On Separation of Distributed Estimation and Control for LTI Systems

Anthony J. Savas*, Shinkyu Park*, H. Vincent Poor, and Naomi E. Leonard

***Abstract*— The separation principle in a centralized estimation and control problem gives us the flexibility to design a feedback controller independent of the state estimator. However, the same principle does not hold when the estimation and control are distributed over a network of agents. In this case, the estimator may need to be redesigned when the controller is revised, which can be computationally expensive. We investigate a weaker notion of the separation principle in the distributed estimation and control of linear time-invariant (LTI) systems. As a main contribution, applying the small-gain theorem, we characterize the notion using matrix inequalities and compute a set of feedback controllers that agents in the network can adopt without redesigning the estimator. We also analyze how the frequency of information exchange between neighboring agents affects the characterization. We illustrate our analytical results through simulations of a multi-vehicle system problem.**

I. Introduction

Consider the problem of designing a network of agents for estimation and control of a discrete-time linear time-invariant (LTI) system. Each agent can only partially observe the LTI system output and only partially control the LTI system state; thus, it cannot stabilize the LTI system on its own. However, suppose that jointly the agents can fully observe the system output and fully control the system state, and each agent can communicate over the network with its neighbors. The problem is to design the estimation dynamics and feedback control law for each agent, given its own set of information, so that jointly the agents stabilize the LTI system. Such a design problem arises in a wide range of control engineering domains including formation control of networked robots, efficient power transfer in large-scale power systems, and traffic flow control in transportation networks.

An interesting aspect of the problem is that each agent computes its control input based on its own set of information, which is different from the set of information available to other agents. Hence, the problem setting imposes a *non-classical information structure* [1], under which changing the feedback controller of an agent to improve the system performance negatively affects the network's estimation performance and requires redesigning the estimator. Such a redesign can be computationally expensive.

In a centralized counterpart, where a single agent estimates and controls the system's state, the *separation principle* gives the agent the freedom to adopt any stabilizing feedback controller independent of the estimator design [2]. However, because of the non-classical information structure, the same separation principle does not hold in our problem unless every agent can broadcast its locally available information to every other agent. When this is not possible, the estimator and controller across the network need to be jointly designed.

A large body of literature, including [3]–[7] and references therein, presents methods and performance analysis for design of networks to estimate the state of LTI systems, where consensus-type algorithms are adopted to exchange and fuse state estimates among neighboring agents. Recent work [8] proposes a control framework based on distributed estimation to design a network of agents to stabilize LTI systems. LMI-based approaches for joint design of a distributed observer and controller are discussed in [9], [10].

In [11] a decentralized approach is discussed for designing a distributed controller for continuous-time LTI systems in which the agents are required to share their local information with neighbors at every time instance over the continuous-time domain. The authors of [12] propose a consensus-based distributed observer over a rate-limited communication network to design linear feedback control systems. The work of [13] investigates a distributed approach to the design of a distributed observer and controller for spatially interconnected systems in the continuous-time domain.

As a complement to the existing literature, we address the following question: *"Under what conditions can the agents change their feedback control without revising their estimation strategies?"* As we have explained above, in our problem setting, the agents cannot arbitrarily change their feedback control without revision of the estimator design. Our main contribution is to characterize and compute a set of state feedback controllers from which the agents can select for a given *fixed* estimator design. Thus, if agents change their feedback control to improve system performance such that the controller is from this set, there is no need for a redesign of the estimator. Further, we show that more frequent information exchange through consensus allows the agents to select from a larger set of controllers.

The paper is organized as follows. In §II, we present our framework and explain the main problem on distributed estimation and control design. In §III, we propose matrix inequality formulations to find a solution to the main problem, and we analyze the effect of the frequency of information exchange on the formulations. In §IV, we illustrate our main results using simulations. We conclude the paper in §V.

Supported by the DoD through the NDSEG program, ONR grant N00014-19-1-2556, funding from KAUST, and Princeton School of Engineering and Applied Science through the generosity of Lydia and William Addy '82.

* indicates equal contributions.

Savas and Leonard are with the Department of Mechanical and Aerospace Engineering, and Poor is with the Department of Electrical and Computer Engineering, Princeton University, Princeton, NJ 08544, USA. {asavas, naomi, poor}@princeton.edu. Park is with the Electrical and Computer Engineering, King Abdullah University of Science and Technology (KAUST), Thuwal 23955, Saudi Arabia. shinkyu.park@kaust.edu.sa

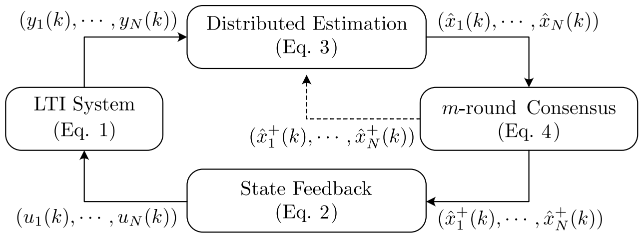


Fig. 1. A diagram illustrating the closed loop consisting of the LTI system, distributed estimation, m-round (linear) consensus, and state feedback.

