Provably Efficient Multi-Agent Reinforcement Learning with Fully Decentralized Communication

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Abstract—A challenge in reinforcement learning (RL) is minimizing the cost of sampling associated with exploration. Distributed exploration reduces sampling complexity in multi-agent RL (MARL). We investigate the benefits to performance in MARL when exploration is fully decentralized. Specifically, we consider a class of online, episodic, tabular $Q$-learning problems under time-varying reward and transition dynamics, in which agents can communicate in a decentralized manner. We show that group performance, as measured by the bound on regret, can be significantly improved through communication when each agent uses a decentralized message-passing protocol, even when limited to sending information up to its $\gamma$-hop neighbors. We prove regret and sample complexity bounds that depend on the number of agents, communication network structure and $\gamma$. We show that incorporating more agents and more information sharing into the group learning scheme speeds up convergence to the optimal policy. Numerical simulations illustrate our results and validate our theoretical claims.

I. INTRODUCTION

Multi-agent reinforcement learning (MARL) is a fast-growing field that extends many of the ideas of single-agent reinforcement learning to tasks involving multiple decision-making agents operating in a shared environment. Similar to the single-agent setting, agents try to maximize their cumulative reward through estimation of a value function, which tabulates the expected cumulative return starting at a given state. In the model-free paradigm, the optimal value function is estimated without a model of the system dynamics. Typically what sets MARL systems apart from a group of single-agent RL decision makers is the consideration of the joint action space; that is, agents must coordinate their efforts in order to converge on the optimal joint policy. We measure this convergence in two ways: sample complexity, which bounds the number of reward samples needed to find an approximately optimal value function, and regret, which bounds the cumulative value function error over time. We denote an algorithm as sample efficient if it has near-optimal sample complexity.

In training MARL algorithms, allowing agents to share value function parameters or reward samples can lead to faster convergence, but it still remains an open question whether this information sharing should be centralized or decentralized. There has been empirical evidence [1], [2], [3] that training using a central controller to aggregate joint states and actions is effective even when the state and action spaces are large. However, as the number of agents increases, the number of joint states and actions becomes exponentially large, demanding more storage space to tabulate scenarios, increasing the space complexity. Due to the distributed nature of real-world MARL applications and the combinatorial complexity of MARL, there has been increased interest in developing theory for decentralized model-free algorithms that allow agents to train and to perform optimally with space complexity remaining polynomial in the number of agents [4]–[6], particularly when the problem of optimizing the joint action factors as optimizing the individual agent actions.

A key underlying trade-off in RL is known as exploration versus exploitation; an efficient exploration strategy is always necessary to discover new scenarios while capitalizing on experience from prior scenarios. In the episodic model-free $Q$-learning literature [7], [8], choosing a time-varying, Hoeffding-style exploration bonus leads to $\tilde{O}(\sqrt{HT\text{SAT}})$ regret. Here, $S$ is the number of states, $A$ is the number of actions, $H$ is the number of steps per episode, and $T$ is the total number of reward samples. The exploration bonus is chosen to match the value function error up to a constant factor.

Multi-agent reinforcement learning changes the landscape of exploration because multiple agents interact with a shared environment simultaneously and share information. A naive application of the single-agent result running in parallel gives $\tilde{O}(M\sqrt{HT\text{SAT}})$ regret, where $M$ is the number of agents. In this paper, we show how tabular $Q$-learning can give $\tilde{O}(\sqrt{MH^4\text{SAT}})$ regret by providing an optimal exploration strategy that considers communication among the agents to accelerate online learning.

Our results apply to tasks in which each agent in a network learns an optimal value function under the episodic Markov decision process (MDP) paradigm. We focus on the scenario where agents interact with their respective MDP in parallel, and there is no coupling between joint actions and the joint reward. Agents are allowed to perform one action and one round of message passing per time step. To this end, we investigate the principle of exploration by proxy. Specifically, we ask the question: if agents operate in similar environments, can they share samples in such a way that enables discovery of the optimal policy faster than individual agents operating in parallel? To this end, we propose an algorithm that permits full decentralization of the learning process in which agents explore proportional to the amount of information they receive from other agents. Agents use a message-passing communication scheme where an agent can

$\tilde{O}$ ignores log terms.
send and receive information up to its $\gamma$-hop neighbors.

The contributions of this paper are as follows: (1) we provide a novel multi-agent Q-learning algorithm in which agents use message passing to share information, and (2) we show that group performance, as measured by the bound on regret, improves upon the Hoeffding-style exploration strategy in the single agent setting by a factor of $\sqrt{1/M}$. Moreover, our regret takes into consideration the network structure and communication threshold $\gamma$, suggesting that even mild communication leads to improvement in the regret. As far as we know, this paper provides the first multi-agent regret bound for decentralized tabular Q-learning for a general network.

