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# Control of Agreement and Disagreement Cascades with Distributed Inputs

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***Abstract*— For a group of autonomous communicating agents to carry out coordinated objectives, it is paramount that they can distinguish meaningful input from disturbance, and come rapidly and reliably to agreement or disagreement in response to that input. We study how opinion formation cascades through a group of networked decision makers in response to a distributed input signal. Using a nonlinear opinion dynamics model with dynamic feedback modulation of an attention parameter, we prove how the triggering of an opinion cascade and the collective decision itself depend on both the distributed input and node agreement and disagreement centrality indices, determined by the spectral properties of the network graph. Moreover, we show how the attention dynamics introduce an implicit threshold that distinguishes between distributed inputs that trigger cascades and ones that are rejected as disturbance.**

## I. INTRODUCTION

Emerging technologies rely on network communications and sensor input to make coherent collective decisions. For example, autonomous multi-robot teams must cooperate to move as a group, avoid collisions, and perform collective tasks in potentially dynamic and uncertain environments. These objectives necessarily involve on-the-fly collective decision making about context-dependent options, such as which of multiple available paths to take, in which direction to turn, or how to distribute available tasks among team members. There is urgent need for a unified design framework that enables autonomous teams to rapidly and reliably coordinate decisions across different contexts in a distributed manner.

Mathematical models of networked opinion dynamics, e.g. [1]–[5], can be useful for this purpose, in part due to their analytical tractability. However, most existing models rely on a linear weighted-average opinion updating process, which imposes limits on the range of behaviors exhibited. Notably, special network structure or asymmetry is needed to produce solutions other than consensus, whereas applications that require groups to split among locations or tasks warrant more generically enabled disagreement solutions.

We present new results for the nonlinear opinion dynamics model [6]–[9], which provides an analytically tractable generalization of models that rely on linear weighted-averaging by applying a saturation function to opinion exchanges. The saturation makes the opinion update process fundamentally nonlinear, which has a number of important consequences. First, the model yields multi-stability of disagreement solutions as well as agreement solutions, each in easily identifiable parameter regimes, even for homogeneous agents. Second, opinions form through a bifurcation in which the neutral solution becomes unstable and agreement or disagreement solutions become stable, independent of the number of agents or options [6] and across network topologies [9]. This means solutions are reached rapidly and reliably – strength of opinions grow nonlinearly even with little or no input.

Our contributions yield a rigorous and systematic method for designing distributed inputs to control opinion formation and opinion cascades. We specialize the model to opinions on two options here, but results extend naturally to an arbitrary number of options. First, using Lyapunov-Schmidt reduction methods [10, Chapter VII], we prove that opinions generically form through a supercritical pitchfork bifurcation where the two stable branches are either agreement solutions or disagreement solutions, which we can fully characterize. Second, we prove that the agreement (disagreement) centrality of a node, which depends only on the spectral properties of the network adjacency matrix, determines the influence an input to the node has on the agreement (disagreement) bifurcation behavior. Third, when the opinion dynamics are coupled with the feedback attention dynamics introduced in [6], sufficiently large inputs can trigger an opinion cascade, depending on where in the network they are introduced. We show how agreement and disagreement centrality indices predict the sensitivity of opinion cascades to distributed inputs: The more aligned the input vector is with the centrality vector, the smaller the inputs need to be to trigger a cascade.

We present the model in Section II and review Lyapunov-Schmidt reduction in Section III. We prove the pitchfork bifurcations and the role of distributed input on opinion formation behavior for constant attention in Section IV and for dynamic feedback controlled attention in Section V. We conclude in Section VI.

## II. OPINION DYNAMICS MODEL

We study a model of $N_a$ agents communicating over a network and forming opinions on two options through a nonlinear process specialized from the multi-option general model in [6],[7]. As in [9], we specialize to agents that are homogeneous with respect to three fixed parameters in the dynamics: the rate of forgetting (damping coefficient $d > 0$), the edge weight in the communication network ($\gamma \in \mathbb{R}$), and the strength of self-reinforcement of opinion ($\alpha \geq 0$). In [9], we focused on the zero-input setting, i.e., the case in which

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there is no stimulus, evidence or bias that informs the agents about the relative merits of the options. Instead, here, we consider an input $b_i \in \mathbb{R}$, for each agent $i = 1, \dots, N_a$, and allow the inputs to be heterogeneously distributed over the network of agents. We further model heterogeneity over the agents in their attention to network exchange.