II. Problem Description

Consider a discrete-time LTI system given by

$$x(k+1) = Ax(k) + \textstyle\sum_{i=1}^{N} B_i u_i(k), \ x(0) \in \mathbb{R}^n \tag{1a}$$

$$y_i(k) = C_i x(k), \ i \in \{1, \cdots, N\} \tag{1b}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u_i(k) \in \mathbb{R}^{q_i}$ is the i-th input, and $y_i(k) \in \mathbb{R}^{r_i}$ is the i-th output of the system. We design a network of N agents where each agent i assesses its associated system output $y_i(k)$, communicates over a fixed directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, and assigns control input $u_i(k)$. Each vertex i in $\mathbb{V} = \{1, \cdots, N\}$ represents agent i and each edge $(j, i) \in \mathbb{E}$ indicates that agent j can transmit information to agent i. We define the neighborhood set $\mathbb{N}_i = \{j \in \mathbb{V} \,|\, (j, i) \in \mathbb{E}\}$ to specify a subset of agents that can transmit information to agent i.

We assume that $\mathcal{G}$ is strongly connected and (1) is jointly controllable and observable: pairs (A, B) and (C, A) are controllable and observable, respectively, where $B = (B_1, \cdots, B_N)$ and $C = (C_1^T, \cdots, C_N^T)^T$. However, individual agents do not necessarily have full controllability or observability of the system: For every i in $\mathbb{V}$, pairs (A, B_i) and (C_i, A) may not be controllable and observable, respectively. Hence, without communication with others, each agent can neither estimate the full state of the system nor stabilize it.

Our main goal is to design the network dynamics so that each agent i computes an estimate $\hat{x}_i(k)$ of $x(k)$, exchanges and fuses the estimate with those of its neighbors, and uses the fused state estimate to compute control input $u_i(k)$ for joint system stabilization. To design the network, we implement three components for each agent: state feedback, distributed estimation, and linear consensus (see Fig. 1 for an illustration of the closed loop consisting of the three components and the LTI system):

State feedback: Let $\hat{x}_i(k)$ be the state estimate of agent i. The agent computes $u_i(k)$ according to

$$u_i(k) = K_i \hat{x}_i(k). \tag{2}$$

Note that the agent uses only its own state estimate to compute $u_i(k)$. Throughout the paper, we assume that gain matrices $\{K_i\}_{i \in \mathbb{V}}$ make $A + \sum_{i \in \mathbb{V}} B_i K_i$ Schur stable, i.e., has eigenvalues inside the unit circle in the complex plane.

Distributed estimation: Agent i computes $\hat{x}_i(k)$ by recursively updating it according to

$$\begin{aligned} &\hat{x}_i(k+1) = A\hat{x}_i(k) + L_i \left(y_i(k) - C_i \hat{x}_i(k)\right) \\ &+ \textstyle\sum_{j \in \mathbb{V}} B_j K_j \hat{x}_i(k) + \sum_{j \in \mathbb{N}_i} W_{ij} \left(\hat{x}_j(k) - \hat{x}_i(k)\right), \end{aligned} \tag{3}$$

where K_i is the control gain matrix in (2) and $W_{ij} \in \mathbb{R}^{n \times n}$, $L_i \in \mathbb{R}^{n \times r_i}$ are the parameters that need to be determined. Eq. 3, which is motivated by the existing distributed estimation approaches proposed, for instance, in [3]–[5], [8], updates $\hat{x}_i(k)$ using the partial output $y_i(k)$, the information $\{\hat{x}_j(k)\}_{j \in \mathbb{N}_i}$ from the agent's neighbors, and the estimate $\sum_{j \in \mathbb{V}} B_j K_j \hat{x}_i(k)$ of the control input applied to (1).[1]

m-round linear consensus: When additional information exchange is allowed, the agents exchange their state estimates and fuse the exchanged information using m-round linear consensus, which takes place between two consecutive estimation updates defined by (3). Letting $\{\hat{x}_j(k)\}_{j \in \mathbb{V}}$ be the state estimates of the agents at the beginning of the linear consensus, the output $\hat{x}_i^+(k)$ at agent i is determined by

$$\hat{x}_i^+(k) = \textstyle\sum_{j \in \mathbb{V}} \bar{P}_{ij} \hat{x}_j(k). \tag{4}$$

The parameter $\bar{P}_{ij} \geq 0$ is the i, j-th element of a matrix defined as $\bar{P} = P^m \in \mathbb{R}^{N \times N}$, where m is a non-negative integer and P is a stochastic matrix that conforms with $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, i.e., $(j, i) \notin \mathbb{E} \Leftrightarrow P_{ij} = 0$. Hence, $\hat{x}_i^+(k)$ denotes agent i's updated state estimate after applying the m rounds of linear consensus with matrix P. We refer to m and P as the total number of *consensus rounds* and the *consensus matrix*, respectively. The outcome $\hat{x}_i^+(k)$ is then fed into (2) and (3) to update each agent's state estimate, $\hat{x}_i(k) = \hat{x}_i^+(k)$, as illustrated in Fig. 1.

The work of [4]–[6] presents technical conditions on the system (1) and the graph $\mathcal{G}$ under which there are parameters $\{W_{ij}\}_{i,j \in \mathbb{V}}$, $\{L_i\}_{i \in \mathbb{V}}$ that ensure convergence of $\hat{x}_i(k)$ to $x(k)$ for every i in $\mathbb{V}$, when there is no state feedback, i.e., $K_i = 0, \ \forall i \in \mathbb{V}$. More recent work in [8] describes how to jointly compute $\{K_i\}_{i \in \mathbb{V}}, \{W_{ij}\}_{i,j \in \mathbb{V}}, \{L_i\}_{i \in \mathbb{V}}$ to stabilize (1) using (2) and (3).