The paper is organized as follows. Section II discusses related work, including multi-agent multi-armed bandits, online reinforcement learning, and decentralized reinforcement learning. Section III introduces mathematical preliminaries. Section IV details the main contribution of the paper, an algorithm and regret bound for multi-agent tabular Q-learning where agents communicate in real time equipped with a message passing protocol. Section V explores simulation results. We conclude in Section VI.

II. RELATED WORK

Multi-Agent Multi-Armed Bandits. In multi-armed bandits a single agent repeatedly takes an action from a fixed set of actions, i.e., a fixed state, and receives a numerical reward associated with the taken action with the goal of maximizing the cumulative reward. Multi-agent bandit problems, in which a group of agents are faced with the same problem simultaneously, have received increased attention from both academia and industry due to their capacity to handle decentralized optimization with communication. A number of algorithms have been proposed to improve individual and group performance by leveraging shared information [9]. Consensus-based approaches [10] and imitation-based approaches [11] suffer from having to share a large amount of data or requiring a synchronization phase. Message passing [12], [13] has been proposed as a promising approach that overcomes these shortcomings. Our work generalizes this approach to stochastic, time-varying state transitions by extending message passing to improve group performance in MARL.

Episodic Q-learning. Online Q-learning has become a popular approach when agents do not have access to a generator/simulator. Jin et al. [7] provide both $O(\sqrt{H^{3}SAT})$ and $O(\sqrt{H^{3}SAT})$ theoretical regret bounds for episodic Q-learning under time-varying dynamics, i.e. state transitions and reward structures. The extra $\sqrt{H}$ factor can be attributed to accurate estimation of the moments of the empirical value function. The authors of [8] improve this regret bound to $O(\sqrt{H^{3}SAT})$, which is proved in [7], [14] to be minimax-optimal in the single-agent case.

Fully Decentralized Multi-Agent Reinforcement Learning. Theoretical analysis of decentralized reinforcement learning is still a new and growing field. Several works provide a decentralized actor-critic approach to convergence guarantees in the cooperative setting [15], [16], although no finite-time sample complexity results are provided. The authors of [6] provide $O(1/\varepsilon^2)$ sample complexity analysis for cooperative actor-critic under a general utility function by empirically estimating the state-action occupancy measures; our method differs by being value-based only and does not require differentiating the cumulative reward with respect to the occupancy measures. The paper [5] provides a finite-sample PAC bound in the cooperative and competitive batch (i.e. not online) settings. In contrast, the papers [17] and [18] provide a finite sample analysis for MARL in zero-sum Markov games, but emphasize only the competitive setting. The paper [19] considers decentralized Q-learning, but also for competitive MARL only. For continuous state and action spaces, [20] provides a decentralized multi-agent regret bound for linear-quadratic systems for unknown dynamics and a one-directional communication from the agent controlling the unknown system to the other agents, suggesting that ideas from discrete Q-learning algorithms (such as ours) can be readily extended to continuous domains. Paper [4] provides multi-agent regret bounds for cooperative RL for parallel MDPs (see Section III). A linear function approximation and a central server are used to perform least-squares value iteration with shared transition samples. Our work complements these results by providing the first decentralized multi-agent regret bound for tabular Q-learning, with message passing eliminating the need for a central server. In the sequel, we show further that our bound matches centralized benchmarks.

III. MATHEMATICAL PRELIMINARIES

A. Single-Agent Online Q-Learning

Consider an MDP described by tuple $(S, A, P, R, \gamma_d)$, where $S$ is a finite state space, $A$ is a finite action space, $P$ is the transition matrix, $R$ is the reward, and $\gamma_d$ is a discounting factor. We take $\gamma_d = 1$ since the horizon $H$ is finite. The goal is to maximize the cumulative reward over $K$ episodes of length $H$. We consider time-varying state transition dynamics, such that $P_h(x'|x,a)$ denotes the probability of transitioning to state $x'$ from state $x$ by taking action $a$ at step $h \in H$. In the online learning setting, we consider the transitions $P_h(x'|x,a)$ to be unknown in advance; this is significantly more complex compared to some of the offline algorithms that are able to simulate transitions from all state action pairs in a batch update. Rewards $r_h(x,a)$ can vary with time $h \in [H]$.

