}
\end{equation*}
$$

Repeating the above step $t$ times we have

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t)\right)\right) \leq 1 \tag{46}
\end{equation*}
$$

Note that we have

$$
\begin{align*}
& \mathrm{P}\left(\exp \left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{i}^{2}}{2} N_{k}^{i}(t)\right) \geq \exp (2 \kappa \vartheta)\right)  \tag{47}\\
& =\mathrm{P}\left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t) \geq 2 \kappa \vartheta\right)  \tag{48}\\
& =\mathrm{P}\left(\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \frac{2 \kappa \vartheta}{\lambda \sqrt{N_{k}^{i}(t)}}+\frac{\sigma_{k}^{2}}{2} \lambda \sqrt{N_{k}^{i}(t)}\right) . \tag{49}
\end{align*}
$$

Fix a constant $\zeta>1$. Then $1 \leq N_{k}^{i}(t) \leq \zeta^{D_{t}}$ where $D_{t}=\frac{\log \left(\left(d_{i}(G)+1\right) t\right)}{\log \zeta}$. For $\lambda_{l}=\frac{2}{\sigma_{k}} \sqrt{\frac{\kappa \vartheta}{\zeta^{l-1 / 2}}}$ and $\zeta^{l-1} \leq N_{k}^{i}(t) \leq \zeta^{l}$ we have

$$
\begin{equation*}
\frac{2 \kappa \vartheta}{\lambda_{l}} \sqrt{\frac{1}{N_{k}^{i}(t)}}+\frac{\sigma_{k}^{2}}{2} \lambda_{l} \sqrt{N_{k}^{i}(t)}=\sigma_{k} \sqrt{\kappa \vartheta}\left(\sqrt{\frac{\zeta^{l-1 / 2}}{N_{k}^{i}(t)}}+\sqrt{\frac{N_{k}^{i}(t)}{\zeta^{l-1 / 2}}}\right) \leq \sqrt{\vartheta} \tag{50}
\end{equation*}
$$

where $\kappa=\frac{1}{\sigma_{k}^{2}\left(\zeta^{\frac{1}{4}}+\zeta^{-\frac{1}{4}}\right)^{2}}$.
Then we have

$$
\begin{align*}
\left\{\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \sqrt{\vartheta}\right\} & \subset \cup_{l=1}^{D_{t}}\left\{\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \frac{2 \kappa \vartheta}{\lambda_{l} \sqrt{N_{k}^{i}(t)}}+\frac{\sigma_{k}^{2}}{2} \lambda_{l} \sqrt{N_{k}^{i}(t)}\right\}  \tag{51}\\
& =\cup_{l=1}^{D_{t}}\left\{\lambda_{l} Z_{k}^{i}(t)-\frac{\lambda_{l}^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t) \geq 2 \kappa \vartheta\right\} \tag{52}
\end{align*}
$$

Recall from the Markov inequality that $\mathrm{P}(Y \geq a) \leq \frac{\mathbb{E}(Y)}{a}$ for any positive random variable $Y$. Thus from (52) and Markov inequality we get,

$$
\begin{equation*}
\mathrm{P}\left(\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \sqrt{\vartheta}\right) \leq \sum_{l=1}^{D_{t}} \exp (-2 \kappa \vartheta) \tag{53}
\end{equation*}
$$

Then we have,

$$
\begin{equation*}
\mathrm{P}\left(\frac{Z_{k}^{i}(t)}{N_{k}^{i}(t)} \geq \sqrt{\frac{\vartheta}{N_{k}^{i}(t)}}\right) \leq \sum_{l=1}^{D_{t}} \exp (-2 \kappa \vartheta) \tag{54}
\end{equation*}
$$

Substituting $\vartheta=2 \sigma_{k}^{2}(\xi+1) \log t$ we get

$$
\begin{equation*}
\mathrm{P}\left(\left|\widehat{\mu}_{k}^{i}(t)-\mu_{k}\right|>\sigma_{k} \sqrt{\frac{2(\xi+1) \log t}{N_{k}^{i}(t)}}\right) \leq \frac{\log \left(\left(d_{i}(G)+1\right) t\right)}{\log \zeta} \exp \left(-\frac{4(\xi+1) \log t}{\left(\zeta^{\frac{1}{4}}+\zeta^{-\frac{1}{4}}\right)^{2}}\right) \tag{55}
\end{equation*}
$$

Note that $\forall \zeta>1$ we have

$$
\begin{equation*}
\frac{4}{\left(\zeta^{\frac{1}{4}}+\zeta^{-\frac{1}{4}}\right)^{2}} \geq 1-\frac{(\zeta-1)^{2}}{16} \tag{56}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\mathrm{P}\left(\left|\widehat{\mu}_{k}^{i}(t)-\mu_{k}\right|>\sigma_{k} \sqrt{\frac{2(\xi+1) \log t}{N_{k}^{i}(t)}}\right) \leq \frac{\log \left(\left(d_{i}(G)+1\right) t\right)}{\log \zeta} \frac{1}{t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)}} \tag{57}
\end{equation*}
$$

This concludes the proof of Lemma 3.

Lemma 4. Let $\zeta=1.3, \xi \geq 1.1, d_{i} \geq 0$ and $t \in[T]$. Then we have

$$
\begin{equation*}
\sum_{t=1}^{T-1} \frac{1}{\log \zeta} \frac{\log \left(\left(d_{i}+1\right) t\right)}{\left.t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)} \leq 12 \log \left(3\left(d_{i}+1\right)\right)+3\left(\log \left(d_{i}+1\right)+1\right)\right) ~(1)} \tag{58}
\end{equation*}
$$

Proof. For $\zeta=1.3$ we have $\frac{1}{\log \zeta}<8.78$. Further $(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)>2$ and $\forall t \geq 3$ we see that $\frac{\log \left(\left(d_{i}+1\right) t\right)}{t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)}}$ is monotonically decreasing. Thus we have

$$
\begin{equation*}
\sum_{t=1}^{T-1} \frac{\log \left(\left(d_{i}+1\right) t\right)}{t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)}} \leq 1.362 \log \left(3\left(d_{i}+1\right)\right)+\int_{3}^{T-1} \frac{\log \left(\left(d_{i}+1\right) t\right)}{t^{2}} d t \tag{59}
\end{equation*}
$$

Let $z=\left(d_{i}+1\right) t$. Then we have

$$
\begin{align*}
\int_{3}^{T-1} \frac{\log \left(\left(d_{i}+1\right) t\right)}{t^{2}} d t & =\left(d_{i}+1\right) \int_{3\left(d_{i}+1\right)}^{\left(d_{i}+1\right)(T-1)} \frac{\log z}{z^{2}} d z  \tag{60}\\
& =\left(d_{i}+1\right)\left[-\frac{\log z}{z}-\frac{1}{z}\right]_{3\left(\left(d_{i}+1\right)\right.}^{\left(d_{i}+1\right)(T-1)} \tag{61}
\end{align*}
$$

Thus we have

$$
\begin{align*}
\int_{3}^{T-1} \frac{\log \left(\left(d_{i}+1\right) t\right)}{t^{2}} d t & \leq\left(d_{i}+1\right)\left[\frac{\log \left(d_{i}+1\right)}{3\left(d_{i}+1\right)}+\frac{1}{3\left(d_{i}+1\right)}\right]  \tag{62}\\
& =\frac{1}{3} \log \left(d_{i}+1\right)+\frac{1}{3} \tag{63}
\end{align*}
$$

Recall that For $\zeta=1.3$ we have $\frac{1}{\log \zeta}<8.78$. Thus the proof of Lemma 4 follows from (59) and (63).

Now we prove Theorem 1 as follows. Recall that group regret can be given as $\operatorname{Reg}_{G}(T)=$ $\sum_{i=1}^{N} \sum_{k>1} \Delta_{k} \cdot \mathbb{E}\left[n_{k}^{i}(t)\right]$. Thus using Lemmas 2,3 and 4 we obtain

$$
\begin{align*}
& \operatorname{Reg}_{G}(T) \leq 8(\xi+1) \sigma_{k}^{2}\left(\sum_{i=1}^{N}\left(1-p_{i} p\right)+\bar{\chi}(G) p_{\max } p\right)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)  \tag{64}\\
& +5 N \sum_{k>1} \Delta_{k}+4 \sum_{i=1}^{N}\left(3 \log \left(3\left(d_{i}(G)+1\right)\right)+\left(\log \left(d_{i}(G)+1\right)\right)\right) \sum_{k>1} \Delta_{k} \tag{65}
\end{align*}
$$

## B Proof of Theorem 2

In this section we consider the case where agents pass messages up to $\gamma$ hop neighbors with each hop adding a delay of 1 time step. Let $\mathcal{C}_{\gamma}$ be a non overlapping clique covering of $G_{\gamma}$. For any $\mathcal{C} \in \mathcal{C}_{\gamma}$ and $i, j \in \mathcal{C}$ let $\gamma_{i}=\max _{j \in \mathcal{C}} d(i, j)$ be the maximum distance (in graph $G$ ) between agent $i$ and any other agent $j$ in the same clique in graph $G_{\gamma}$. Let $\mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}$ is a random variable that takes value 1 if at time $\tau$ agent $i$ receives the message initiated by agent $j$ at time $\tau^{\prime}$. Recall
that each communicated message fails with probability $1-p$ and each agent $i$ incorporates the messages it receives from its neighbors with probability $p_{i}$.

We follow an approach similar to proof of Theorem 1. We star by providing a tail probability bound similar to Lemma 3.