In our work, similar to [8], we investigate the problem of designing the state feedback and distributed estimation for the system stabilization. However, our work is distinct from [8] in that we investigate the case where the gain matrices $\{K_i\}_{i \in \mathbb{V}}$ can be designed independently of the parameters $\{W_{ij}\}_{i,j \in \mathbb{V}}$, $\{L_i\}_{i \in \mathbb{V}}$. In particular, our main result establishes technical conditions that ensure the stability of (1) when $\{K_i\}_{i \in \mathbb{V}}$ and $\{W_{ij}\}_{i,j \in \mathbb{V}}, \{L_i\}_{i \in \mathbb{V}}$ are computed using two decoupled numerical methods. As discussed in §III-C, our result can be applied to the scenario where each agent needs to update its control gain without re-computing the parameters of the distributed estimation over the entire network for which the analysis of [8] cannot be directly

[1]According to (2), the control input applied to (1) is given by $\sum_{j \in \mathbb{V}} B_j K_j \hat{x}_j(k)$ which depends on every agent's state estimate.

applied.[2]

Also our work allows communication among agents to take place at discrete-time instances, different from the continuous-time problem setting investigated in [11], [13]. This makes our proposed framework and main theorems not only technically distinct but also applicable to a wide range of engineering problems in which agents can only exchange information for a limited number of times over any given finite time interval.

Below, we formalize our main problem and provide a motivating example.

Problem 1: For fixed $m \geq 0$, compute the parameters $\{W_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ and identify the set of state feedback gains $\{K_i\}_{i\in\mathbb{V}}$ for which the control inputs determined by (2) and (3) stabilize the system (1).

Example 1: Consider a system of N vehicles moving on the plane where each vehicle can apply a force to control its motion and can observe its position. The system is thus jointly controllable and jointly observable. The state and parameters of the system's model (1) are given as

$$x = (p_1^T, v_1^T, \cdots, p_N^T, v_N^T)^T \tag{5a}$$

$$A = I_N \otimes \begin{pmatrix} I_2 & 0.5 I_2 \\ 0 & I_2 \end{pmatrix} \tag{5b}$$

$$B_i = e_i \otimes \begin{pmatrix} 0 \\ I_2 \end{pmatrix}, \; C_i = e_i^T \otimes \begin{pmatrix} I_2 & 0 \end{pmatrix}, \; i \in \{1, \cdots, N\} \tag{5c}$$

where $p_i, v_i \in \mathbb{R}^2$ are, respectively, the position and velocity of the i-th vehicle and e_i is a canonical basis in $\mathbb{R}^N$ whose elements are all zero except the i-th element, which is 1.

The authors of [14] study the design of feedback controllers to improve robustness of a vehicle platoon by allowing the controller of each vehicle to assess state information of the vehicles other than the ones nearby, e.g., front and rear. Motivated by such approach, in this example we consider the scenario where each vehicle is controlled using state estimates of all the vehicles. In particular, we will consider that the control input $u_i(k) = (u_i^x(k), u_i^y(k))$ of vehicle i is defined by

$$\begin{aligned} u_i^x(k) = -\textstyle\sum_{j\in\mathbb{V}} (k_p(\hat{p}_i^x(k) - \hat{p}_j^x(k) - d_{ij}) \\ + k_v(\hat{v}_i^x(k) - \hat{v}_j^x(k))) \end{aligned} \tag{6a}$$

$$\begin{aligned} u_i^y(k) = -\textstyle\sum_{j\in\mathbb{V}} (k_p(\hat{p}_i^y(k) - \hat{p}_j^y(k)) \\ + k_v(\hat{v}_i^y(k) - \hat{v}_j^y(k))) \end{aligned} \tag{6b}$$

where k_p, k_v are control gains and d_{ij} denotes a desired distance the vehicles i and j needs to maintain along the platoon's moving direction (x-axis in this case). The position and velocity estimates $(\hat{p}_i^x, \hat{p}_i^y)$, $(\hat{v}_i^x, \hat{v}_i^y)$ are computed by the distributed estimation (3).

When the leading vehicle detects an obstacle on the road, the feedback control (6) needs to be changed to avoid collision while platooning. A solution to Problem 1 can be used to find such feedback control without recomputing the parameters $\{W_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ of (3).

[2] As we describe in §IV-A, finding the parameters for (3) involves finding a solution to a large-scale bilinear matrix inequality, which can be computationally expensive. For this reason, whenever possible, it is preferred not to re-compute the parameters of (3) when the state feedback is revised.

III. Parameter Design for State Feedback and Distributed Estimation

We present matrix inequality formulations to compute the parameters of (3) that result in the stability of (1), and discuss the existence of solutions to the formulations. We start with $m = 0$, i.e., no linear consensus and address the general case, where $m \geq 0$, in §III-B.

We begin by defining the estimation error as $\tilde{x}_i(k) = x(k) - \hat{x}_i(k)$. Using (1), (2)-(4), the state equation for $\tilde{x}_i(k)$ can be derived as follows:

$$\begin{aligned} \tilde{x}_i(k+1) = (A - L_i C_i)\, \tilde{x}_i(k) + \textstyle\sum_{j\in\mathbb{V}} B_j K_j \left(\tilde{x}_i(k) - \tilde{x}_j(k)\right) \\ + \textstyle\sum_{j\in\mathbb{N}_i} W_{ij} \left(\tilde{x}_j(k) - \tilde{x}_i(k)\right). \end{aligned} \tag{7}$$

In what follows, we cast (7) as a feedback interconnection of two components – the *control component* (8) and *estimation component* (9) defined below – and find sufficient conditions for the interconnection to attain the convergence $\lim_{k\to\infty} \|\tilde{x}_i(k)\|_2 = 0, \; \forall i \in \mathbb{V}$. We will use this result to address Problem 1.