The expected discounted cumulative reward is controlled by the choice of policy, $\pi$, which we take to be a deterministic mapping from states to actions. The goal is to find an optimal policy $\pi^*_h$, at every step $h$. The expected return under a policy $\pi$ is the value function:

$$V^\pi_h(x) := \mathbb{E} \left[ \sum_{h'=h}^{H} r_{h'}(x_{h'}, \pi_{h'}(x_{h'})) \mid x_h = x \right].$$

In Q-learning, the action-value function is the expected cumulative reward conditioned also on initial action $a$:

$$Q^\pi_h(x,a) := r_h(x,a) + \mathbb{E}[P_h V^\pi_{h+1}(x,a)],$$

where $P_h(x'|x,a)$ is the probability of transitioning to state $x'$ from state $x$ by taking action $a$ at step $h$. The value function $V^\pi_h(x)$ is the expected cumulative reward from time $h$ onward under policy $\pi$, starting from state $x$. The action-value function $Q^\pi_h(x,a)$ is the expected cumulative reward from time $h$ onward under policy $\pi$, starting from state $x$ and taking action $a$.
where \([P_h V_{h+1}](x, a) := E_{x' \sim P(\cdot|x,a)}[V_{h+1}(x')]\) denotes the conditional one-step expectation for any \(V\), and \([P_h^k V_{h+1}](x, a) := V_{h+1}(x_{h+1}^k)\) is a stochastic approximation to \([P_h V_{h+1}](x, a)\) at episode \(k\). The optimal value-function is \(V^*(x) := V^*(x) = \sup_{\pi} V^\pi(x)\) for all \(x \in S\), i.e., the argmax over expected returns from one-step transitions.

This formulation allows for a dynamic programming approach to solve iteratively for the optimal value function with standard Q-learning update as follows:

\[
Q_h(x, a) \leftarrow (1 - \alpha_t)Q_h(x, a) + \alpha_t [r_h(x, a) + V_{h+1}(x') + b_t],
\]

where \(\alpha_t\) is the learning rate and \(b_t\) is an exploration bonus proportional to \(t\), the number of times a state-action pair has been visited. In the online setting, each state-action pair is visited once per step since we assume no access to a simulator. As a result, the Q-table is updated highly asynchronously, and careful weighting of the learning rate \(\alpha_t\) and exploration parameter \(b_t\) are required to ensure that samples are weighted optimally. For more information about how \(\alpha_t\) and \(b_t\) are chosen, see Jin et al. [7]. To measure convergence of the iterates, let the regret be

\[
\text{Regret}(K) = \sum_{k=1}^{K} [V^*_h(x'_h^k) - V^n_{h+k}(x'_h^k)].
\]

Intuitively, the regret is the cumulative value error measured at the initial state of each episode, using the value function for step 1 of \(H\). Jin et al. provide an \(O(\sqrt{T})\) bound in the single agent case, i.e., the number of samples required to achieve an \(\epsilon\)-optimal policy is of order \(1/\epsilon^2\).

**B. Cooperative Multi-Agent Reinforcement Learning**

We consider parallel MDPs [4], [21], which are defined as a collection \({\text{MDP}}(S_i, A_i, P_i, r_i, \gamma_i)\}_{i=1}^{M}\), where each agent \(i \in [M]\) has access to identical state space \(S_i = S_j\) and action space \(A_i = A_j\), \(\forall i, j \in [M]\). The joint state space is \(S = S_1 \times S_2 \times \cdots \times S_M\). The joint action space is similarly \(A = A_1 \times A_2 \times \cdots \times A_M\). Again we take \(\gamma_d = 1\). The local reward functions \(r_i\) and transitions \(P_i\) can be different but only depend on the local state and action information of each agent. Thus, each agent interacts in parallel with their corresponding MDP, and there is no coupling between the state and actions chosen by an agent and the reward received by any other agent. Here we let \(r = r_1 = \cdots = r_M\) and \(P = P_1 = \cdots = P_M\), such that every agent interacts with the same MDP!

The paper [4] provides a multi-agent regret bound under heterogeneous reward and transition structure using estimation of the feature covariance and bias. In contrast, our framework permits direct sharing of transition-reward tuples where the Q-function can be tabulated directly. We let the \(M\) agents communicate over a network \(G\) with edge set \(E\). In the episodic setting, each agent \(m \in [M]\) gets a potentially time-varying local reward \(r_h(x_{m, h}, a_{m, h}^k)\) for the state-action pair it chooses at step \(h\) of episode \(k\). In the Parallel MDP construction, we assume that average total reward and transition probability are decoupled:

\[
r_{tot}(x, a) := \frac{1}{M} \sum_{i=1}^{M} r_i h(x_i, a_i, h)\]
\[
P(x'|x, a) := \prod_{i=1}^{M} P_i(x_i'|x_i, a_i).
\]

The objective is then to maximize the global (average total) reward through optimization of local (individual agent) rewards. The cumulative value error is measured by the group regret:

\[
\text{Regret}_G(K) := \sum_{m=1}^{M} \text{Regret}_m(K)
\]

Here, we show that the local optimizations are provably accelerated through communication.