Agent interactions are encoded in graph $G = (V, E)$ where $V = \{1, \dots, N_a\}$ is the index set of vertices. Vertex $i \in V$ represents agent $i$, and edge set $E \subseteq V \times V$ represents agent interactions. $A = (a_{ik})$, $i, k \in V$, is the unweighted graph adjacency matrix with elements satisfying $a_{ik} = 1$ if and only if $e_{ik} \in E$, and $a_{ik} = 0$ otherwise. We let $a_{ii} = 0$ for all $i \in V$. $G$ is an *undirected* graph if $a_{ik} = a_{ki}$ for all $i, k \in V$. Let $\lambda_q$, $q = 1, \dots, N_a$, be the eigenvalues of $A$ and $W(\lambda_q)$ the generalized eigenspace associated to $\lambda_q$. We define $\lambda_{max}$ and $\lambda_{min}$ to be the $\lambda_q$ with largest and smallest real parts, respectively, and $\mathbf{v}_{max}$ ($\mathbf{w}_{max}$) and $\mathbf{v}_{min}$ ($\mathbf{w}_{min}$) to be the corresponding unit right (left) eigenvectors.

With two options, the opinion of each agent $i$ is a real-valued scalar $x_i \in \mathbb{R}$. The sign of $x_i$ corresponds to agent $i$ favoring option 1 ($x_i > 0$) or favoring option 2 ($x_i < 0$). The magnitude of the opinion variable $x_i$ describes the strength of agent $i$'s commitment. The vector of agents' opinions $\mathbf{x} = (x_1, \dots, x_{N_a}) \in \mathbb{R}^{N_a}$ is the *network opinion state*.

Agent $i$ has a *neutral* opinion when $x_i = 0$, and we say it is *opinionated* otherwise. Furthermore we say that any pair of agents $i, k \in V$ *agree* (*disagree*) when they are opinionated and favor the same option (different options), i.e. $\operatorname{sign}(x_i) = \operatorname{sign}(x_k)$ ($\operatorname{sign}(x_i) \neq \operatorname{sign}(x_k)$ ). The group is in an *agreement state* when all agents agree, and in a *disagreement state* when at least one pair of agents disagree.

Each agent $i$ updates its own opinion state $x_i$ in continuous time according to the nonlinear update rule:

$$\dot{x}_i = -dx_i + u_i S\left(\alpha x_i + \gamma \sum_{\substack{k \neq i \\ k=1}}^{N_a} a_{ik} x_k\right) + b_i. \quad (1)$$

The rule has three parts: a damping term with coefficient $d > 0$, a nonlinear interaction term that includes inter-agent exchanges with weight $\gamma \in \mathbb{R}$ and a self-reinforcement term with weight $\alpha \geq 0$, and an additive input $b_i \in \mathbb{R}$.

The nonlinearity applied to the inter-agent exchanges and self-reinforcement is defined by an odd sigmoidal saturating function $S$ which satisfies $S(0) = 0$, $S'(0) = 1$, and $\operatorname{sign}(S''(z)) = -\operatorname{sign}(z)$. This is motivated from the literature and means that agent $i$ is more influenced by opinion fluctuations in its neighbors when their average opinion is close to neutral, and as neighbors' opinions grow large on average their influence levels off. In simulations and analysis throughout this paper we use $S = \tanh$. We purposely leave the sigmoid more general in (1) because the results in this paper generalize to arbitrary odd sigmoidal functions with minor modifications in the algebraic details of the proofs.

We begin by specializing a result from [9].

**Proposition II.1** ([9], Theorem 1)**.** *The following hold true for* (1) *with $u_i := u \geq 0$ and $b_i = 0$ for all $i = 1, \dots, N_a$:*
***A. Cooperation leads to agreement:*** *Let $G$ be a connected undirected graph. If $\gamma > 0$, the neutral state $\mathbf{x} = 0$ is a locally exponentially stable equilibrium for $0 < u < u_a$ and unstable for $u > u_a$, with $u_a = \frac{d}{\alpha + \gamma \lambda_{max}}$. At $u = u_a$, branches of agreement equilibria, $x_i \neq 0$, $\operatorname{sign}(x_i) = \operatorname{sign}(x_k)$ for all $i, k \in V$, emerge in a steady-state bifurcation off of $\mathbf{x} = \mathbf{0}$ along $W(\lambda_{max})$;*
***B. Competition leads to disagreement:*** *Let $G$ be a connected undirected graph. If $\gamma < 0$ the neutral state $\mathbf{x} = \mathbf{0}$ is a locally exponentially stable equilibrium for $0 < u < u_d$ and unstable for $u > u_d$, with $u_d = \frac{d}{\alpha + \gamma \lambda_{min}}$. At $u = u_d$, branches of disagreement equilibria, $\operatorname{sign}(x_i) = -\operatorname{sign}(x_k)$ for at least one pair $i, k \in V$, $i \neq k$, emerge in a steady-state bifurcation off of $\mathbf{x} = \mathbf{0}$ along $W(\lambda_{min})$.*