Lemma 5. Let $d_{i}\left(G_{\gamma}\right)$ be the degree of agent $i$ in graph $G_{\gamma}$. For any $\sigma_{k}>0$ some constant $\zeta>1$

$$
\begin{equation*}
\mathrm{P}\left(\left|\widehat{\mu}_{k}^{i}(t)-\mu_{k}\right|>\sigma_{k} \sqrt{\frac{2(\xi+1) \log t}{N_{k}^{i}(t)}}\right) \leq \frac{\log \left(\left(d_{i}\left(G_{\gamma}\right)+1\right) t\right)}{\log \zeta} \frac{1}{t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)}} . \tag{66}
\end{equation*}
$$

Proof. For all $k$ let $X_{k}^{i}(t)$ for all $i, t$ be iid copies of $X_{k}$. Then we have $X_{t}^{i} \mathbf{1}\left\{A_{i}(t)=k\right\}=X_{k}^{i}(t) \mathbf{1}\left\{A_{i}(t)=k\right\}$. Recall that reward distribution of arm $k$ has mean $\mu_{k}$ and variance proxy $\sigma_{k}$. Thus $\forall i, t$ we have

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda\left(X_{k}^{i}(t)-\mu_{k}\right)\right)\right) \leq \exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2}\right) \tag{67}
\end{equation*}
$$

Define local history at every agent $i$ as follows

$$
\begin{equation*}
\mathcal{H}_{t}^{i}:=\sigma\left(X_{\tau^{\prime}}^{i}, A_{i}\left(\tau^{\prime}\right), X_{\tau^{\prime}}^{j} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}, A_{j}\left(\tau^{\prime}\right) \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}, \forall \tau^{\prime}, \tau \in[t], j \in \mathcal{N}_{i}\left(G_{\gamma}\right)\right) . \tag{68}
\end{equation*}
$$

Since $\mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}$ for $j \in \mathcal{N}_{i}\left(G_{\gamma}\right)$ is a $\mathcal{H}_{\tau-1}^{i}$ measurable random variable, we have $\forall \tau^{\prime} \leq \tau$

$$
\begin{align*}
& \mathbb{E}\left(\exp \left(\lambda\left(X_{\tau^{\prime}}^{j}-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{69}\\
& =\mathbb{E}\left(\exp \left(\lambda\left(X_{k}^{j}\left(\tau^{\prime}\right)-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{70}\\
& \leq \exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2} \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) . \tag{71}
\end{align*}
$$

Define a new random variable such that $\forall \tau>0$ and $\tau^{\prime} \leq \tau$

$$
\begin{align*}
Y_{k}^{i}(\tau)= & \sum_{j=1}^{N} \sum_{\tau^{\prime}=1}^{\tau}\left(X_{k}^{j}\left(\tau^{\prime}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right.  \tag{72}\\
& \left.-\mathbb{E}\left[X_{k}^{j}\left(\tau^{\prime}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\} \mid \mathcal{H}_{\tau-1}^{i}\right]\right)  \tag{73}\\
= & \sum_{j=1}^{N} \sum_{\tau^{\prime}=1}^{\tau}\left(X_{k}^{j}\left(\tau^{\prime}\right)-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\} . \tag{74}
\end{align*}
$$

Note that $\mathbb{E}\left(Y_{k}^{i}(\tau)\right)=\mathbb{E}\left(Y_{k}^{i}(\tau) \mid \mathcal{H}_{\tau-1}^{i}\right)=0$. Let $Z_{k}^{i}(t)=\sum_{\tau=1}^{t} Y_{k}^{i}(\tau)$. For any $\lambda>0$

$$
\begin{align*}
& \mathbb{E}\left(\exp \left(\lambda Y_{k}^{i}(\tau)\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{75}\\
& =\mathbb{E}\left(\exp \left(\lambda \sum_{j=1}^{N} \sum_{\tau^{\prime}=1}^{\tau}\left(X_{k}^{j}\left(\tau^{\prime}\right)-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{76}\\
& =\mathbb{E}\left(\prod_{j=1}^{N} \prod_{\tau^{\prime}=1}^{\tau} \exp \left(\lambda\left(X_{k}^{j}\left(\tau^{\prime}\right)-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right) \tag{77}
\end{align*}
$$

$$
\begin{align*}
& \stackrel{(a)}{=} \prod_{j=1}^{N} \prod_{\tau^{\prime}=1}^{\tau} \mathbb{E}\left(\exp \left(\lambda\left(X_{k}^{j}\left(\tau^{\prime}\right)-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) \mid \mathcal{H}_{\tau-1}^{i}\right)  \tag{78}\\
& \leq \prod_{j=1}^{N} \prod_{\tau^{\prime}=1}^{\tau} \exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2} \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right)  \tag{79}\\
& =\exp \left(\frac{\lambda^{2} \sigma_{k}^{2}}{2} \sum_{j=1}^{N} \sum_{\tau^{\prime}=1}^{\tau} \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right) \tag{80}
\end{align*}
$$

Equality (a) follows from the fact that $\forall \tau^{\prime} \leq \tau$ random variables $\left\{\exp \left(\lambda\left(X_{k}^{j}\left(\tau^{\prime}\right)-\mu_{k}\right) \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau}\right.\right.\right.$ are conditionally independent with respect to $\mathcal{H}_{\tau-1}^{i}$. Since $\mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\}, \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}$ are $\mathcal{H}_{\tau-1}^{i}$ measurable, and so

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda Y_{k}^{i}(\tau)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} \sum_{j=1}^{N} \sum_{\tau^{\prime}=1}^{\tau} \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}\right)\left|\mid \mathcal{H}_{\tau-1}^{i}\right) \leq 1\right. \tag{81}
\end{equation*}
$$

Let $N_{k}^{i}(t)=\sum_{\tau=1}^{t} \sum_{\tau^{\prime}=1}^{\tau} \sum_{j=1}^{N} \mathbf{1}\left\{A_{i}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\}$. Further, using the tower property of conditional expectation we have

$$
\begin{equation*}
\mathbb{E}\left(\left.\exp \left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t)\right) \right\rvert\, \mathcal{H}_{t-1}^{i}\right) \leq \exp \left(\lambda Z_{k}^{i}(t-1)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t-1)\right) \tag{82}
\end{equation*}
$$

Repeating the above step $t$ times we have

$$
\begin{equation*}
\mathbb{E}\left(\exp \left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t)\right)\right) \leq 1 \tag{83}
\end{equation*}
$$

Note that we have

$$
\begin{align*}
& \mathrm{P}\left(\exp \left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{i}^{2}}{2} N_{k}^{i}(t)\right) \geq \exp (2 \kappa \vartheta)\right)  \tag{84}\\
& =\mathrm{P}\left(\lambda Z_{k}^{i}(t)-\frac{\lambda^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t) \geq 2 \kappa \vartheta\right)  \tag{85}\\
& =\mathrm{P}\left(\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \frac{2 \kappa \vartheta}{\lambda \sqrt{N_{k}^{i}(t)}}+\frac{\sigma_{k}^{2}}{2} \lambda \sqrt{N_{k}^{i}(t)}\right) \tag{86}
\end{align*}
$$

Fix a constant $\zeta>1$. Then $1 \leq N_{k}^{i}(t) \leq \zeta^{D_{t}}$ where $D_{t}=\frac{\log \left(\left(d_{i}\left(G_{\gamma}\right)+1\right) t\right)}{\log \zeta}$. For $\lambda_{l}=\frac{2}{\sigma_{k}} \sqrt{\frac{\kappa \vartheta}{\zeta^{1-1 / 2}}}$ and $\zeta^{l-1} \leq N_{k}^{i}(t) \leq \zeta^{l}$ we have

$$
\begin{equation*}
\frac{2 \kappa \vartheta}{\lambda_{l}} \sqrt{\frac{1}{N_{k}^{i}(t)}}+\frac{\sigma_{k}^{2}}{2} \lambda_{l} \sqrt{N_{k}^{i}(t)}=\sigma_{k} \sqrt{\kappa \vartheta}\left(\sqrt{\frac{\zeta^{l-1 / 2}}{N_{k}^{i}(t)}}+\sqrt{\frac{N_{k}^{i}(t)}{\zeta^{l-1 / 2}}}\right) \leq \sqrt{\vartheta} \tag{87}
\end{equation*}
$$

where $\kappa=\frac{1}{\sigma_{k}^{2}\left(\zeta^{\frac{1}{4}}+\zeta^{-\frac{1}{4}}\right)^{2}}$.

Then we have

$$
\begin{align*}
\left\{\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \sqrt{\vartheta}\right\} & \subset \cup_{l=1}^{D_{t}}\left\{\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \frac{2 \kappa \vartheta}{\lambda_{l} \sqrt{N_{k}^{i}(t)}}+\frac{\sigma_{k}^{2}}{2} \lambda_{l} \sqrt{N_{k}^{i}(t)}\right\}  \tag{88}\\
& =\cup_{l=1}^{D_{t}}\left\{\lambda_{l} Z_{k}^{i}(t)-\frac{\lambda_{l}^{2} \sigma_{k}^{2}}{2} N_{k}^{i}(t) \geq 2 \kappa \vartheta\right\} \tag{89}
\end{align*}
$$

Recall from the Markov inequality that $\mathrm{P}(Y \geq a) \leq \frac{\mathbb{E}(Y)}{a}$ for any positive random variable $Y$. Thus from (89) and Markov inequality we get,

$$
\begin{equation*}
\mathrm{P}\left(\frac{Z_{k}^{i}(t)}{\sqrt{N_{k}^{i}(t)}} \geq \sqrt{\vartheta}\right) \leq \sum_{l=1}^{D_{t}} \exp (-2 \kappa \vartheta) \tag{90}
\end{equation*}
$$

Then we have,

$$
\begin{equation*}
\mathrm{P}\left(\frac{Z_{k}^{i}(t)}{N_{k}^{i}(t)} \geq \sqrt{\frac{\vartheta}{N_{k}^{i}(t)}}\right) \leq \sum_{l=1}^{D_{t}} \exp (-2 \kappa \vartheta) \tag{91}
\end{equation*}
$$