**IV. MULTI AGENT RL WITH FULL COMMUNICATION**

**A. Algorithm**

We consider a multi-agent extension to the online Q-learning algorithm provided by [7]. Consider \(M\) agents operating in a parallel MDP for \(K\) episodes of length \(H\). In the single-agent case, each agent makes one update corresponding to the state-action pair \((x^k_{m, h}, a^k_{m, h})\) visited at each step \(h\) of episode \(k\), for a total of \(T = HK\) samples. In the multi-agent case, at each time \(\tau\), each agent \(m \in [M]\) sees a state-action pair \((x^k_{m, h}, a^k_{m, h})\), reward \(r_h(x^k_{m, h}, a^k_{m, h})\), and next state \(x^k_{m, h+1}\) and exchanges messages with its neighbors. We consider a message-passing protocol in which each agent \(m \in [M]\) sends to its neighbors a message \(m^{k}_{\tau} := (h, k, m, x^k_{m, h}, a^k_{m, h}, x^k_{m, h+1}, r^k_{h})\) containing step, episode, agent id, current state, current action, next state and current reward. Each neighbor then forwards the message to its neighbors. All messages older than \(\gamma\) are excluded. Here \(0 \leq \gamma \leq D(G)\), where \(D(\cdot)\) is graph diameter. The message-passing protocol allows each agent to send information up to \(\gamma\)-hop neighbors. For \(\gamma = 0\), we recover \(M\) copies of the single-agent case with no communication and group regret \(O(M\sqrt{T})\).

We make several definitions for the Q-learning update.

**Definition 1.** (Number of state-action visitations)

\[
h^k_{m, h}(x, a) := \sum_{\ell=1}^{k} 1[x^\ell_{m, h} = x, a^\ell_{m, h} = a]
\]

**Definition 2.** (Number of state-action observations)

\[
h^k_{m, h}(x, a) := \sum_{j=1}^{M} \sum_{\ell=1}^{k} 1[x^\ell_{j, h} = x, a^\ell_{j, h} = a] I[(m, j) \in E]
\]

Consider the power graph \(G_x\) of \(G\): nodes \(m\) and \(m'\) share an edge if and only if \(d(m, m') \leq \gamma\). Let \(G_x(m)\) be
the neighbors of $m$ in $G_\gamma$. Define set $\mathcal{V}^k_{m,h}(x,a) = \mathcal{V}$,  
\[  \mathcal{V} = \left\{ \bigcup_{i \in G_\gamma(m)} (r_{i,k-1},r_{i,k-1}+1) \right\} \ni k : N^k_{i,k}(x,a) > 0 \]

.o.w.

to be the set of all new reward and next-state tuples available to each agent $m$ for state-action pair $(x,a)$ for any agent across the network at step $h$ and episode $k$. Define $U^k_{m,h}$ to be the set of reward and next-state tuples taken by and observed by agent $m$. Clearly, $U^k_{m,h}(x,a) = \{(r_{m,k-1},r_{m,k-1}+1)\}$. The update rule for $Q^k_{m,h}(x,a)$ is  
\[  Q^k_{m,h} = U^k_{m,h} \cup \left\{ \bigcup_{m' \in G} \mathcal{V}^{k'}_{m',d(m,m')+1} \right\}  
\]

where $\mathcal{V}^{k}_{m,h}(x,a) = \emptyset$ if $N^{k}_{m,h}(x,a) = 0$ and $h < 0$.

Assume for fixed $(x,a,h,k)$, $Q^k_{m,h}(x,a) > 0$. Let $t_m = N^{k}_{m,h}(x,a)$. The Q-learning update is  
\[  Q^k_{m,h}(x,a) := \sum_{(r',x) \in \mathcal{V}^{k}_{m,h}(x,a)} (1-\alpha_m)Q^k_{m,h}(x,a) \]
\[  + \alpha_m t_m r + V^{k+1}_{m,h+1}(x') + b_{m,t} \]

The online multi-agent Q-learning algorithm with Hoeffding-style upper confidence bound is provided in Algorithm 1 where $C(m)$ denotes the clique size of agent $m$ in the clique cover of $G_\gamma$.