## III. Lyapunov-Schmidt Reduction

To systematically characterize the equilibria of the opinion dynamics model as a function of parameters, we leverage Lyapunov-Schmidt reduction and its use in computing bifurcation diagrams. Consider the $n-$dimensional dynamical system $\dot{\mathbf{y}} = \Phi(\mathbf{y}, \mathbf{p})$, where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth parameterized vector field, $\mathbf{y} \in \mathbb{R}^n$ is a state vector, and $\mathbf{p} \in \mathbb{R}^m$ is a vector of parameters. Let $\mathbf{r}_s \in \mathbb{R}^n$, $s = 1 \dots N$. The $N^{th}$ order derivative of $\Phi$ at $(\mathbf{y}^*, \mathbf{p}^*)$ is

$$(d^N \Phi)_{\mathbf{y}^*, \mathbf{p}^*}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_N} \Phi\left(\mathbf{y}^* + \sum_{s=1}^{N} t_s \mathbf{r}_s, \mathbf{p}^*\right)\Bigg|_{t_1 = \dots = t_N = 0} . \quad (2)$$

The equilibria of $\dot{\mathbf{y}} = \Phi(\mathbf{y}, \mathbf{p})$ are the level sets $\Phi(\mathbf{y}, \mathbf{p}) = 0$, which defines the *bifurcation diagram* of the system.

The Jacobian of the system is the matrix $J$ with elements $J_{ij} = \frac{\partial \Phi(\mathbf{y}, \mathbf{p})}{\partial y_{ij}}$. When $J$ evaluated at an equilibrium point $(\mathbf{y}^*, \mathbf{p}^*)$ is degenerate (i.e. has rank $n - m$ where $0 < m < n$), the local bifurcation diagram can be described using $m$ variables and the point is a *singular point*. The *Lyapunov-Schmidt reduction* of $\Phi(\mathbf{y}, \mathbf{p})$ is an $m$-dimensional system of equations that captures the structure of the local bifurcation diagram of the system near $(\mathbf{y}^*, \mathbf{p}^*)$. The procedure for deriving the Lyapunov-Schmidt reduction [10, Chapter VII] involves projecting the Taylor expansion of $\Phi(\mathbf{y}, \mathbf{p})$ onto the kernel of its Jacobian at the singularity. The Implicit Function Theorem is used to solve for $n - m$ variables as function of the remaining $m$, thus approximating the local vector field in the directions orthogonal to the kernel.

The *normal form* for a bifurcation is the simplest equation that captures all qualitative features of the bifurcation diagram. Systems with an odd state symmetry $\Phi(-\mathbf{y}, \mathbf{p}) = -\Phi(\mathbf{y}, \mathbf{p})$ often exhibit a *pitchfork bifurcation*. A normal form for the pitchfork bifurcation universal unfolding is

$$\dot{y} = p_1 y \pm y^3 + p_2 + p_3 y^2 \quad (3)$$

where $y \in \mathbb{R}$ is the reduced state, $p_1$ is a *bifurcation parameter* and $p_2$, $p_3$ are *unfolding parameters*. When $p_2 = p_3 = 0$, the symmetric pitchfork normal form is recovered in (3). When one of the unfolding parameters is nonzero, it follows from unfolding theory [10, Chapter III] that the bifurcation diagram changes locally to one of four possible topologically distinct configurations (see Fig. 1).