To cast (7) into two components, let $W_{ij} = W_{ij}^{\mathrm{C}} + W_{ij}^{\mathrm{E}}$ and define[3]

$$\begin{aligned} \tilde{v}_i(k) = \textstyle\sum_{j\in\mathbb{V}} B_j K_j (\tilde{x}_i(k) - \tilde{x}_j(k)) \\ - \textstyle\sum_{j\in\mathbb{N}_i} W_{ij}^{\mathrm{C}} (\tilde{x}_i(k) - \tilde{x}_j(k)) \end{aligned} \tag{8}$$

$$\begin{aligned} \tilde{x}_i(k+1) = (A - L_i C_i)\, \tilde{x}_i(k) \\ - \textstyle\sum_{j\in\mathbb{N}_i} W_{ij}^{\mathrm{E}} (\tilde{x}_i(k) - \tilde{x}_j(k)) + \tilde{v}_i(k). \end{aligned} \tag{9}$$

Note that the feedback interconnection of (8) and (9) is equivalent to (7). In (8), the first term $\sum_{j\in\mathbb{V}} B_j K_j (\tilde{x}_i(k) - \tilde{x}_j(k))$ denotes the difference between the control input applied to the system and its estimate by agent i, and the second term $\sum_{j\in\mathbb{N}_i} W_{ij}^{\mathrm{C}} (\tilde{x}_i(k) - \tilde{x}_j(k))$ is adopted to counteract the error in the control input estimation. In §IV, we present LMI-based optimization to find the best selection of W_{ij}^{C}. Note that (9) is equivalent to (7) except we represent the control input estimation error term by $\tilde{v}_i(k)$ and adopt $\{W_{ij}^{\mathrm{E}}\}_{i,j\in\mathbb{V}}$ in place of $\{W_{ij}\}_{i,j\in\mathbb{V}}$.

We use the small-gain theorem [15, Chapter 5.4] to specify conditions on the parameter selection that ensure the convergence in (7). To this end, we represent (9) as an LTI system with state $\tilde{x}(k) = (\tilde{x}_1(k), \cdots, \tilde{x}_N(k)) \in \mathbb{R}^{nN}$ and input $\tilde{v}(k) = (\tilde{v}_1(k), \cdots, \tilde{v}_N(k)) \in \mathbb{R}^{nN}$ as follows:

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{v}(k), \tag{10}$$

where $\tilde{A} \in \mathbb{R}^{nN\times nN}$ is defined as

$$\tilde{A} = \mathrm{diag}(A - L_1 C_1, \cdots, A - L_N C_N) + W^{\mathrm{E}} \tag{11}$$

[3] We represent W_{ij} as a sum of the two terms W_{ij}^{C}, W_{ij}^{E} to use them as design parameters in (8) and (9), respectively.

and W^{E} is a block matrix whose i,j-th block element is

$$[W^{\mathrm{E}}]_{ij} = \begin{cases} W^{\mathrm{E}}_{ij} & \text{if } j \in \mathbb{N}_i\backslash\{i\} \\ -\sum_{l\in\mathbb{N}_i\backslash\{i\}} W^{\mathrm{E}}_{il} & \text{if } j = i \\ 0_n & \text{otherwise.} \end{cases} \quad (12)$$

Also, we rewrite (8) as

$$\tilde{v}(k) = \tilde{D}\tilde{x}(k), \quad (13)$$

where $\tilde{D}$ is a block matrix whose i,j-th block element is

$$[\tilde{D}]_{ij} = \begin{cases} -B_jK_j + W^{\mathrm{C}}_{ij} & \text{if } j \in \mathbb{N}_i\backslash\{i\} \\ \sum_{l\in\mathbb{V}\backslash\{i\}} B_lK_l - \sum_{l\in\mathbb{N}_i\backslash\{i\}} W^{\mathrm{C}}_{il} & \text{if } j = i \\ -B_jK_j & \text{otherwise.} \end{cases}$$

A. Matrix Inequality Formulation for Stability

Let $\tilde{G}$ be the (input-to-state) transfer function matrix of (10). As an application of the small-gain theorem [15, Chapter 5.4], the feedback interconnection of the estimation component (10) and the control component (13) is L_2-stable if it holds that $\|\tilde{G}\|_{H_\infty}\|\tilde{D}\|_2 < 1$.

Adopting the bounded real lemma for discrete-time LTI systems [16], [17], we establish the following equivalence:

$$\|\tilde{G}\|_{H_\infty} < \gamma \Leftrightarrow \begin{pmatrix} -X & X\tilde{A} & I_{nN} & 0_{nN} \\ \tilde{A}^TX & -X & 0_{nN} & X \\ I_{nN} & 0_{nN} & -\gamma I_{nN} & 0_{nN} \\ 0_{nN} & X & 0_{nN} & -\gamma I_{nN} \end{pmatrix} \prec 0 \quad (14)$$

where γ is a positive real number and $X \in \mathbb{R}^{nN\times nN}$ is a symmetric and positive-definite matrix. In the following proposition, we provide a sufficient condition under which a solution X, γ exists for (14).