Let $t_m = N^{k}_{m,h}(x,a)$ and suppose $(x,a)$ was previously taken at step $h$ of episodes $k_1, \ldots, k_{t_m} < k$, for each agent $m \in [M]$. Let $t = \max_{m \in [M]} t_m$ and $k_1, \ldots, k_t < k$ denote the episodes where $(x,a)$ was visited for any $m \in [M]$. Let $N^{k}_{m,h}(x,a) := \sum_{m \in [M]} N^{k}_{m,h}(x,a)$ represent the total number of times $(x,a)$ is seen by all agents. This makes the episode-wise $Q$-function determined as:
\[  Q^{k}_{m,h}(x,a) := \alpha^0_{m,t}H \]
\[  + \sum_{(r',x) \in \mathcal{V}^{k}_{m,h}(x,a)} \alpha^i_{m,t} r + V^k_{m,h+1}(x') + b_{m,t} \]

As in [7], the Q-learning update is highly asynchronous. Thus, the regret bound depends on optimal choice of $\alpha_m$, $b_{m,t}$, and the clique structure under $G_\gamma$.

**B. Convergence proof**

Let $(x,a)$ be a state-action pair, $m$ an arbitrary agent (estimating the value function and Q-function) and $(h,k)$ a step and episode index, respectively.

**Definition 3. (Estimated policy performance gap)**
\[  \phi^{k}_{m,h} := (V^k_{m,h} - V^*_{m,h})(x^k_{m,h}) \]

**Definition 4. (Value estimation error)**
\[  \phi^{k}_{m,h} := (V^k_{m,h} - V^*_{m,h})(x^k_{m,h}) \]

**Definition 5. (Value gap due to modeling error)**
\[  \xi^{k}_{m,h} := (\mathbb{P}_h - \hat{\mathbb{P}}_h)(V^*_{m,h+1} - V^*_{m,h+1})(x^k_{m,h},a^k_{m,h}) \]

Note that $\xi^{k}_{m,h}$ is a martingale difference sequence.

**Assumption 1. (Episodic length bounds message life). Assume $0 \leq \gamma \leq \min(D(G), H)$.

Lemmas 1 and 2 are reproduced from [7].

**Lemma 1. (Recursion on $(Q)$)** For any $(m,x,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]$, episode $k \in [K]$, let $t = N^k_{m,h}(x,a)$, and $(x,a)$ be observed by agent $m$ at episodes $k_1, \ldots, k_t < k$. Then
\[  (Q^k_{m,h} - Q^*_{m,h})(x,a) = \alpha^0_{m,t}H - Q^k_{m,h}(x,a) \]
\[  + \sum_{i=1}^{t} \alpha^i_{m,t}(V^k_{m,h+1} - V^*_{m,h+1})(x^k_{m,h}) \]
\[  + \sum_{i=1}^{t} (\mathbb{E}^k_{m,h} - \mathbb{P}^k_{m,h}V^*_{m,h+1})(x^k_{m,h},a^k_{m,h}) + b_{m,t}. \]

**Lemma 3. (Bound on $\xi^{k}_{m,h}$ accumulation)**
\[  \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{h=1}^{H} \xi^{k}_{m,h} \leq \sqrt{2H^3MT\log(2/\rho)} \]

Proof. (Based on Dubey & Pentland [4]). We know from Def. 4 that $\xi^{k}_{m,h}$ represents a martingale difference sequence. A straightforward application of the Azuma-Hoeffding inequality gives
\[  \mathbb{P} \left( \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{h=1}^{H} \xi^{k}_{m,h} > \varepsilon \right) \leq \exp \left( -\frac{\varepsilon^2}{2MH^3K} \right). \]

Solving for $\varepsilon$ and bounding the summation completes the proof.

**Lemma 4. (Clique bound on $Q$-error)** Let $C_{\gamma}$ be a partition of $G_\gamma$. If for every clique $C \in C_{\gamma}$, we assign exploration bonus $b_{m,t} = b_{C,t} = c\sqrt{H^3/(\omega C)}$, where $m \in C$, we have bounded error rate $e_{m,t} = \epsilon_{\omega C,t} = 2\sum_{m \in C} \sum_{i=1}^{t} \alpha^i_{C,t} b_i \leq 4c\sqrt{C(\omega H^3)/t}$, where $t_m = N^k_{m,h}(x,a)$ is the number of times $(x,a)$ has been seen by step $h$ in episode $k$ by agent $m$. Further, there exists an absolute constant $c > 0$ such that, for any $p \in (0,1)$, with probability at least $1 - p$, we have simultaneously for all $(x,a,h,k,m) \in \mathcal{S} \times \mathcal{A} \times [H] \times [K] \times [M]$:
\[  0 \leq \sum_{m \in C} (Q^k_{m,h} - Q^*_{m,h})(x,a) \leq |C|\alpha^0_{\omega C,H} \]
\[  + \sum_{m \in C} \sum_{t=1}^{t_m} (V^k_{m,h+1} - V^*_{m,h+1})(x^k_{m,h+1}) \leq e_{\omega C,t}. \]