## IV. Constant Attention: Sensitivity to Input

In this section, we investigate how a vector of constant inputs $\mathbf{b}$ informs the outcome of the opinion formation process (1) when attention is constant and $u_i := u \in \mathbb{R}$ for all $i = 1, \ldots, N_a$. The Jacobian of (1) evaluated at $\mathbf{x} = 0$ is

$$J_x = (u\alpha - d)\mathcal{I} + u\gamma A \tag{4}$$

with identity matrix $\mathcal{I}$. The dynamics (1) in vector form are

$$\dot{\mathbf{x}} = -d\mathbf{x} + u\mathbf{S}\left((\alpha\mathcal{I} + \gamma A)\mathbf{x}\right) + \mathbf{b} := F(\mathbf{x}, u, \mathbf{b}) \tag{5}$$

where $\mathbf{S}(\mathbf{y}) = (S(y_1), \ldots, S(y_n))$, $\mathbf{y} \in \mathbb{R}^n$, and $\mathbf{b} = (b_1, \ldots, b_{N_a})$. The following theorem generalizes [11, Theorem 1] to describe bifurcations of the opinion dynamics of homogeneous agents. The theorem shows that any bifurcation of $\mathbf{x} = 0$ of (1) that is generated by a simple eigenvalue of the adjacency matrix $A$ must be a pitchfork bifurcation.

**Theorem IV.1** (Pitchfork Bifurcation)**.** *Consider* (1) *and define* $u^* = \frac{d}{\alpha + \lambda\gamma}$*, where* $\lambda$ *is a simple real eigenvalue of adjacency matrix* $A$ *for a strongly connected graph* $G$*. Let* $\mathbf{v} = (v_1, \ldots, v_{N_a})$ *and* $\mathbf{w} = (w_1, \ldots, w_{N_a})$ *be right and left unit eigenvectors, respectively, corresponding to* $\lambda$*. Assume that (i) for all eigenvalues* $\xi \neq \lambda$ *of* $A$*,* $\mathrm{Re}[\xi] \neq \lambda$*; (ii)* $\alpha + \lambda\gamma \neq 0$*; (iii)*$\langle \mathbf{w}, \mathbf{v}^3 \rangle \neq 0$*. Let* $f(z, u, \mathbf{b})$ *be the Lyapunov-Schmidt reduction of* $F(\mathbf{x}, u, \mathbf{b})$ *at* $(0, u^*, 0)$*.*
***A.*** *Bifurcation problem* $f(z, u, 0) = 0$ *has a symmetric pitchfork singularity at* $(z, u, \mathbf{b}) = (0, u^*, 0)$*. For values of* $u > u^*$ *and sufficiently small* $|u - u^*|$*, two branches of equilibria branch off from* $\mathbf{x} = 0$ *in a pitchfork bifurcation along a manifold tangent at* $\mathbf{x} = 0$ *to* $\mathrm{span}\{\mathbf{v}\}$*. When* $\mathrm{sign}\{\langle \mathbf{w}, \mathbf{v}^3 \rangle / \langle \mathbf{w}, \mathbf{v} \rangle\}(\alpha + \lambda\gamma) > 0$ *(*$< 0$*) the bifurcation happens supercritically (subcritically) with respect to* $u$*.*
***B.*** *Bifurcation problem* $f(z, u, \mathbf{b}) = 0$ *is an* $N_a$*-parameter unfolding of the symmetric pitchfork, and* $\frac{\partial f}{\partial b_i}(z, u, \mathbf{b}) = w_i$*.*

*Proof.* The eigenvalues of $J_x$ (4) are $\mu = u\alpha - d + u\gamma\lambda$, and so, at $u = u^*$, $J_x$ has a single zero eigenvalue. Observe that the left and right null eigenvectors of $J_x$ are precisely $\mathbf{w}$ and $\mathbf{v}$. Following the procedure outlined in [10, Chapter I, 3.(e)] we derive $f(z, u, \mathbf{b})$. We derive the coefficients of the polynomial expansion of $f(z, u, \mathbf{b})$ [10, Chapter I, Equations 3.23(a)-(e)] through third order in the state variable. Note that $(d^2F)_{0,u^*,0}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0$ for any $\mathbf{v}_i$ because $S''(0) = 0$, which implies that $f_{zz} = 0$ by [10, Chapter I, Equation 3.23(b)]. Also, $f_z(0, u^*, 0) = 0$ by [10, Chapter I, Equation 3.23(a)]. The nonzero coefficients in the expansion are

$$f_{xxx} = \langle \mathbf{w}, (d^3F)_{0,u^*,0}(\mathbf{v}, \mathbf{v}, \mathbf{v}) \rangle = -2d(\alpha + \lambda\gamma)^2 \langle \mathbf{w}, \mathbf{v}^3 \rangle$$

$$f_{b_i} = \left\langle \mathbf{w}, \frac{\partial F}{\partial b_i}(0, u^*, 0) \right\rangle = w_i$$