Proposition 1: Suppose that there are $\{W^{\mathrm{E}}_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ for which $\tilde{A}$ given in (11) is Schur stable. Then, a solution $X = X^T \succ 0$, $\gamma > 0$ exists for (14).

Proof: Based on [18, Lemma 5.1 (iii)] and by applying the Schur complement, (14) can be equivalently expressed as

$$X \prec \gamma^2 I_{nN} \quad (15a)$$

$$\tilde{A}^TX\tilde{A} - X + I_{nN} - \tilde{A}^TX\left(X - \gamma^2 I_{nN}\right)^{-1}X\tilde{A} \prec 0 \quad (15b)$$

Note that the last term in (15b) goes to zero as γ tends to infinity. Since $\tilde{A}$ is assumed to be Schur stable, there is $X = X^T \succ 0$ satisfying $\tilde{A}^TX\tilde{A} - X \prec -I_{nN}$. Consequently, the same X satisfies (15) for sufficiently large γ. This completes the proof. ■

Remark 1: The results of [4]–[7] from the distributed estimation literature address the existence of the parameters $\{W^{\mathrm{E}}_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ for which $\tilde{A}$ is Schur stable when the system (1) is jointly observable and the graph $\mathcal{G}$ is strongly connected. The results of [5], [6] are based on a state augmentation idea and those of [4], [7] leverage the structure in the system model (1) and the graph connectivity. Therefore, in conjunction with the results from those references, Proposition 1 implies that the joint observability of (1) and the strong connectivity of $\mathcal{G}$ are sufficient for (14) to have a solution.

Based on Proposition 1, in the following proposition, we address Problem 1. Given $\gamma_2 > 0$, define

$$\mathbb{K}_{\gamma_2} = \{\{K_i\}_{i\in\mathbb{V}} \,|\, A + \textstyle\sum_{i\in\mathbb{V}} B_iK_i \text{ is Schur stable}, \min_{\{W^{\mathrm{C}}_{ij}\}_{i,j\in\mathbb{V}}} \|\tilde{D}\|_2 < \gamma_2\}.$$

Proposition 2: Suppose given parameters $\{W^{\mathrm{E}}_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ of the estimation component $\tilde{G}$ satisfy $\|\tilde{G}\|_{H_\infty} < \gamma$, e.g., (14) has a solution with the same γ. Assuming that $\mathbb{K}_{\gamma^{-1}}$ is nonempty, for each state feedback gains $\{K_i\}_{i\in\mathbb{V}}$ belonging to $\mathbb{K}_{\gamma^{-1}}$ there is $\{W^{\mathrm{C}}_{ij}\}_{i,j\in\mathbb{V}}$ such that the control inputs determined by (2) and (3) stabilize the system (1).

Proof: Since, under the assumptions on $\|\tilde{G}\|_{H_\infty}$ and $\|\tilde{D}\|_2$, the small-gain theorem holds, we have that $\lim_{k\to\infty}\|\tilde{x}(k)\|_2 = 0$. By applying (2) to (1a), we obtain

$$x(k+1) = (A + \textstyle\sum_{i\in\mathbb{V}} B_iK_i)x(k) - \sum_{i\in\mathbb{V}} B_iK_i\tilde{x}_i(k).$$

Since $A + \sum_{i\in\mathbb{V}} B_iK_i$ is Schur stable and the last term converges to zero as k tends to infinity, the state $x(k)$ converges to zero. This completes the proof. ■

Note that when γ in the statement of Proposition 2 is too large, $\mathbb{K}_{\gamma^{-1}}$ would be an empty set. In other words, when the H_∞-norm of the estimation component is too large, there is no control gain that stabilizes (1) while satisfying the inequality $\min_{\{W^{\mathrm{C}}_{ij}\}_{i,j\in\mathbb{V}}}\|\tilde{D}\|_2 < \gamma^{-1}$ for the small-gain theorem to hold. In §III-B, we study the effect of m-round linear consensus on the state feedback design. We show that with more frequent information exchange (m large), the agents have more options for selecting state feedback gains that stabilize (1) and satisfy the inequality condition for the small-gain theorem to hold, i.e., the set $\mathbb{K}_{\gamma^{-1}}$ becomes large as m grows.

B. Effect of Linear Consensus on Stability

Suppose that the agents are allowed to fuse their state estimates using m-round linear consensus (4) with a consensus matrix P. For concise presentation, we assume that m is an even number and $\mathcal{G}$ is undirected (and we select P to be symmetric). With $Q = P \otimes I_n$, define $\tilde{x}'(k) = Q^{m/2}\tilde{x}(k)$ and $\tilde{v}'(k) = Q^{m/2}\tilde{v}(k)$.[4] By using (1), (2)-(4) and following similar steps to obtain (10) and (13), we can derive the state equations for the control and estimation components as follows:

$$\tilde{v}'(k) = Q^{m/2}\tilde{D}Q^{m/2}\tilde{x}'(k) \quad (16)$$

$$\tilde{x}'(k+1) = Q^{m/2}\tilde{A}Q^{m/2}\tilde{x}'(k) + \tilde{v}'(k) \quad (17)$$

We refine the definition of $\mathbb{K}_{\gamma_2}$ as follows: Given $\gamma_2 > 0$,

$$\mathbb{K}_{\gamma_2,m} = \{\{K_i\}_{i\in\mathbb{V}} \,|\, A + \textstyle\sum_{i\in\mathbb{V}} B_iK_i \text{ is Schur stable}, \min_{\{W^{\mathrm{C}}_{ij}\}_{i,j\in\mathbb{V}}} \|Q^{m/2}\tilde{D}Q^{m/2}\|_2 < \gamma_2\}.$$

We extend Proposition 2 to the case where the agents are allowed to use m rounds of linear consensus.