Proof. First, consider a clique $C \in C_{\gamma}$. Assume $\gamma > 0$. Fix $(x,a) \in S \times A \times [H]$. Let $k_t$ be the episode where $(x,a)$
is seen for the $i$th time (by any agent), and $k_i = K + 1$ if $(x, a)$ hasn’t been seen for the $i$th time yet. Thus, $k_i$ is a random variable and a stopping time. Next, let $\mathcal{I}_i$ where

$\mathcal{I}_i = \{k_i \leq K\}$. For any collection of $|\mathcal{I}_i| = K$. Let $\mathcal{I}_G \subset \mathcal{I}_i$ for an agent’s martingale is connected by $\mathcal{I}_G$. Let $I_{G/C} = |\mathcal{I}_G|/I_{C}$, i.e. $|I_{G/C}| = |\mathcal{I}_G| - |\mathcal{I}_C|$. By the Azuma-Hoeffding inequality, for fixed $(x, a, h)$, we have for any collection of fixed data-sets $I_{\mathcal{C}}$ of size $\tau \in [0, MK]\forall m$, with probability $1 - p/(MSAH)$:

$$\sum_{m \in \mathcal{C}} \sum_{i=1}^{\tau} \alpha_{m, \tau, m} \cdot X_{m,i} \leq \sum_{m \in \mathcal{C}} \sum_{i=1}^{\tau} \alpha_{m, \tau, m} \cdot X_{m,i} + \sum_{m \in \mathcal{C}} \sum_{i=1}^{\tau} \alpha_{m, \tau, m} \cdot X_{m,i} \leq \sum_{m \in \mathcal{C}} \sum_{i=1}^{\tau} \alpha_{m, \tau, m} \cdot X_{m,i} + \sum_{m \in \mathcal{C}} \sum_{i=1}^{\tau} \alpha_{m, \tau, m} \cdot X_{m,i} \leq \sqrt{\frac{|\mathcal{C}|H^3t}{\tau C}} = \sqrt{\frac{|\mathcal{C}|H^3t}{\tau C}}$$

where $t = p/(MSAH)$. This follows since $\{\alpha_{m, \tau, m}\}_{i=1}^{M}$ is dominated pointwise by $\{\alpha_{m, \tau, m}\}_{i=1}^{N}$. Since $\tau \in [0, |\mathcal{C}| K]$, the above bound also holds for $\tau_C = t := N_{m,h}(x, a)$, which is a random variable. Since $\{k_i \leq K\}$ for any $i \leq N_{m,h}(x, a)$, we have that with probability $1 - p$, for all $(m, a, h) \in |\mathcal{M}|$, with $\sum_{m \in \mathcal{C}} \sum_{i=1}^{\tau} \alpha_{m, \tau, m} \cdot X_{m,i} \leq \sqrt{\frac{|\mathcal{C}|H^3t}{\tau C}}$. Thus, picking exploration bonus $b_{C,t} = c\sqrt{\frac{H^3t}{\tau C}}$.

$$\sum_{m \in \mathcal{C}} (Q_{m,h}^k - Q_{m,h}^*) \leq 2\sqrt{\frac{|\mathcal{C}|H^3t}{\tau C}}, \forall m \in \mathcal{C}$$. If the clique structure is not known, we can bound $\epsilon_{m,t} \leq \sqrt{N(m)H^3t}/t \leq \epsilon_{C,t}$, where $N(m)$ represents neighbors of node $m$ in $G$.

**Remark 1.** This analysis allows exploration by proxy. That is, all agents in a clique $\mathcal{C}$ are able to explore proportional to the size of the clique, and share the resulting samples. If $d_{C}^2 \leq |\mathcal{C}|$, then $\tau - \tau = 0$. So assume $d_{C}^2 > |\mathcal{C}|$. We show that the number of samples used by clique $\mathcal{C}$ is approximately generated by $\mathcal{C}$, and the concentration is bounded by the left hand term.

**Lemma 5.** (Single agent cumulative value error). For fixed $h$, we have

$$\sum_{k=1}^{K} \delta_{m,k} \leq O\left(\sum_{k=1}^{K} (\epsilon_{m,k} + \epsilon_{m,k+1})\right)$$

**Proof.** See equation 4.8 of [7].

**Theorem 1.** (Hoeffding regret bound for parallel MDP with communication) There exists an absolute constant $c > 0$
such that, for any $p \in (0, 1)$, if we choose $b_{m,t} = b_{c,t} = c/\sqrt{H/S_{t}}/(|C|/t)$, then with probability $1 - p$, the group regret of multi-agent $Q$-learning with Algorithm 1 is at most $O(\sqrt{MH^2SAT})$, where $t := \log(SATM/p)$.