$$f_{\hat{u}x} = \left\langle \mathbf{w}, \left( d\frac{\partial F}{\partial \hat{u}} \right)_{0,u^*,0} (\mathbf{v}, \mathbf{v}) \right\rangle = (\alpha + \lambda\gamma)\langle \mathbf{w}, \mathbf{v} \rangle$$

where $\hat{u} = u - u^*$ and $\langle \cdot, \cdot \rangle$ denotes the standard vector inner product. Also, observe that we can align the left and right eigenvectors to satisfy $\langle \mathbf{w}, \mathbf{v} \rangle = k_1 > 0$ (the inner product is nonzero by duality). Then $\langle \mathbf{w}, \mathbf{v}^3 \rangle := k_2 = \sum_{i=1}^{N_a} w_i v_i^3$. The Lyapunov-Scmidt reduction of (1) about $(0, u^*, 0)$ is thus

$$\dot{z} = k_1(\alpha + \lambda\gamma)\hat{u}z - 2k_2 d(\alpha + \lambda\gamma)^2 z^3 + \langle \mathbf{w}, \mathbf{b} \rangle + h.o.t. \tag{6}$$

Part A of the lemma follows by (6), by the recognition problem for the pitchfork bifurcation [10, Chapter II, Proposition 9.2], as well as by the definition of a center manifold. Part B follows by the definition of an unfolding and by (6). □

From Theorem IV.1 we can describe many of the bifurcations of $\mathbf{x} = 0$ of (5) from the spectrum of $A$. In particular, if $A$ has $N \leq N_a$ simple eigenvalues $\lambda_q$, we expect $\mathbf{x} = 0$ to exhibit $N$ distinct pitchfork bifurcations at critical values of the parameter $u_q^* = d/(\alpha + \lambda_q\gamma)$. Locally near the bifurcation point the left eigenvector $\mathbf{w}_q$ corresponding to $\lambda_q$ informs the sign structure of the emergent equilibria, as explored in [9]. For undirected graphs we can deduce the direction in which the bifurcation branches appear.

**Corollary IV.1.1.** *Suppose* $G$ *is an undirected graph. When* $u_q^* = d/(\alpha + \lambda_q\gamma) > 0 (< 0)$ *the pitchfork bifurcation at* $u_q^*$ *happens supercritically (subcritically).*

*Proof.* Let $\mathbf{v}_q$ and $\mathbf{w}_q$ be the right and left eigenvectors of $A$ corresponding to $\lambda_q$. For an undirected graph, $\mathbf{w}_q = \mathbf{v}_q$. Then $\langle \mathbf{w}_q, \mathbf{v}_q \rangle = \langle \mathbf{v}_q, \mathbf{v}_q \rangle > 0$ and $\langle \mathbf{w}_q, \mathbf{v}_q^3 \rangle = \langle \mathbf{v}_q, \mathbf{v}_i^3 \rangle = \sum_{k=1}^{N_a} (\mathbf{v}_q)_k^4 > 0$. The criticality condition from Theorem IV.1 becomes $(\alpha + \lambda_i\gamma) > 0 (< 0)$ for supercritical (subcritical) pitchfork bifurcation. Since $d > 0$ the result follows. □

Using these general results on the bifurcation behavior of the opinion dynamics, the next theorem establishes that the agreement and disagreement bifurcations in Proposition II.1 are supercritical pitchfork bifurcations in which $\mathbf{x} = 0$ loses stability and new branches of locally stable solutions appear.

**Theorem IV.2** (Agreement and Disagreement Pitchforks)**.** *Consider* (1) *and let* $u_i := u \geq 0$*. The agreement and disagreement bifurcations in Proposition II.1 are supercritical pitchfork bifurcations. Additionally, the two steady-state solutions, which appear for* $u > u_a(u_d)$*, are locally exponentially stable for* $|u - u_a|(|u - u_d|)$ *sufficiently small.*

*Proof.* Supercriticality of the bifurcating branches of equilibria follows for the undirected case from Corollary IV.1.1. For a directed graph and $\gamma > 0$ it follows from the Perron-Frobenius theorem that $\mathbf{v}_{max}$ and $\mathbf{w}_{max}$ have strictly positive components, i.e., $\langle \mathbf{w}_{max}, \mathbf{v}_{max} \rangle > 0$ and $\langle \mathbf{w}_{max}, \mathbf{v}_{max}^3 \rangle > 0$. Supercriticality then follows from Theorem IV.1. The two nontrivial fixed points are locally exponentially stable by analytic continuity of eigenvalues: $N_a - 1$ negative eigenvalues are shared with $\mathbf{x} = 0$ and the bifurcating eigenvalue is negative by [10, Chapter I, Theorem 4.1] because $\partial f/\partial z > 0$ for the Lyapunov-Schmidt reduction (6). □