Recall that $\forall \zeta>1$ we have

$$
\begin{equation*}
\frac{4}{\left(\zeta^{\frac{1}{4}}+\zeta^{-\frac{1}{4}}\right)^{2}} \geq 1-\frac{(\zeta-1)^{2}}{16} \tag{92}
\end{equation*}
$$

Substituting $\vartheta=2 \sigma_{k}^{2}(\xi+1) \log t$ we get

$$
\begin{equation*}
\mathrm{P}\left(\left|\widehat{\mu}_{k}^{i}(t)-\mu_{k}\right|>\sigma_{k} \sqrt{\frac{2(\xi+1) \log t}{N_{k}^{i}(t)}}\right) \leq \frac{\log \left(\left(d_{i}\left(G_{\gamma}\right)+1\right) t\right)}{\log \zeta} \frac{1}{t^{(\xi+1)\left(1-\frac{(\zeta-1)^{2}}{16}\right)}} . \tag{93}
\end{equation*}
$$

This concludes the proof of Lemma 5.
We prove a Lemma similar to Lemma 2 for message-passing as follows.
Lemma 6. Let $\bar{\chi}\left(G_{\gamma}\right)$ is the clique number of graph $G_{\gamma}$. Let $\eta_{k}=\left(\frac{8(\xi+1) \sigma_{k}^{2}}{\Delta_{k}^{2}}\right) \log T$. Then we have

$$
\begin{align*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right] & \leq\left(\sum_{i=1}^{N}\left(1-p_{i} p^{\gamma_{i}}\right)+\bar{\chi}\left(G_{\gamma}\right) \max _{i \in[N]} p_{i} p^{\gamma_{i}}\right) \eta_{k}+N(\gamma+1)+  \tag{94}\\
& +\sum_{i=1}^{N} \sum_{t=1}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right] \tag{95}
\end{align*}
$$

Proof. Note that for each suboptimal arm $k>1$ we have

$$
\begin{equation*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right]=\sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)=\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) \tag{96}
\end{equation*}
$$

Let $\tau_{k, \mathcal{C}}$ denote the maximum time step when the total number of times arm $k$ has been played by all the agents in clique $\mathcal{C}$ is at most $\eta_{k}+|\mathcal{C}|$ times. This can be stated as $\tau_{k, \mathcal{C}}:=\max \{t \in[T]$ : $\left.\sum_{i \in \mathcal{C}} n_{k}^{i}(t) \leq \eta_{k}+|\mathcal{C}|\right\}$. Then, we have that $\eta_{k}<\sum_{i \in \mathcal{C}} n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right) \leq \eta_{k}+|\mathcal{C}|$.

For each agent $i \in \mathcal{C}$ let

$$
\bar{N}_{k}^{i}(t):=\sum_{j \in \mathcal{C}} \sum_{\tau=1}^{t} \sum_{\tau^{\prime}=1}^{\tau} \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\},
$$

denote the sum of the total number of times agent $i$ pulled arm $k$ and the total number of observations it received from agents in its clique about arm $k$ until time $t$. Define $\bar{\tau}_{k, \mathcal{C}}^{i}:=\max \{t \in[T]:$ $\left.\bar{N}_{k}^{i}(t) \leq \eta_{k}\right\}$. For each agent $i \in[N]$ let $\bar{\tau}_{k, \mathcal{C}}^{i}=\max \left\{\tau_{k, \mathcal{C}}+\gamma_{i}-1, \bar{\tau}_{k, \mathcal{C}}^{i}\right\}$.

Note that $N_{k}^{i}(t) \geq \bar{N}_{k}^{i}(t), \forall t$, hence for all $i \in \mathcal{C}$ we have $N_{k}^{i}(t)>\eta_{k}, \forall t>\bar{\tau}_{k, \mathcal{C}}^{i}$. Here we consider that $\bar{\tau}_{k, \mathcal{C}}^{i} \geq \tau_{k, \mathcal{C}}, \forall i$. From regret results it follows that regret for this case is greater than the regret for the case where $\bar{\tau}_{k, \mathcal{C}}^{i}<\tau_{k, \mathcal{C}}$ for some (or all) $i$.

We analyse the expected number of times agents pull suboptimal arm $k$ as follows,

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{97}\\
& =\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}+\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\}+\sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\bar{\tau}_{k, \mathcal{C}}^{i}}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{98}\\
& \leq \sum_{\mathcal{C} \in \mathcal{C}_{\gamma}}\left(\eta_{k}+|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{99}\\
& +\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\bar{\tau}_{k, \mathcal{C}}^{i}}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\} \mathbf{1}\left\{N_{k}^{i}(t-1)>\eta_{k}\right\} . \tag{100}
\end{align*}
$$

Taking expectation we have

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)  \tag{101}\\
& \leq \sum_{\mathcal{C} \in \mathcal{C}_{\gamma}}\left(\eta_{k}+2|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathrm{P}\left(A_{i}(t)=k\right)  \tag{102}\\
& +\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}^{i}}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) . \tag{103}
\end{align*}
$$

Case 1. For agent $i$ we have that $\tau_{k, \mathcal{C}}+\gamma_{i}-1 \geq \bar{\tau}_{k, \mathcal{C}}^{i}$ then we have $\bar{\tau}_{k, \mathcal{C}}^{i}=\tau_{k, \mathcal{C}}+\gamma_{i}-1$. Then we have $\sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\} \leq \gamma_{i}-1$

Case 2. For agent $i$ we have that $\tau_{k, \mathcal{C}}+\gamma_{i}-1<\bar{\tau}_{k, \mathcal{C}}^{i}$ then we have $\bar{\tau}_{k, \mathcal{C}}^{i}=\bar{\tau}_{k, \mathcal{C}}^{i}$.

$$
\begin{align*}
& \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{104}\\
& =\tilde{N}_{k}^{i}\left(\bar{\tau}_{k, \mathcal{C}}^{i}\right)-\sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}-\sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \sum_{\tau=1}^{t} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau, t}\right\}  \tag{105}\\
& \leq \tilde{N}_{k}^{i}\left(\bar{\tau}_{k, \mathcal{C}}^{i}\right)-\sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}-\sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}+\gamma_{i}-1} \sum_{\tau=1}^{t} \mathbf{1}\left\{A_{j}(\tau)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau, t}\right\} . \tag{106}
\end{align*}
$$

Taking the expectation we have

$$
\begin{align*}
\sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{\bar{\tau}_{k, \mathcal{C}}^{i}} \mathrm{P}\left(A_{i}(t)=k\right) & \leq|\mathcal{C}| \eta_{k}-\eta_{k}+\sum_{i \in \mathcal{C}}\left(\gamma_{i}-1\right)-\sum_{i \in \mathcal{C}} p_{i} p^{\gamma_{i}} \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathrm{P}\left(A_{j}(t)=k\right)  \tag{107}\\
& =|\mathcal{C}| \eta_{k}-\eta_{k}+\sum_{i \in \mathcal{C}}\left(\gamma_{i}-1\right)-\sum_{i \in \mathcal{C}} p_{i} p^{\gamma_{i}} \sum_{j \neq i, j \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbb{E}\left(n_{k}^{j}\left(\tau_{k, \mathcal{C}}\right)\right)  \tag{108}\\
& \leq\left(|\mathcal{C}|-1-\left(\sum_{j \in \mathcal{C}} p_{j} p^{\gamma_{j}}-\max _{i \in[N]} p_{i} p^{\gamma_{i}}\right)\right) \eta_{k}+\sum_{i \in \mathcal{C}}\left(\gamma_{i}-1\right) . \tag{109}
\end{align*}
$$

Substituting these results to (103) we get

$$
\begin{align*}
\sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) & \leq \sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}}\left(|\mathcal{C}|-1-\left(\sum_{j \in \mathcal{C}} p_{j} p^{\gamma_{j}}-\max _{i \in[N]} p_{i} p^{\gamma_{i}}\right)\right) \eta_{k}+\sum_{i \in[N]}\left(\gamma_{i}-1\right)  \tag{110}\\
& +\sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}}\left(\eta_{k}+2|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}^{i}}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right)  \tag{111}\\
& \leq\left(\sum_{i=1}^{N}\left(1-p_{i} p^{\gamma_{i}}\right)+\bar{\chi}\left(G_{\gamma}\right) \max _{i \in[N]} p_{i} p^{\gamma_{i}}\right) \eta_{k}+\sum_{i \in[N]} \gamma_{i}+N  \tag{112}\\
& +\sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}^{i}}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) \tag{113}
\end{align*}
$$

This concludes the proof of Lemma 6.
Now we prove Theorem 2 as follows. Thus using Lemmas 4,5 and 6 we obtain

$$
\begin{align*}
\operatorname{Reg}_{G}(T) & \leq 8(\xi+1) \sigma_{k}^{2}\left(\sum_{i=1}^{N}\left(1-p_{i} p^{\gamma_{i}}\right)+\bar{\chi}\left(G_{\gamma}\right) \max _{i \in[N]} p_{i} p^{\gamma_{i}}\right)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)  \tag{114}\\
& +\left(\sum_{i=1}^{N} \gamma_{i}+4 N\right) \sum_{k>1} \Delta_{k}+4 \sum_{i=1}^{N}\left(3 \log \left(3\left(d_{i}\left(G_{\gamma}\right)+1\right)\right)+\left(\log \left(d_{i}\left(G_{\gamma}\right)+1\right)\right)\right) \sum_{k>1} \Delta_{k} \tag{115}
\end{align*}
$$

## C Proof of Theorem 3

Agents receive information from their neighbors with a stochastic time delay. Let $\mathcal{N}_{D}$ be the maximum number of outstanding arm pulls by all the agent. We start by proving a result similar to Lemma 2.