[4]For odd m, we let $\tilde{x}'(k) = Q^{\lfloor m/2\rfloor}\tilde{x}(k)$ and $\tilde{v}'(k) = Q^{\lfloor m/2\rfloor}\tilde{v}(k)$.

$$M_1=\begin{pmatrix} -X^* & X^*\left(A-\sum_{i\in\mathbb{V}} L_i^* C_i\right) & I_n & 0_n \\ \left(A-\sum_{i\in\mathbb{V}} L_i^* C_i\right)^T X^* & -X^* & 0_n & X^* \\ I_n & 0_n & -\gamma^* I_n & 0_n \\ 0_n & X^* & 0_n & -\gamma^* I_n \end{pmatrix},\; M_2=\begin{pmatrix} -X^* & 0_n & I_n & 0_n \\ 0_n & -X^* & 0_n & X^* \\ I_n & 0_n & -\gamma^* I_n & 0_n \\ 0_n & X^* & 0_n & -\gamma^* I_n \end{pmatrix} \quad (18)$$

Theorem 1: For any given $\gamma_2 > 0$, there is $m^* \geq 0$ for which $\mathbb{K}_{\gamma_2,m}$ is non-empty for $m \geq m^*$, and for $m_1 \geq m_2$, it holds that $\mathbb{K}_{\gamma_2,m_2} \subseteq \mathbb{K}_{\gamma_2,m_1}$. For sufficiently large m, there are parameters $\{W_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ for which the LTI system (1) is stable with any $\{K_i\}_{i\in\mathbb{V}}$ belonging to $\mathbb{K}_{\gamma^{-1},m}$, where γ is the H_∞-norm associated with (17).

Proof: Notice that since $\tilde{D}(\mathbf{1}_N \otimes I_n) = 0$ holds, $\|Q^{m/2}\tilde{D}Q^{m/2}\|_2$ converges to zero as m tends to infinity. Also we note that since Q is a stochastic matrix, $\|Q^{m_1/2}\tilde{D}Q^{m_1/2}\|_2 \leq \|Q^{m_2/2}\tilde{D}Q^{m_2/2}\|_2$ if $m_1 \geq m_2$. Hence, given fixed $\gamma_2 > 0$, we can find m^* for which $\mathbb{K}_{\gamma_2,m}$ is non-empty for all $m \geq m^*$. It remains to show that for sufficiently large m, we can find $\{W_{ij}^{\mathrm{E}}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ for which $\|\tilde{G}'\|_{H_\infty}\|Q^{m/2}\tilde{D}Q^{m/2}\|_2 < 1$ holds, where $\tilde{G}'$ is the (input-to-state) transfer function of (17).

Since the system (1) is jointly observable, we can find $\{L_i^*\}_{i\in\mathbb{V}}$ such that $A-\sum_{i\in\mathbb{V}} L_i^* C_i$ is Schur stable. Consider the matrix M_1 defined in (18). Suppose that we select the matrices $\{L_i^*\}_{i\in\mathbb{V}}$ and a symmetric positive-definite matrix $X^* \in \mathbb{R}^{n\times n}$ that satisfy $M_1 \prec 0$ with smallest $\gamma^* > 0$.[5] Also we can verify that

$$\frac{1}{N}\mathbf{1}\mathbf{1}^T \otimes M_1 + \left(I_{nN} - \frac{1}{N}\mathbf{1}\mathbf{1}^T\right) \otimes M_2 \prec 0 \qquad (19)$$

where M_1 and M_2 are defined in (18).

Define $W_{ij}^{\mathrm{E}} = P_{ij}A$ and $L_i = NL_i^*$ and express (17) as

$$\tilde{x}'(k+1) = (P^{m+1}\otimes A - Q^{m/2}\bar{L}\bar{C}Q^{m/2})\tilde{x}'(k) + \tilde{v}'(k) \quad (20)$$

with $\bar{L} = \mathrm{diag}(NL_1^*, \cdots, NL_N^*)$, $\bar{C} = \mathrm{diag}(C_1, \cdots, C_N)$. By adopting (14) with $\tilde{A} = P^{m+1}\otimes A - Q^{m/2}\bar{L}\bar{C}Q^{m/2}$ and $X = I_N \otimes X^*$, we have that

$$\begin{pmatrix} -X & X\tilde{A} & I_{nN} & 0_{nN} \\ \tilde{A}^T X & -X & 0_{nN} & X \\ I_{nN} & 0_{nN} & -\gamma^* I_{nN} & 0_{nN} \\ 0_{nN} & X & 0_{nN} & -\gamma^* I_{nN} \end{pmatrix} \prec 0 \Rightarrow \|\tilde{G}'\|_{H_\infty} < \gamma^*. \qquad (21)$$