Proof. Expanding the group regret defined in section III we have

$$\sum_{m=1}^{M} \text{Regret}_m(K) = \sum_{m=1}^{M} \sum_{k=1}^{K} [V^*_{m,1}(x^k_t) - V^{\pi_{m,k}}_{m,1}(x^k_t)]$$

$$\leq \sum_{m=1}^{M} \sum_{k=1}^{K} k\sum_{i=1}^{K} \alpha_{m,i,t}^k + e_{m,t} + \phi_{m,h+1}^k + \xi_{m,h+1}^k$$

for $t := n_{m,h}(x^k_{m,h}, q^k_{m,h})$ and applying Lemma 3 see [7] for the complete performance gap is straightforward:

$$\sum_{m \in C} \delta_{m,h}^k \leq (V^k_h - V^{\pi_{m,k}}_{m,h})(x^k_{m,h}) + \frac{H}{t} \sum_{i=1}^{K} \alpha_{m,i,t}^k$$

$$\sum_{m \in C} \sum_{k=1}^{K} \delta_{m,h}^k \leq (V^k_h - V^{\pi_{m,k}}_{m,h})(x^k_{m,h}) + \frac{H}{t} \sum_{i=1}^{K} \alpha_{m,i,t}^k$$

for $t = N_{C,m}^k(x,a)$. Let $C_\gamma$ denote a clique covering of $G_\gamma$. From Lemma 3

$$\sum_{k=1}^{K} \delta_{m,h}^k \leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

$$\leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

$$\leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

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where $[1]$ comes from the fact that for any $(x,a,k,h)$, $N^{G/c,k}_{C,h}(x,a) \leq |k| |C| (d^*_C - |C|)$, where $d^*_C$ is the max degree in the clique. Expression [2] comes from Hölder’s Inequality, and where $d^{\text{eff}}_{G_\gamma} := 1/\sum_{C \in C_\gamma} (1/(d^*_C - |C|))$. In the worst case $N_{C,h}^k = |C|^2/k/SA$ and $\omega(G_\gamma) := \sup_{C \in C_\gamma} |C|$. Finally, the total multi-agent regret is bounded as

$$\sum_{m=1}^{M} \sum_{k=1}^{K} \delta_{m,h}^k \leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

$$\leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

$$\leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

$$\leq \sum_{m \in C_\gamma} \sum_{k=1}^{K} \delta_{m,h}^k$$

with probability $1 - p$. 

Remark 2. (On sample complexity) Note that Algorithm 1 achieves a high probability regret of $\sum_{m=1}^{M} \sum_{k=1}^{K} [V^*_{m,1}(x^k_{m,1}) - V^{\pi_{m,k}}_{m,1}(x^k_{m,1})] \leq O(\sqrt{MH^2SAT})$. Suppose that we add the assumption also that $x_t^{k} \leftarrow x_{nom}^{k}$ for some nominal initial state $x_{nom}^{k}$. Then, by dividing by $M$ and $K$, we see that for any agent $m$ and episode $k$, $V^*_{m,1}(x^k_{m,1}) - \frac{1}{MK} \sum_{m=1}^{M} \sum_{k=1}^{K} V^{\pi_{m,k}}_{m,1}(x^k_{m,1})$
policy is computed offline using cies to the optimal joint policy for networks that are trials.

Consider \( \rho \rightarrow \rho \rightarrow \text{Dir} \), i.e., \( P \rightarrow \text{Dir} \) fixed. We assume time-invariant dynamics, i.e., \( P = P = \ldots = P = P \). Results shown are the average of 5 trials.

We investigate the convergence rates of the learned policies to the optimal joint policy for networks that are \( r \)-ary trees for a fixed branching factor \( r = 3 \). The optimal joint policy is computed offline using \( T = 1000 \) iterations of \( \epsilon \)-greedy tabular Q-learning, with \( \epsilon = 0.2 \) and discount factor \( \gamma = 0.95 \). Then, we assume the converged Q-function holds for all agents simultaneously. In the following experiments, we run Algorithm 1 for \( K = 1000 \) episodes of \( H = 5 \) steps. At intervals of 10 episodes, we “roll out” the accumulated reward under the policy \( \pi_k \) implied by the Q-function at that episode (i.e., \( V^{\pi_k} \)), for a total of \( H \) steps. We initialize the rollout procedure to start at a fixed nominal initial state so that the maximum cumulative reward is identical across trajectories. We perform two experiments.