Lemma 7. Let $\bar{\chi}(G)$ is the clique number of graph $G$. Let $\eta_{k}=\left(\frac{8(\xi+1) \sigma_{k}^{2}}{\Delta_{k}^{2}}\right) \log T$. Then we have

$$
\begin{align*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right] & \leq \bar{\chi}(G) \eta_{k}+\mathbb{E}\left[\mathcal{N}_{D}\right]+2 N+  \tag{116}\\
& +\sum_{i=1}^{N} \sum_{t=1}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right] \tag{117}
\end{align*}
$$

Proof. Let $\mathcal{C}$ be a non overlapping clique covering of $G$. Note that for each suboptimal arm $k>1$ we have

$$
\begin{equation*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right]=\sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)=\sum_{\mathcal{C} \in \mathfrak{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) . \tag{118}
\end{equation*}
$$

Let $\tau_{k, \mathcal{C}}$ denote the maximum time step such that the total number of arm pulls shared by agents in clique $\mathcal{C}$ from arm $k$ is at most $\eta_{k}+|\mathcal{C}|$. For each agent $i \in \mathcal{C}$ let $D_{i}\left(\tau_{k, \mathcal{C}}\right)$ be the number of outstanding messages by agent $i$ from arm $k$ at time $\tau_{k, \mathcal{C}}$. This can be stated as $\tau_{k, \mathcal{C}}:=\max \{t \in[T]$ : $\left.\sum_{i \in \mathcal{C}} n_{k}^{i}(t) \leq \eta_{k}+\sum_{i \in \mathcal{C}} D_{i}\left(\tau_{k, \mathcal{C}}\right)+|\mathcal{C}|\right\}$. Then, we have that $\eta_{k}+\sum_{i \in \mathcal{C}} D_{i}\left(\tau_{k, \mathcal{C}}\right)<\sum_{i \in \mathcal{C}} n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right) \leq$ $\eta_{k}+\sum_{i \in \mathcal{C}} D_{i}\left(\tau_{k, \mathcal{C}}\right)+|\mathcal{C}|$.

Note that for all $i \in \mathcal{C}$ we have $N_{k}^{i}(t)>\eta_{k}, t>\tau_{k, \mathcal{C}}$.
We analyse the expected number of times agents pull suboptimal arm $k$ as follows,

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{119}\\
& =\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{120}\\
& \leq \sum_{\mathcal{C} \in \mathcal{C}}\left(\eta_{k}+\sum_{i \in \mathcal{C}} D_{i}\left(\tau_{k, \mathcal{C}}\right)+2|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{T-1} \mathbf{1}\left\{A_{i}(t+1)=k\right\} \mathbf{1}\left\{N_{k}^{i}(t)>\eta_{k}\right\} . \tag{121}
\end{align*}
$$

Taking expectation we have

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)  \tag{122}\\
& \leq \bar{\chi}\left(G_{\gamma}\right) \eta_{k}+\mathbb{E}\left[\max _{t \in[T]} \sum_{i=1}^{N} D_{i}(t)\right]+2 N+\sum_{i=1}^{N} \sum_{t=1}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) \tag{123}
\end{align*}
$$

The proof of Lemma 7 follows from Lemma 1 and (123).

We upper bound the expected number of outstanding messages by any agent using results by Joulani, Gyorgy and Szepesvári (2013) as follows.

Lemma 8. . Let $D_{\text {total }}$ be the maximum number of outstanding messages by all the agent at any time step $t \in[T]$ and let $\mathbb{E}[\tau]$ be the expected delay of any message. Then with probability at least $1-\frac{1}{T}$ we have

$$
\begin{equation*}
\mathbb{E}\left[D_{\text {total }}\right] \leq N \mathbb{E}[\tau]+2 \log T+2 \sqrt{N \mathbb{E}[\tau] \log T} \tag{124}
\end{equation*}
$$

Proof. The proof directly follows from Lemma 2 by Joulani, Gyorgy and Szepesvári (2013).
From Lemmas 7, 3, 4 and 8 we obtain with probability at least $1-\frac{1}{T}$

$$
\begin{align*}
\operatorname{Reg}_{G}(T) & \leq 8(\xi+1) \sigma_{k}^{2} \bar{\chi}(G)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)  \tag{125}\\
& +(N \mathbb{E}[\tau]+2 \log T+2 \sqrt{N \mathbb{E}[\tau] \log T}) \sum_{k>1} \Delta_{k}  \tag{126}\\
& +5 N \sum_{k>1} \Delta_{k}+4 \sum_{i=1}^{N}\left(3 \log \left(3\left(d_{i}(G)+1\right)\right)+\left(\log \left(d_{i}(G)+1\right)\right)\right) \sum_{k>1} \Delta_{k} \tag{127}
\end{align*}
$$

## D Proof of Theorem 4

We first restate the result for clarity.
Theorem 7. Algorithm 1 obtains, with probability at least $1-\delta$, cumulative group regret of

$$
\operatorname{Reg}_{G}(T)=\mathcal{O}\left(K T N \gamma \epsilon+\psi\left(G_{\gamma}\right) \sum_{k \neq k^{\star}} \frac{\log T}{\Delta_{k}} \log \left(\frac{K \psi\left(G_{\gamma}\right) \log T}{\delta}\right)+N \Delta_{k}+\frac{N \log (N \gamma \log T)}{\Delta_{k}}\right)
$$

Proof. We decompose the regret based on the dominating set and epoch. Let $\mathcal{I} \subseteq \mathcal{V}$ be an dominating set of $G_{\gamma}$ and $M_{i}$ be the number of epochs run for the subgraph covered by agent $i$. Observe that the total regret can be written as,

$$
\begin{equation*}
\operatorname{Reg}_{G}(T)=\sum_{i \in \mathcal{I}}\left(\sum_{k=1}^{K} \sum_{t=1}^{T} \Delta_{k} \cdot\left(\mathrm{P}\left(A_{i}(t)=k\right)+\sum_{j \in \mathcal{N}_{i}\left(G_{\gamma}\right)} \mathrm{P}\left(A_{j}(t)=k\right)\right)\right) . \tag{128}
\end{equation*}
$$

First, observe that $A_{j}(t)=A_{i}(t-d(i, j))$ for all $j \in \mathcal{N}_{i}\left(G_{\gamma}\right)$ and all $t \in[d(i, j), T]$. Rearranging the above, we have,

$$
\begin{align*}
\operatorname{Reg}_{G}(T) & \leqslant \sum_{i \in \mathcal{I}}\left(\sum_{k=1}^{K} \Delta_{k} \cdot\left(\sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)+\sum_{j \in \mathcal{N}_{i}\left(G_{\gamma}\right)}\left(\sum_{t=1}^{T-d(i, j)} \mathrm{P}\left(A_{i}(t)=k\right)+d(i, j)\right)\right)\right)  \tag{129}\\
& \leqslant \sum_{i \in \mathcal{I}}\left(\sum_{k=1}^{K} \Delta_{k} \cdot\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| \cdot\left(\sum_{t=1}^{T-\gamma} \mathrm{P}\left(A_{i}(t)=k\right)+\gamma\right)\right) \tag{130}
\end{align*}
$$

$$
\begin{equation*}
=\sum_{i \in \mathcal{I}}\left(\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| \sum_{k=1}^{K} \Delta_{k}\left(\sum_{t=1}^{T-\gamma} \mathrm{P}\left(A_{i}(t)=k\right)\right)\right)+N \gamma \sum_{k=1}^{K} \Delta_{k} . \tag{131}
\end{equation*}
$$

Now, observe that we run two algorithms in tandem for each subgraph of $G$ induced by $\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)$. Let us split the total number of rounds of the game into epochs that run arm elimination and the intermittent periods of running UCB1. We denote the cumulative regret in the $i^{\text {th }}$ induced subgraph from rounds $\gamma$ to $T$ as $\operatorname{Reg}_{\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}(T)$, and analyse it separately.

$$
\begin{equation*}
\operatorname{Reg}_{\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}(T) \leqslant\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| \sum_{k=1}^{K}\left(\Delta_{k}\left(\sum_{t \leq T-\gamma: t \in \mathcal{M}_{i}} \mathrm{P}\left(A_{i}(t)=k\right)+\sum_{t \leq T-\gamma: t \notin \mathcal{M}_{i}} \mathrm{P}\left(A_{i}(t)=k\right)\right)\right) . \tag{133}
\end{equation*}
$$

Here $\mathcal{M}_{i}$ denotes the rounds in which arm elimination is played in the agents in the $i^{\text {th }}$ induced subgraph. Since each UCB1 period after each epoch is of length $2 \gamma$, we have at most $2 \gamma M_{i}$ rounds of isolated UCB1. We analyse the second term in the bound first. By the standard analysis of the UCB1 algorithm (Auer, Cesa-Bianchi and Fischer, 2002), we have that the leader agent, i.e. agent $i$, incurs $\mathcal{O}(K \log T / \Delta)$ regret. We therefore have,

$$
\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| \sum_{k=1}^{K}\left(\Delta_{k}\left(\sum_{t \notin \mathcal{M}_{i}} \mathrm{P}\left(A_{i}(t)=k\right)\right)\right) \leqslant\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| \cdot \sum_{k=1}^{K}\left(\left(1+\frac{\pi^{2}}{3}\right) \Delta_{k}+\frac{8 \log \left(2 \gamma M_{i}\right)}{\Delta_{k}}\right) .
$$

Now, we analyse the first term in the regret bound. By Theorem 8, we have that with probability at least $1-\delta$ simultaneously for each induced subgraph corresponding to agent $i \in \mathcal{I}$,

$$
\sum_{k=1}^{K}\left(\Delta_{k}\left(\sum_{m \in \mathcal{M}_{i}} \mathbb{E}\left[n_{k}^{i}(m)\right]\right)\right)=\mathcal{O}\left(\gamma \epsilon \cdot K T\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|+\sum_{k>1} \frac{\log T}{\Delta_{k}} \log \left(\frac{K \psi\left(G_{\gamma}\right)}{\delta} \log T\right)\right)
$$