Since $\lim_{m\to\infty} P^{m/2} = \lim_{m\to\infty} P^{m+1} = \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$, when m tends to infinity, the matrices in (19) and (21) are identical up to permutation. With the same choice of X^*, γ^* as in (19), the inequality (21) holds for sufficiently large m. Therefore, the H_∞-norm associated with (20) approaches γ^*, which is the H_∞-norm of the centralized estimator, as a number of rounds of linear consensus is allowed. In conjunction with the fact that $\lim_{m\to\infty}\|Q^{m/2}\tilde{D}Q^{m/2}\|_2 = 0$, we conclude that when m is sufficiently large, we can compute $\{W_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ for which $\|\tilde{G}'\|_{H_\infty}\|Q^{m/2}\tilde{D}Q^{m/2}\|_2 < 1$ holds for any $\{K_i\}_{i\in\mathbb{V}}$ belonging to $\mathbb{K}_{(\gamma^*)^{-1},m}$. ■

Theorem 1 implies that when sufficiently frequent information exchange is allowed, $\mathbb{K}_{\gamma^{-1},m}$ will be nonempty and we can always find gains $\{K_i\}_{i\in\mathbb{V}}$ to stabilize the system. Furthermore, by increasing m, we can design the parameters $\{W_{ij}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ to allow the agents to adopt gains $\{K_i\}_{i\in\mathbb{V}}$ from a larger set of gain matrices, and hence potentially achieve better control performance.

[5] By similar arguments in Proposition 1 such matrices exist for some γ^*.

C. Extension

Suppose that (2) consists of two linear parts:

$$u_i(k) = K_i^{(1)}\hat{x}_i(k) + K_i^{(2)}\hat{x}_i(k). \qquad (22)$$

In Example 1, for instance, the first part $K_i^{(1)}\hat{x}_i(k)$ can be designed to maintain a desired formation of the multiple vehicles and the second part $K_i^{(2)}\hat{x}_i(k)$ would be used, whenever necessary, for the vehicles to keep a safe distance from obstacles nearby. In this case, we can represent (7) as a feedback interconnection of the following two components and perform the stability analysis using Proposition 2 and Theorem 1:

$$\tilde{v}_i(k) = \textstyle\sum_{j\in\mathbb{V}} B_j K_j^{(2)}(\hat{x}_i(k) - \hat{x}_j(k))) - \sum_{j\in\mathbb{N}_i} W_{ij}^{\mathrm{C}}(\hat{x}_i(k) - \hat{x}_j(k)) \qquad (23)$$

and

$$\tilde{x}_i(k+1) = (A - L_iC_i)\,\tilde{x}_i(k) + \textstyle\sum_{j\in\mathbb{V}} B_j K_j^{(1)}(\tilde{x}_i(k) - \tilde{x}_j(k)) - \sum_{j\in\mathbb{N}_i} W_{ij}^{\mathrm{E}}(\tilde{x}_i(k) - \tilde{x}_j(k)) + \tilde{v}_i(k). \qquad (24)$$

The small-gain theorem can be used to establish the stability results if it holds that

$$\{K_i^{(2)}\}_{i\in\mathbb{V}} \in \mathbb{K}_{\gamma^{-1}}$$

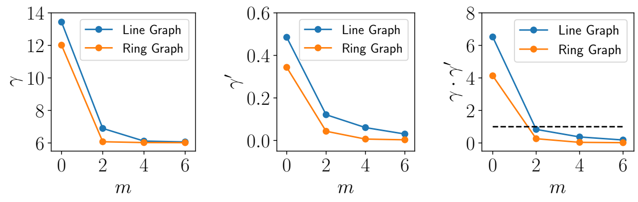
where γ is the H_∞-norm associated with (24). In this case, the design of the parameters $\{W_{ij}^{\mathrm{E}}\}_{i,j\in\mathbb{V}}$, $\{L_i\}_{i\in\mathbb{V}}$ for the estimation component will depend on the control gains $\{K_i^{(1)}\}_{i\in\mathbb{V}}$ of (22), but not on $\{K_i^{(2)}\}_{i\in\mathbb{V}}$.

IV. Simulations

Using simulations, we discuss the effect of the number of consensus rounds m on establishing the stability. We begin with explaining methods to compute the parameters of (16) and (17). For illustration purposes, from Example 1, we adopt (5) as the system model and (6) as state feedback applied to the system.

A. Parameter Design

Recall that the control gains $\{K_i\}_{i\in\mathbb{V}}$ are assumed to be given, and the design parameters in our framework are P, $\{W_{ij}^E\}_{i,j\in\mathbb{V}}$, $\{W_{ij}^C\}_{i,j\in\mathbb{V}}$, and $\{L_i\}_{i\in\mathbb{V}}$. We select P to be a stochastic matrix with smallest second eigenvalue and conforming with $\mathcal{G}$ to allow the agents to fuse the estimates as fast as possible. For simplicity, we choose $W_{ij}^E = P_{ij}A$ as motivated by the approach of [5].



(a) (b) (c)

Fig. 2. Plots of the optimal value of (a) γ for the estimation component design (25), (b) γ' for the control component design (26), and (c) the product $\gamma \cdot \gamma'$, where the critical value of 1 is drawn as a dotted line.

We solve the following two optimization formulations to compute $\{L_i\}_{i\in\mathbb{V}}$ and $\{W_{ij}^C\}_{i,j\in\mathbb{V}}$ that minimize the H_∞-norm of (17) and the 2-norm of (16), respectively.

$$\begin{aligned}&\text{minimize}_{\gamma, X, \{L_i\}_{i\in\mathbb{V}}} \gamma \\ &\text{subject to (21)}\end{aligned} \tag{25}$$

$$\begin{aligned}&\text{minimize}_{\gamma', \{W_{ij}^C\}_{i,j\in\mathbb{V}}} \gamma' \\ &\text{subject to} \begin{pmatrix} \gamma' I_{nN} & Q^{m/2}\tilde{D}Q^{m/2} \\ \left(Q^{m/2}\tilde{D}Q^{m/2}\right)^T & \gamma' I_{nN}\end{pmatrix} \succ 0\end{aligned} \tag{26}$$

where $\gamma > 0$, $X = X^T \succ 0$, and $Q = P \otimes I_n$. Note that (25) is non-convex optimization having the bilinear matrix inequality (21) as a constraint, whereas (26) can be cast as LMI-based convex optimization.