In the first experiment, the number of agents is held fixed at \( M = 13 \). We vary the message life parameter \( \gamma \) over the range \( \{0, 1, \ldots, D(G, \gamma) = 4\} \). The performance difference of the rolled-out value function versus rolled-out offline Q function is plotted in Figure 1. As expected, the performance deficit decreases (i.e., performance improves) as agents gain access to more samples. Under the message-passing scheme \( \gamma = 0 \), i.e., no communication among the agents, agents struggle to reduce the total average performance error over 1000 episodes. For larger \( \gamma \), the performance increase appears to plateau around \( \gamma = 2 \). We leave as an open question whether there is an optimal communication bandwidth for learning; see e.g. [4] for an investigation on this topic.

In the second experiment, we hold the message life fixed at \( \gamma = 2 \), representing communication between 2-hop neighbors. We then vary the number of agents \( M \) from 2 up to 10 in order to investigate the relationship between the number of agents and the rate of convergence to the optimal average group performance. As expected, performance improves as more agents are added to the optimization, although communication is limited by pairwise path length. When the number of agents is reduced, the dynamic programming scheme must be solved more sequentially rather than in parallel. Thus, exploration is more costly and leads to suboptimal trajectories.

V. Simulations

In this section we provide numerical simulations to illustrate results and validate theoretical bounds. For all simulations, we consider \( S = 10 \) states and \( A = 2 \) actions. Each agent operates on the same MDP (i.e. parallel MDPs), where the rewards are uniformly distributed in [0, 1]. For each state-action pair \( (x, a) \), the transition probabilities \( P(x, a) := \text{Dir}(\rho) \), the Dirichlet distribution with shape parameter \( \rho \). As \( \rho \rightarrow 0 \) transitions become deterministic. For simplicity, we consider \( \rho = 0.01 \) fixed. We assume time-invariant dynamics, i.e., \( P = P = \ldots = P = P \). Results shown are the average of 5 trials.

We investigate the convergence rates of the learned policies to the optimal joint policy for networks that are \( r \)-ary trees for a fixed branching factor \( r = 3 \). The optimal joint policy is computed offline using \( T = 1000 \) iterations of \( \epsilon \)-greedy tabular Q-learning, with \( \epsilon = 0.2 \) and discount factor \( \gamma = 0.95 \). Then, we assume the converged Q-function holds for all agents simultaneously. In the following experiments, we run Algorithm 1 for \( K = 1000 \) episodes of \( H = 5 \) steps. At intervals of 10 episodes, we “roll out” the accumulated reward under the policy \( \pi_k \) implied by the Q-function at that episode (i.e., \( V^{\pi_k} \)), for a total of \( H \) steps. We initialize the rollout procedure to start at a fixed nominal initial state so that the maximum cumulative reward is identical across trajectories. We perform two experiments.

In the first experiment, the number of agents is held fixed at \( M = 13 \). We vary the message life parameter \( \gamma \) over the range \( \{0, 1, \ldots, D(G, \gamma) = 4\} \). The performance difference of the rolled-out value function versus rolled-out offline Q function is plotted in Figure 1. As expected, the performance deficit decreases (i.e., performance improves) as agents gain access to more samples. Under the message-passing scheme \( \gamma = 0 \), i.e., no communication among the agents, agents struggle to reduce the total average performance error over 1000 episodes. For larger \( \gamma \), the performance increase appears to plateau around \( \gamma = 2 \). We leave as an open question whether there is an optimal communication bandwidth for learning; see e.g. [4] for an investigation on this topic.

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VI. Conclusion

Optimal exploration in online reinforcement learning is a key consideration that impacts sample complexity. We investigate the benefits of fully decentralized exploration in MARL using regret as a metric. We provide a multi-agent extension of the UCB-Hoeffding algorithm provided in [7] where the agents are equipped with a message passing scheme. We prove a regret bound that is \( O(\sqrt{MH^2 \delta^2}) \), an improvement over the single-agent setting. Specifically, we consider general network \( G \) and show that the regret also depends on the clique structure of the power graph \( G_c \), in addition to the number of agents. This key result suggests that the dense network structure, higher message life \( \gamma \), and higher number of agents \( M \) all reduce the average regret incurred by each agent, as shown in the simulations.

While the assumption of time-varying dynamics demands samples that are polynomial in the episode length \( H \), cooperative estimation of the optimal value functions allow parallel experience generation and exploration by proxy. When the initial state is fixed, our regret bound corresponds to \( O(1/(\sqrt{M}e^2)) \) sample complexity. Further work may involve providing a multi-agent minimax-optimal regret bound for general network structure using a Bernstein-style or similar UCB bonus as shown in [7], [8]. Further, the message life \( \gamma \) should be optimized with respect to the number of agents and network structure by considering communication cost and privacy.

References