Summing over each leader agent, we have that with probability at least $1-\delta$,

$$
\sum_{i \in \mathcal{I}} \sum_{k=1}^{K}\left(\Delta_{k}\left(\sum_{m \in \mathcal{M}_{i}} \mathbb{E}\left[n_{k}^{i}(m)\right]\right)\right)=\mathcal{O}\left(\gamma \epsilon \cdot K T N+\sum_{k>1} \frac{\log T}{\Delta_{k}} \log \left(\frac{K \psi\left(G_{\gamma}\right)}{\delta} \log T\right)\right) .
$$

Next, observe that for all $i,\left|\mathcal{M}_{i}\right| \leq \log (M T)$ by Lemma 9. Replacing this result in the UCB1 regret for each leader, and summing over all $i \in \mathcal{I}$, we have,

$$
\operatorname{Reg}_{G}(T)=\mathcal{O}\left(\gamma \epsilon \cdot K T N+\sum_{k>1} \psi\left(G_{\gamma}\right) \frac{\log T}{\Delta_{k}} \log \left(\frac{K \psi\left(G_{\gamma}\right) \log T}{\delta}\right)+N \Delta_{k}+\frac{N \log (N \gamma \log T)}{\Delta_{k}}\right)
$$

Lemma 9. For any leader $i$, let $L^{i}(m)$ denote the length of the $m^{\text {th }}$ epoch of arm elimination. Then, we have that $L^{i}(m)$ satisfies,

$$
2^{2 m-2} \lambda \leq L^{i}(m) \leq K 2^{2 m-2} \lambda .
$$

Furthermore, the number of arm elimination epochs for agent $i$ satisfies $M_{i} \leq \log _{2}(T-2 \gamma)$.

Proof. The proof closely follows the proof of Lemma 2 in Gupta, Koren and Talwar (2019). For any leader $i$, let $\hat{k}$ be the optimal arm under $r^{i}(m)$, therefore $r_{\star}^{i}(m)-r_{\hat{k}}^{i}(m) \leq 0$ and therefore $\Delta_{\hat{k}}^{i}(m)=2^{-m}$, and therefore $L^{i}(m+1) \geq n_{\hat{k}}^{i}(m+1)=\lambda\left(\Delta_{\hat{k}}^{i}(m)\right)^{-2} \geq 2^{2 m} \lambda$. Next, observe that $\Delta_{k}^{i}(m) \geq 2^{-m}$ for each arm $k$, and therefore $n_{k}^{i}(m+1) \leq 2^{2 m} \lambda$, giving the upper bound.

For the second part, observe that $\sum_{m=1}^{M_{i}} L^{i}(m) \leq T-2 \gamma M_{i} \leq T-2 \gamma$, and that $L^{i}(m) \geq \frac{2^{2 m-2} \lambda}{\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|}$. Summing over $m \in\left[M_{i}\right]$ and taking the logarithm provides us with the result.

Lemma 10. Denote $\mathcal{E}$ to be the event for which,

$$
\left\{\forall m, i, k,\left|r_{k}^{i}(m)-\mu_{k}\right| \leq 2 \gamma \epsilon+\frac{\Delta_{k}^{i}(m-1)}{16} \bigwedge \sum_{\substack{t \in \mathcal{M}_{i}(m) \\ j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}} X_{k}^{j}(t+d(i, j)) \leq 2 n_{k}^{i}(m)\right\}
$$

Then, we have that $\mathrm{P}(\mathcal{E}) \geq 1-\delta$.
Proof. Recall that at each step in the epoch, the leader agent picks an arm $k$ with probability $p_{k}^{i}(m)=\frac{n_{k}^{i}(m)}{L^{2}(m)}$, and let $X_{k}^{j}(t)$ denote whether agent $j$ picks arm $k$ at time $t$. Let $C_{j \rightarrow i}(t)=$ $\tilde{r}_{j \rightarrow i}(t)-r_{j}(t)$ denote the corruption in the transmitted reward from agent $j$ when it reaches agent $i$, and $\mathcal{M}_{i}(m)=\left[T_{i}(m-1)+1, \cdots, T_{i}(m)\right]$ denote the $L^{i}(m)$ steps in the $m^{\text {th }}$ epoch for the arm elimination algorithm run by the leader $i$. We then have,

$$
r_{k}^{i}(m)=\frac{1}{n_{k}^{i}(m)}\left(\sum_{\substack{t \in \mathcal{M}_{i}(m) \\ j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}} X_{k}^{j}(t+d(i, j)) \cdot\left(r_{j}(t+d(i, j))+C_{j \rightarrow i}(t+d(i, j))\right)\right)
$$

For simplicity, let

$$
A_{k}^{i}(m)=\sum_{\substack{t \in \mathcal{M}_{i}(m) \\ j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}} X_{k}^{j}(t+d(i, j)) \cdot r_{j}(t+d(i, j)), B_{k}^{i}(m)=\sum_{\substack{t \in \mathcal{M}_{i}(m) \\ j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}} X_{k}^{j}(t+d(i, j)) \cdot C_{j \rightarrow i}(t+d(i, j)) .
$$

We can bound the first summation by a multiplicative version of the Chernoff-Hoeffding bound (Angluin and Valiant, 1979) as each $r_{j}$ is bounded within $[0,1]$ and $X_{k}^{i}$ is a random variable in $\{0,1\}$ with mean $p_{k}^{i}(m) L^{i}(m) \mu_{k} \leq n_{k}^{i}(m)$. We obtain that with probability at least $1-\beta / 2$,

$$
\left|\frac{A_{k}^{i}(m)}{n_{k}^{i}(m)}-\mu_{i}\right| \leq \sqrt{\frac{3 \log \left(\frac{4}{\beta}\right)}{n_{k}^{i}(m)}}
$$

To bound the second term, we must construct a filtration that ensures that the corruption is measurable. For the set $\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)$, consider an order $\sigma$ of the $N$ agents, such that $\sigma[1]=i$, followed by the agents at distance 1 from $i$, then the agents at distance 2 , and so on until distance $\gamma$, and next consider the ordering $\left\{\tilde{r}_{\tau}\right\}_{\tau=1}^{\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| t}$ of the rewards generated by all agents within $\mathcal{M}_{i}(m)$ where $\tilde{r}_{\tau}$ is the reward obtained by agent $j=\left(\sigma(\tau) \bmod \left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|\right)$ during the round $\left\lfloor\frac{\tau}{\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|}\right\rfloor+$ $d(i, j)$, and similarly consider an identical ordering of the pulled arms $\left\{\widetilde{X}_{\tau}\right\}_{\tau=1}^{\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| t}$. Now consider
the filtration $\left\{\mathcal{F}_{t}\right\}_{t=1}^{T\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|}$ generated by the two stochastic processes of $\tilde{r}$ and $\widetilde{X}$. Clearly, the corruption $C_{\sigma(j) \rightarrow i}(t)$ is deterministic conditioned on $\mathcal{F}_{t-1}$. Moreover, we have that the pulled arm satisfies, for all $\tau \in\left[\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| t\right]$ that $\mathbb{E}\left[\tilde{X}_{\tau} \mid \mathcal{F}_{\tau-1}\right]=p_{k}^{i}(m)$. Furthermore, since the corruption in each round is bounded and deterministic, we have that the sequence $Z_{\tau}=\left(\widetilde{X}_{\tau}-p_{k}^{i}(m)\right) \cdot \widetilde{C}_{\tau}$ (where $\widetilde{C}_{\tau}$ is the corresponding ordering of corruptions) is a martingale difference sequence with respect to $\left\{\mathcal{F}_{\tau}\right\}_{\tau=1}^{T}$. Now, consider the slice of $\left[\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right| t\right]$ that is present within $B_{k}^{i}(m)$, and let the corresponding indices be given by the set $\widetilde{\mathcal{M}}_{i}(m)$. Using the fact that the observed rewards are bounded, we have that,

$$
\sum_{\tau \in \widetilde{\mathcal{M}}_{i}(m)} \mathbb{E}\left[Z_{\tau}^{2} \mid \mathcal{F}_{\tau-1}\right] \leq \sum_{\tau \in \widetilde{\mathcal{M}}_{i}(m)}\left|\widetilde{C}_{\tau}\right| \cdot \mathbb{V}\left(Z_{\tau}\right) \leq p_{k}^{i}(m) \cdot \sum_{\tau \in \widetilde{\mathcal{M}}_{i}(m)} \widetilde{C}_{\tau} \leq \gamma C L^{i}(m) .
$$

We then have by Freedman's inequality that with probability at least $1-\frac{\beta}{4}$,

$$
\frac{B_{k}^{i}(m)}{n_{k}^{i}(m)} \leq \frac{p_{k}^{i}(m)}{n_{k}^{i}(m)}\left(\sum_{\tau \in \widetilde{\mathcal{M}}_{i}(m)} \widetilde{C}_{\tau}+\frac{\gamma C L^{i}(m)+\log (4 / \beta)}{n_{k}^{i}(m)}\right) \leq 2 \gamma \epsilon+\sqrt{\frac{\log (4 / \beta)}{16 n_{k}^{i}(m)}}
$$

The last inequality follows from the fact that $n_{k}^{i}(m) \geq \lambda \geq 16 \ln (4 / \beta)$. With the same probability, we can derive a bound for the other tail. Now, observe that since each $X_{k}^{i}$ is a random variable with mean $p_{k}^{i}$, we have by the multiplicative Chernoff-Hoeffding bound that the probability that the sum of $L^{i}(m)$ i.i.d. bernoulli trials with mean $p_{k}^{i}(m)$ is greater than $2 p_{k}^{i}(m) \cdot L^{i}(m)=2 n_{k}^{i}(m)$ is at most $2 \exp \left(-n_{k}^{i}(m) / 3\right) \leq 2 \exp (-\lambda / 3) \leq \beta$.