B. Simulation Results

Consider Example 1 with $N = 4$ and two types of communication graphs: undirected line and ring graphs. For each graph, we compute the parameters $\{L_i\}_{i\in\mathbb{V}}$ and $\{W_{ij}^C\}_{i,j\in\mathbb{V}}$ by solving (25) and (26) for $m = 0, 2, 4, 6$, where the parameters of the state feedback (6) are given by $k_p = 0.1438$, $k_v = 0.2128$, and $d_{ij} = j - i$.

Fig. 2 depicts the minimal costs obtained in the optimization (25) and (26) with the increasing number of consensus rounds m over line and ring graphs. We can observe that as m increases, both norms of the estimation and control components decrease. In conjunction with Theorem 1, this suggests that when the agents are allowed to communicate more frequently (when m increases), they would have more flexibility in revising the state feedback gains without recomputing the parameters of the estimator. We can also observe that since the ring graph has one additional edge between agents 1 and 4, which allows the network to have more paths for the agents to share state estimates, the consensus on the state estimates takes place faster over the ring graph. Consequently, the ring graph attains smaller norms for both components than does the line graph. Fig. 2(c) shows that $m \geq 2$ consensus rounds are needed to satisfy the small-gain theorem for the control gain we select.

V. Conclusions

We investigated the design of a network of agents for the estimation and control of LTI systems. Since the separation principle does not hold, the estimator and controller need to be jointly designed, which involves finding a solution to a large-scale optimization. This could be a disadvantage if the agents need to change their controllers without re-solving the optimization. We presented matrix inequality formulations to characterize the conditions under which the design of estimation and control can be decoupled. We also showed how the frequency of information exchange between agents affects the establishment of the conditions, which we illustrated through simulations using a multi-vehicle system example.

References

[1] A. Mahajan, N. C. Martins, M. C. Rotkowitz, and S. Yüksel, "Information structures in optimal decentralized control," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 1291–1306.

[2] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, 3rd ed. Belmont, MA, USA: Athena Scientific, 2005, vol. I.

[3] R. Olfati-Saber, "Distributed Kalman filter with embedded consensus filters," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005, pp. 8179–8184.

[4] A. Mitra and S. Sundaram, "Distributed observers for LTI systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3689–3704, 2018.

[5] S. Park and N. C. Martins, "Design of distributed LTI observers for state omniscience," *IEEE Transactions on Automatic Control*, vol. 62, no. 2, pp. 561–576, 2017.

[6] L. Wang and A. S. Morse, "A distributed observer for a time-invariant linear system," *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 2123–2130, 2018.

[7] Álvaro Rodríguez del Nozal, P. Millán, L. Orihuela, A. Seuret, and L. Zaccarian, "Distributed estimation based on multi-hop subspace decomposition," *Automatica*, vol. 99, pp. 213–220, 2019.

[8] L. Wang, D. Fullmer, F. Liu, and A. S. Morse, "Distributed control of linear multi-channel systems: Summary of results," in *2020 American Control Conference (ACC)*, 2020, pp. 4576–4581.

[9] M. Razeghi-Jahromi and A. Seyedi, "Stabilization of networked control systems with sparse observer-controller networks," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1686–1691, 2015.

[10] J. Xiong and W. Liu, "H_∞ observer-based control for large-scale systems with sparse observer communication network," *Numerical Algebra, Control and Optimization*, vol. 10, no. 3, pp. 331–343, 2020.

[11] T. Kim, D. Lee, and H. Shim, "Decentralized design and plug-and-play distributed control for linear multi-channel systems," *arXiv 2011.09735*, 2021.

[12] F. C. Rego, Y. Pu, A. Alessandretti, A. P. Aguiar, A. M. Pascoal, and C. N. Jones, "A distributed Luenberger observer for linear state feedback systems with quantized and rate-limited communications," *IEEE Transactions on Automatic Control*, vol. 66, no. 9, pp. 3922–3937, 2021.

[13] X. Zhang, K. Hengster-Movrić, M. Šebek, W. Desmet, and C. Faria, "Distributed observer and controller design for spatially interconnected systems," *IEEE Transactions on Control Systems Technology*, vol. 27, no. 1, pp. 1–13, 2019.

[14] Y. Zheng, S. E. Li, K. Li, and W. Ren, "Platooning of connected vehicles with undirected topologies: Robustness analysis and distributed h-infinity controller synthesis," *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 5, pp. 1353–1364, 2018.

[15] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.

[16] G. Pipeleers, B. Demeulenaere, J. Swevers, and L. Vandenberghe, "Extended LMI characterizations for stability and performance of linear systems," *Systems & Control Letters*, vol. 58, no. 7, pp. 510–518, 2009.

[17] R. J. Caverly and J. R. Forbes, "LMI properties and applications in systems, stability, and control theory," *ArXiv 1903.08599*, 2019.

[18] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 4, pp. 421–448, 1994.