To conclude the proof, we apply each of the above bounds with $\beta=\frac{\delta}{2 K \alpha\left(G_{\gamma}\right) \log T}$ to each epoch and arm. Observe that $\beta \geq 4 \exp \left(-\frac{\lambda}{16}\right)$. Now, since $\log (4 / \beta)=\lambda /(32)^{2}$ we have that,

$$
\mathbb{P}\left(\left|r_{k}^{i}(m)-\mu_{k}\right| \geq 2 \gamma \epsilon+\frac{\Delta_{k}^{i}(m-1)}{16} \bigwedge \sum_{\substack{t \in \mathcal{M}_{i}(m) \\ j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)}} X_{k}^{j}(t+d(i, j)) \geq 2 n_{k}^{i}(m)\right) \leq \frac{\delta}{2 K \alpha\left(G_{\gamma}\right) \log T}
$$

The proof concludes by a union bound over all epochs, arms and agents in $\mathcal{I}$.
Lemma 11. If the event $\mathcal{E}$ (Lemma 10) occurs then for each $i \in \mathcal{I}, m \in \mathcal{M}_{i}$,

$$
-2 \gamma \epsilon-\frac{\Delta_{\star}^{i}(m-1)}{8} \leq r_{\star}^{i}(m)-\mu_{\star} \leq 2 \gamma \epsilon
$$

Proof. Observe that $r_{\star}^{i}(m) \geq r_{k^{\star}}^{i}(m)-\frac{1}{16} \Delta_{k^{\star}}^{i}(m-1)$. This fact coupled with the fact that $\mathcal{E}$ holds provides the lower bound. The upper bound is obtained by observing that,

$$
r_{\star}^{i}(m) \leq \max _{i}\left\{\mu_{i}+2 \gamma \epsilon+\frac{\Delta_{k}^{i}(m-1)}{16}-\frac{\Delta_{k}^{i}(m-1)}{16}\right\} \leq \mu_{\star}+2 \gamma \epsilon .
$$

Lemma 12. If the event $\mathcal{E}$ (Lemma 10) occurs then for each $i \in \mathcal{I}, m \in \mathcal{M}_{i}$,

$$
\Delta_{k}^{i}(m) \geq \frac{\Delta_{k}}{2}-6 \gamma \epsilon \sum_{n=1}^{m} 8^{n-m}-\frac{3}{4} 2^{-m}
$$

Proof. We first bound $\Delta_{k}^{i}(m) \leq 2\left(\Delta_{k}+2^{-m}+2 \gamma \epsilon \cdot \sum_{n=1}^{m} 8^{n-m}\right)$ under $\mathcal{E}$ by induction. Observe that when $m=1$ we have that trivially $\Delta_{k}^{i}(1) \leq 1 \leq 2 \cdot 2^{-1}$. Now, if the bound holds for epoch $m-1$ for any agent, we have by Lemma 11,

$$
r_{\star}^{i}(m)-r_{k}^{i}(m)=r_{\star}^{i}(m)-\mu_{\star}+\mu_{\star}-\mu_{k}+\mu_{k}-r_{k}^{i}(m) \leq 4 \gamma \epsilon+\Delta_{k}+\frac{\Delta_{k}^{i}(m-1)}{16}
$$

Replacing the induction hypothesis in the upper bound, we have,

$$
\begin{aligned}
r_{\star}^{i}(m)-r_{k}^{i}(m) & \leq 4 \gamma \epsilon+\Delta_{k}+\frac{1}{8}\left(\Delta_{k}+2^{-(m-1)}+2 \gamma \epsilon \cdot \sum_{n=1}^{m-1} 8^{n-m+1}\right) \\
& \leq 2\left(\Delta_{k}+2^{-m}+2 \gamma \epsilon \cdot \sum_{n=1}^{m} 8^{n-m}\right) .
\end{aligned}
$$

Now, we bound the gaps as,

$$
\Delta_{k}^{i}(m) \geq r_{\star}^{i}(m)-r_{k}^{i}(m) \geq \Delta_{k}-4 \gamma \epsilon-\left(\frac{\Delta_{k^{\star}}^{i}(m-1)}{8}-\frac{\Delta_{k}^{i}(m-1)}{16}\right)
$$

The last inequality follows from Lemma 11 and the event $\mathcal{E}$. Replacing the bound from induction we obtain,

$$
\begin{aligned}
\Delta_{k}^{i}(m) & \geq \Delta_{k}-4 \gamma \epsilon-\left(\frac{6 \gamma \epsilon}{8} \sum_{n=1}^{m} 2^{n-m}+\frac{3}{8} 2^{-(m-1)}+\frac{\Delta_{k}}{8}\right) \\
& \geq \frac{\Delta_{k}}{2}-6 \gamma \epsilon \sum_{n=1}^{m} 8^{n-m}-\frac{3}{4} 2^{-m} .
\end{aligned}
$$

Theorem 8. The cumulative regret for all agents within each independent set corresponding to leader $i \in \mathcal{I}$ satisfy simultaneously, with probability at least $1-\delta$,

$$
\sum_{m=1}^{\mathcal{M}_{i}} \sum_{k=1}^{K} \Delta_{k} \mathbb{E}\left[n_{k}^{i}(m)\right]=\mathcal{O}\left(\log \left(\frac{K \psi\left(G_{\gamma}\right)}{\delta} \log (T)\right) \log (T)\left(\sum_{k=1}^{K} \frac{1}{\Delta_{k}}\right)+\gamma \epsilon \cdot K T \cdot\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|\right) .
$$

Proof. We bound the regret in each epoch $m \in \mathcal{M}_{i}$ for each arm $k \neq k^{\star}$ based on three cases.
Case 1. $0 \leq \Delta_{k} \leq 4 / 2^{m}$ : We have that $n_{k}^{i}(m) \leq \lambda 2^{2(m-1)}$ since $\Delta_{k}^{i}(m-1) \geq 2^{m-1}$, and hence,

$$
\Delta_{k} \mathbb{E}\left[n_{k}^{i}(m)\right] \leq \frac{4 \lambda}{\Delta_{k}^{2}} \cdot \Delta_{k}=4 \lambda \cdot \frac{1}{\Delta_{k}} .
$$

Case 2. $\Delta_{k}>4 / 2^{m}$ and $\gamma \epsilon \sum_{n=1}^{m} 8^{n-m} \leq \Delta_{k} / 64$ : We have by Lemma 12 ,

$$
\Delta_{k}^{i}(m) \geq \frac{\Delta_{k}}{2}-6 \gamma \epsilon \sum_{n=1}^{m} 8^{n-m}-\frac{3}{4} 2^{-m} \geq \Delta_{k}\left(\frac{1}{2}-\frac{3}{32}-\frac{3}{8}\right)=\frac{\Delta_{k}}{32}
$$

Therefore, we have that $n_{k}^{i}(m) \leq \frac{1024 \lambda}{\Delta_{k}^{2}}$, and hence the regret is,

$$
\Delta_{k} \mathbb{E}\left[n_{k}^{i}(m)\right] \leq \frac{1024 \lambda}{\Delta_{k}^{2}} \cdot \Delta_{k}=1024 \lambda \cdot \frac{1}{\Delta_{k}}
$$

Case 3. $\Delta_{k}>4 / 2^{m}$ and $\gamma \epsilon \sum_{n=1}^{m} 8^{n-m}>\Delta_{k} / 64$ : This implies that $\Delta_{k} \leq 64 \gamma \epsilon \cdot \sum_{n=1}^{m} 8^{n-m}$. Therefore,

$$
\begin{aligned}
\Delta_{k} \mathbb{E}\left[n_{k}^{i}(m)\right] & \leq 64 \lambda \gamma \epsilon\left(\sum_{n=1}^{m} 8^{n-m}\right) \cdot 2^{2(m-1)} \\
& \leq 64 \lambda \gamma \epsilon\left(\frac{8^{m+1}}{7}\right) \cdot \frac{2^{2(m-1)}}{2^{3 m}} \\
& \leq \frac{512}{7} \gamma \epsilon \cdot L^{i}(m) .
\end{aligned}
$$

Here the last inequality follows from Lemma 9. Putting it together and summing over all epochs and arms, we have with probability at least $1-\delta$ simultaneously for each $i \in \mathcal{I}$,

$$
\sum_{m=1}^{\mathcal{M}_{i}} \sum_{k=1}^{K} \Delta_{k} \mathbb{E}\left[n_{k}^{i}(m)\right] \leq 1024^{2} \log \left(\frac{8 K \psi\left(G_{\gamma}\right)}{\delta} \log (T)\right) \log (T)\left(\sum_{k=1}^{K} \frac{1}{\Delta_{k}}\right)+74 \gamma \epsilon \cdot K T \cdot\left|\mathcal{N}_{i}^{+}\left(G_{\gamma}\right)\right|
$$

## E Proof of Theorem 5

In this section we consider that each agent passes messages upto $\gamma$-hop neighbors. Agents do not use the messages received during last $\bar{\gamma}$ number of time steps.
Lemma 13. Let $\bar{\chi}\left(G_{\gamma}\right)$ is the clique number of graph $G_{\gamma}$. Let $\eta_{k}=\left(\frac{8(\xi+1) \sigma_{k}^{2}}{\Delta_{k}^{2}}\right) \log T$. Then we have

$$
\begin{align*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right] & \leq \bar{\chi}\left(G_{\gamma}\right) \eta_{k}+\left(N-\bar{\chi}\left(G_{\gamma}\right)\right)(\bar{\gamma}+\gamma-1)+2 N+  \tag{134}\\
& +\sum_{i=1}^{N} \sum_{t=1}^{T-1}\left[\mathrm{P}\left(\widehat{\mu}_{1}^{i}(t) \leq \mu_{1}-C_{1}^{i}(t)\right)+\mathrm{P}\left(\widehat{\mu}_{k}^{i}(t) \geq \mu_{k}+C_{k}^{i}(t)\right)\right] \tag{135}
\end{align*}
$$

Proof. Let $\mathcal{C}_{\gamma}$ be a non overlapping clique covering of $G_{\gamma}$. Note that for each suboptimal arm $k>1$ we have

$$
\begin{equation*}
\sum_{i=1}^{N} \mathbb{E}\left[n_{k}^{i}(T)\right]=\sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)=\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right) \tag{136}
\end{equation*}
$$

Let $\tau_{k, \mathcal{C}}$ denote the maximum time step when the total number of times arm $k$ has been played by all the agents in clique $\mathcal{C}$ is at most $\eta_{k}+(|\mathcal{C}|-1)(\bar{\gamma}+\gamma-1)+|\mathcal{C}|$ times. This can be stated as $\tau_{k, \mathcal{C}}:=\max \left\{t \in[T]: \sum_{i \in \mathcal{C}} n_{k}^{i}(t) \leq \eta_{k}+(|\mathcal{C}|-1)(\bar{\gamma}+\gamma-1)+|\mathcal{C}|\right\}$. Then, we have that $\eta_{k}+(|\mathcal{C}|-1)(\bar{\gamma}+\gamma-1)<\sum_{i \in \mathcal{C}} n_{k}^{i}\left(\tau_{k, \mathcal{C}}\right) \leq \eta_{k}+(\mathcal{C}-1)(\bar{\gamma}+\gamma-1)+|\mathcal{C}|$.

For each agent $i \in \mathcal{C}$ let

$$
\bar{N}_{k}^{i}(t):=\sum_{\tau=1}^{t} \mathbf{1}\left\{A_{i}(\tau)=k\right\}+\sum_{j \neq i, j \in \mathcal{C}} \sum_{\tau=1}^{t-\bar{\gamma}} \sum_{\tau^{\prime}=1}^{\tau} \mathbf{1}\left\{A_{j}\left(\tau^{\prime}\right)=k\right\} \mathbf{1}\left\{(i, j) \in E_{\tau^{\prime}, \tau}\right\},
$$

denote the sum of the total number of times agent $i$ pulled arm $k$ and the total number of observations it received from agents in its clique about arm $k$ until time $t$.

Note that for all $i \in \mathcal{C}$ we have $N_{k}^{i}(t)>\eta_{k}, \forall t>\tau_{k, \mathcal{C}}$.
We analyse the expected number of times agents pull suboptimal arm $k$ as follows,

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{137}\\
& =\sum_{\mathcal{C} \in \mathfrak{e}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{\tau_{k, \mathcal{C}}} \mathbf{1}\left\{A_{i}(t)=k\right\}+\sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\bar{\tau}_{k, \mathcal{C}}^{i}}^{T} \mathbf{1}\left\{A_{i}(t)=k\right\}  \tag{138}\\
& \leq \sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}}\left(\eta_{k}+(|\mathcal{C}|-1)(\bar{\gamma}+\gamma-1)+2|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{T-1} \mathbf{1}\left\{A_{i}(t+1)=k\right\} \mathbf{1}\left\{N_{k}^{i}(t)>\eta_{k}\right\} . \tag{139}
\end{align*}
$$

Taking expectation we have

$$
\begin{align*}
& \sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T} \mathrm{P}\left(A_{i}(t)=k\right)  \tag{140}\\
& \leq \sum_{\mathcal{C} \in \mathfrak{C}_{\gamma}}\left(\eta_{k}+(|\mathcal{C}|-1)(\bar{\gamma}+\gamma-1)+2|\mathcal{C}|\right)+\sum_{\mathcal{C} \in \mathfrak{e}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t>\tau_{k, \mathcal{C}}}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) .  \tag{141}\\
& =\bar{\chi}\left(G_{\gamma}\right) \eta_{k}+\left(N-\bar{\chi}\left(G_{\gamma}\right)\right)(\bar{\gamma}+\gamma-1)+2 N+\sum_{\mathcal{C} \in \mathcal{C}_{\gamma}} \sum_{i \in \mathcal{C}} \sum_{t=1}^{T-1} \mathrm{P}\left(A_{i}(t+1)=k, N_{k}^{i}(t)>\eta_{k}\right) \tag{142}
\end{align*}
$$

The proof of Lemma 13 follows from Lemma 1 and (142).
Now we prove Theorem 5 as follows. Thus using Lemmas 4,5 and 13 we obtain

$$
\begin{align*}
\operatorname{Reg}_{G}(T) & \leq 8(\xi+1) \sigma_{k}^{2} \bar{\chi}\left(G_{\gamma}\right)\left(\sum_{k>1} \frac{\log T}{\Delta_{k}}\right)+\left(\left(N-\bar{\chi}\left(G_{\gamma}\right)(\bar{\gamma}+\gamma-1)+5 N\right) \sum_{k>1} \Delta_{k}\right.  \tag{143}\\
& +4 \sum_{i=1}^{N}\left(3 \log \left(3\left(d_{i}\left(G_{\gamma}\right)+1\right)\right)+\left(\log \left(d_{i}\left(G_{\gamma}\right)+1\right)\right)\right) \sum_{k>1} \Delta_{k} \tag{144}
\end{align*}
$$

## F Lower Bounds

Theorem 9 (Minimax Rate). For any multi-agent algorithm $\mathcal{A}$, there exists a $K$-armed environment over $N$ agents with $\Delta_{k} \leq 1$ such that,

$$
\operatorname{Reg}_{G}(\mathcal{A}, T) \geqslant c \sqrt{K N(T+\widetilde{d}(G))}
$$

Furthermore, if $\mathcal{A}$ is an agnostic decentralized policy, there exists a $K$-armed environment over $N$ agents with $\Delta_{k} \leq 1$ for any connected graph $G$ and $\gamma \geq 1$ such that, for some absolute constant $c^{\prime}$

$$
\operatorname{Reg}_{G}(\mathcal{A}, T) \geqslant c^{\prime} \sqrt{\alpha^{\star}\left(G_{\gamma}\right) K N T}
$$

Where $\tilde{d}(G)=\sum_{i=1}^{d^{\star}(G)} \bar{d}_{=i} \cdot i$ denotes the average delay incurred by message-passing across the network $G$, $d_{=i}=\frac{1}{N} \sum_{i, j} \mathbb{1}\{d(i, j)=i\}$ denotes the number of agent pairs that are at distance exactly $i$, and $\alpha^{\star}\left(G_{\gamma}\right)=\frac{N}{1+\bar{d}_{\gamma}}$ is Turan's lower bound (Turán, 1941) on $\alpha\left(G_{\gamma}\right)$.
Proof. Our approach is an extension of the single-agent bandit lower bound (Cesa-Bianchi and Lugosi, 2006). Let $\mathcal{A}$ be a deterministic (multi-agent) algorithm, and let the empirical distribution of arm pulls across all agents be given by $p^{i}(t)=\left(p_{1}^{i}(t), \ldots, p_{K}^{i}(t)\right)$, where $p_{k}(t)=\frac{n_{i}^{k}(T)}{T}$. Consider the random variable $J_{t}^{i}$ drawn according to $p^{i}(t)$ and $\mathrm{P}_{i}$ denote the law of $J_{t}$ when drawn from arm $k$ having parameter $\frac{1+\varepsilon}{2}$ (and other arms with parameter $\frac{1-\varepsilon}{2}$ ). We have,

$$
\mathrm{P}_{k}\left(J_{t}^{i}=j\right)=\mathbb{E}_{k}\left[\frac{n_{i}^{k}(T)}{T}\right] .
$$

Since on pulling any arm $k^{\prime} \neq k$, we obtain regret $\varepsilon$, we therefore have for the group regret,

$$
\begin{aligned}
\mathbb{E}_{k}\left[\sum_{t=1}^{T}\left(N \cdot r_{k}(t)-\sum_{i \in \mathcal{V}} r_{A_{i}}(t)\right)\right] & =\varepsilon \cdot T \cdot \sum_{i \in \mathcal{V}} \mathrm{P}_{k}\left(J_{t}^{i}=k^{\prime}\right) \\
& =\varepsilon \cdot T \cdot \sum_{i \in \mathcal{V}}\left(1-\sum_{k^{\prime} \neq k} \mathrm{P}_{k}\left(J_{t}^{i}=k^{\prime}\right)\right) .
\end{aligned}
$$

By Pinsker's inequality and averaging over all $k \in[K]$, we have for any $i \in \mathcal{V}$,

$$
\frac{1}{K} \sum_{k=1}^{K} \mathrm{P}_{k}\left(J_{t}^{i}=k\right) \leqslant \frac{1}{K}+\frac{1}{K} \sum_{k=1}^{K} \sqrt{\frac{1}{2} \mathrm{KL}\left(\mathrm{P}_{0}, \mathrm{P}_{k}\right)}
$$

We now bound the R.H.S. using the chain rule for KL-divergence. Since we assume that $\mathcal{A}$ is deterministic, we have that the rewards obtained by the agent $i$ until time $t$ from its neighborhood alone determine uniquely the empirical distribution of plays. Here, the analysis diverges from that of the single-agent bandit as a richer set of observations is available to each agent. Denote the set of rewards observed by agent $i$ at instant $\tau$ be given by $\mathcal{O}_{i}(\tau)$. First, observe that since each reward is i.i.d., we have for any $k$,

$$
\mathrm{KL}\left(\mathrm{P}_{0}\left(\mathcal{O}_{i}(\tau)\right), \mathrm{P}_{k}\left(\mathcal{O}_{i}(\tau)\right)\right)=\left|\mathcal{O}_{i}(\tau)\right| \cdot \mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right)
$$

For $k=0$ the above divergence is 0 . When we consider the standard single-agent setting, $\left|\mathcal{O}_{i}(\tau)\right|=$ 1 , recovering the usual bound. Now, by the chain rule, we have that, at round $t$ for any agent $i$, and arm $k \in[K]$,

$$
\begin{aligned}
\mathrm{KL}\left(\mathrm{P}_{0}(t), \mathrm{P}_{k}(t)\right) & =\mathrm{KL}\left(\mathrm{P}_{0}(1), \mathrm{P}_{k}(1)\right)+\sum_{\tau=2}^{t}\left|\mathcal{O}_{i}(\tau)\right| \mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) \\
& =\mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) \mathbb{E}_{0}\left[\sum_{j \in \mathcal{V}} n_{j}^{k}(t-d(i, j))\right] .
\end{aligned}
$$

Replacing this result in the earlier equation, we have by the concavity of KL divergence:

$$
\begin{aligned}
\frac{1}{K} \sum_{k=1}^{K} \mathrm{P}_{k}\left(J_{t}^{i}=k\right) & \leqslant \frac{1}{K}+\frac{1}{K} \sum_{k=1}^{K} \sqrt{\frac{1}{2} \mathrm{KL}\left(\mathrm{P}_{0}, \mathrm{P}_{k}\right)} \\
& \leqslant \frac{1}{K}+\frac{1}{K} \sum_{k=1}^{K} \sqrt{\mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) \mathbb{E}_{0}\left[\sum_{j \in \mathcal{V}} n_{j}^{k}(T-d(i, j))\right]} \\
& \leqslant \frac{1}{K}+\sqrt{\left(\frac{T N-\sum_{j=1}^{d^{\star}(G)} d_{=j}(i) \cdot j}{K}\right) \cdot \mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) .}
\end{aligned}
$$

Now, observe that the KL divergence between Bernoulli bandits can be bounded as

$$
\mathrm{KL}(p, q) \leq \frac{(p-q)^{2}}{q(1-q)} .
$$

Substituting we get,

$$
\frac{1}{K} \sum_{k=1}^{K} \mathrm{P}_{k}\left(J_{t}^{i}=k\right) \leqslant \frac{1}{K}+\sqrt{\frac{4 \varepsilon^{2}\left(N T-\sum_{j=1}^{d^{\star}(G)} d_{=j}(i) \cdot j\right)}{\left(1-\varepsilon^{2}\right) K}} .
$$

Replacing this in the regret and using $\varepsilon \leqslant 1 / 2$, we get that,

$$
\begin{aligned}
& \mathbb{E}_{k}\left[\sum_{t=1}^{T}\left(N \cdot r_{k}(t)-\sum_{i \in \mathcal{V}} r_{A_{i}}(t)\right)\right] \\
& \geqslant \varepsilon \cdot T \cdot \sum_{i \in \mathcal{V}}\left(1-\frac{1}{K}-\sqrt{\frac{4 \varepsilon^{2}\left(N T-\sum_{j=1}^{d^{\star}(G)} d_{=j}(i) \cdot j\right)}{\left(1-\varepsilon^{2}\right) K}}\right) \\
& \geqslant \varepsilon \cdot T \cdot \sum_{i \in \mathcal{V}}\left(\frac{1}{2}-4 \varepsilon \sqrt{\frac{\left(N T-\sum_{j=1}^{d^{\star}(G)} d_{=j}(i) \cdot j\right)}{3 K}}\right) \\
& =\frac{\varepsilon \cdot N T}{2}-\frac{4 \varepsilon^{2} N T}{\sqrt{K}}\left(\sum_{i, j \in \mathcal{V}} T-d(i, j)\right)^{1 / 2}
\end{aligned}
$$

Setting $\varepsilon=c \cdot \sqrt{\frac{K}{N\left(T-\sum_{j=1}^{d^{*}(G)} \bar{d}_{=j} \cdot j\right)}}$ where $c$ is a constant to be tuned later, we have,

$$
\begin{aligned}
\mathbb{E}_{k}\left[\sum_{\tau=1}^{T}\left(N \cdot r_{k, t}-\sum_{i \in \mathcal{V}} r_{A_{i}(t), t}\right)\right] & \geqslant\left(\frac{c}{2}-\frac{4 c^{2}}{\sqrt{3}}\right) \cdot \sqrt{\frac{K N^{2} T^{2}}{N\left(T-\sum_{j=1}^{d^{\star}(G)} \bar{d}_{=j} \cdot j\right)}} \\
& \geqslant 0.027 \sqrt{K N\left(T+\sum_{j=1}^{d^{\star}(G)} \bar{d}_{=j} \cdot j\right) .}
\end{aligned}
$$

This proves the first part of the theorem. Now, when the policies are decentralized and agnostic, the chain rule step can be factored as follows.

$$
\begin{aligned}
\mathrm{KL}\left(\mathrm{P}_{0}(t), \mathrm{P}_{k}(t)\right) & =\mathrm{KL}\left(\mathrm{P}_{0}(1), \mathrm{P}_{k}(1)\right)+\sum_{\tau=2}^{t}\left|\mathcal{O}_{i}(\tau)\right| \mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) \\
& =\mathrm{KL}\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) \mathbb{E}_{0}\left[\sum_{j \in \mathcal{N}_{\gamma}^{+}(G)} n_{j}^{k}(t-d(i, j))\right]
\end{aligned}
$$

Note that here instead of taking the cumulative sum over all $\mathcal{V}$ we select only those agents that are within the $\gamma$-neighborhood of $i$ in $G$, since conditioned on these observations the rewards of the agents are independent of all other rewards (by Assumption), and hence the higher-order KL divergence terms are 0 . Replacing this in the analysis gives us the following decomposition (after similar steps as the first part):

$$
\begin{aligned}
\mathbb{E}_{k}\left[\sum_{t=1}^{T}\left(N r_{k}(t)-\sum_{i \in \mathcal{V}} r_{A_{i}}(t)\right)\right] & \geqslant \frac{N T \varepsilon}{2}-\frac{4 \varepsilon^{2} T}{\sqrt{3 K}} \cdot \sum_{i \in \mathcal{V}}\left(\sum_{j: \mathcal{N}_{\gamma}^{+}(i)} T-d(i, j)\right)^{1 / 2} \\
& \geqslant \frac{N T \varepsilon}{2}-\frac{4 \varepsilon^{2} N^{1 / 2} T}{\sqrt{3 K}} \cdot\left(\sum_{i \in \mathcal{V}} \sum_{j: \mathcal{N}_{\gamma}^{+}(i)} T-d(i, j)\right)^{1 / 2}
\end{aligned}
$$

Setting $\varepsilon=c \cdot \sqrt{\frac{N K}{\sum_{i \in \mathcal{V}} \sum_{j: N_{\gamma}^{+}(i)}^{T-d(i, j)}}}$ where $c$ is a constant to be tuned later, we have,

$$
\begin{aligned}
\mathbb{E}_{k}\left[\sum_{t=1}^{T}\left(N \cdot r_{k}(t)-\sum_{i \in \mathcal{V}} r_{A_{i}}(t)\right)\right] & \geqslant\left(\frac{c}{2}-\frac{4 c^{2}}{\sqrt{3}}\right) \cdot \sqrt{\frac{N^{3} T^{2}}{\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_{i}^{+}\left(G_{\gamma}\right)} T-d(i, j)}} \\
& \geqslant\left(\frac{c}{2}-\frac{4 c^{2}}{\sqrt{3}}\right) \cdot \sqrt{\frac{N^{3} T}{\sum_{i \in \mathcal{V}} 1+d_{i}\left(G_{\gamma}\right)}} \\
& \geqslant \frac{3}{4}\left(\frac{c}{2}-\frac{4 c^{2}}{\sqrt{3}}\right) \sqrt{\alpha^{\star}\left(G_{\gamma}\right) N T} \\
& \geqslant 0.019 \sqrt{\alpha^{\star}\left(G_{\gamma}\right) N T} .
\end{aligned}
$$

The constants in both settings are obtained by optimizing $c$ over $\mathbb{R}$. Extending this to random (instead of deterministic) algorithms is straightforward via Fubini's theorem, see Theorem 2.6 of Bubeck (2010).

## G Pseudo code

```
Algorithm 2: RCL-LF
    Input: Arms \(k \in[K]\), variance proxy upper bound \(\sigma^{2}\), parameter \(\xi\)
    Initialize: \(N_{k}^{i}(0)=\widehat{\mu}_{k}^{i}(0)=C_{k}^{i}(0)=0, \forall k, i\)
    for each iteration \(t \in[T]\) do
        for each agent \(i \in[N]\) do
            /* Sampling phase */
            if \(t=1\) then
                    \(A_{t}^{i} \leftarrow\) RandomARm \(([K])\)
            end
            else
            \(A_{t}^{i} \leftarrow \arg \max _{k} \widehat{\mu}_{k}^{i}(t-1)+C_{k}^{i}(t-1)\)
            end
            /* Send messages */
            \(\operatorname{Create}\left(\mathbf{m}_{t}^{i}:=\left\langle A_{t}^{i}, r_{t}^{i}, i, t\right\rangle\right)\)
            \(\operatorname{SEnd}\left(\mathbf{M}_{t}^{i} \leftarrow \mathbf{M}_{t-1}^{i} \cup \mathbf{m}_{t}^{i}\right)\)
        end
        for each agent \(i \in[N]\) do
            /* Receive messages */
            for each neighbor \(j \in \mathcal{N}_{i}\left(G_{\gamma}\right)\) do
                /* Discard messages with probability \(1-p_{i}\) */
                    for each message \(\mathbf{m} \in \mathbf{M}_{t}^{j}\) do
                    with probability \(p_{i}, \quad \mathbf{M}_{t}^{i} \leftarrow \mathbf{M}_{t}^{i} \cup \mathbf{m}\)
                    with probability \(1-p_{i}, \quad \mathbf{M}_{t}^{i} \leftarrow \mathbf{M}_{t}^{i}\)
            end
            end
            /* Update estimates */
            for each arm \(k \in[K]\) do
            Calculate \(\left(N_{k}^{i}(t), \widehat{\mu}_{k}^{i}(t), C_{k}^{i}(t)\right)\)
            end
        end
    end
```


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[^1]:    ${ }^{1}$ The $\widetilde{\mathcal{O}}$ notation ignores absolute constants and $\log \log (\cdot)$ factors in $T$.