The results presented in this section provide rigorous predictions of the influence of inputs on the opinion formation bifurcation behavior. We define the *node agreement (disagreement) centrality index for node* $i$ *to be* $w_i$, the $i$th

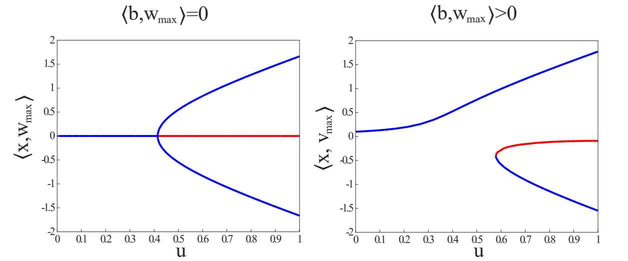


Fig. 1: Symmetric pitchfork bifurcation and its unfolding for opinion dynamics (1) in the agreement regime ($\gamma > 0$) with three agents communicating over an undirected line graph. Blue (red) curves correspond to stable (unstable) equilibria. Vertical axis is the projection of equilibria onto $W(\lambda_{max})$. $d = \alpha = \gamma = 1$. Left: $\mathbf{b} = (0.05, 0, -0.05)$; right: $\mathbf{b} = 0.1\mathbf{v}_{max} + (0.05, 0, -0.05)$. Bifurcation diagrams generated with help of MatCont [12].

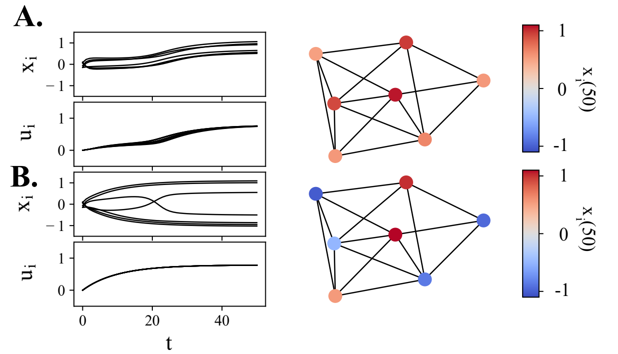


Fig. 2: A. Agreement cascade, $\gamma = 1$, $\underline{u} = u_a - 0.01$, $\bar{u} = u_a + 0.6$; B. Disagreement cascade, $\gamma = -1$, $\underline{u} = u_d - 0.01$, $\bar{u} = u_d + 0.6$. Left) Opinion and attention trajectories. Right) Graph with node $i$ color equal to $x_i(50)$. $d = 1$, $n = 3$, $y_{th} = 0.4$, $\tau_u = 10$, $\alpha = 1$, $d = 1$. For each $i$, $x_i(0) \in \mathcal{N}(0, 0.1)$, $u_i(0) = 0$, $b_i \in \mathcal{N}(0, 0.2)$.

component of $\mathbf{w}_{max}$ ($\mathbf{w}_{min}$) (see also [8]). It follows by Theorem IV.1B and Theorem IV.2 that the influence of an input $b_i$ to node $i$ on the network opinion formation behavior is exactly node $i$'s agreement or disagreement centrality $w_i$. This allows us to predict in which direction the agreement or disagreement pitchfork unfolds as a function of the locations and strengths of distributed inputs. If $\langle \mathbf{b}, \mathbf{w}_{max} \rangle = 0$ the pitchfork does not unfold. If $\langle \mathbf{b}, \mathbf{w}_{max} \rangle < 0$ ($\langle \mathbf{b}, \mathbf{w}_{max} \rangle > 0$) the pitchfork unfolds in a such a way that it exhibits a lower (upper) smooth branch of equilibria. For example, in Fig. 1 the diagram on the left receives a nonzero input which is orthogonal to $\mathbf{w}_{max}$, and the symmetry of the pitchfork bifurcation is unbroken. On the right, $\langle \mathbf{b}, \mathbf{w}_{max} \rangle = 0.1$ and near the singular point of the symmetric diagram, the unfolded diagram favors the positive solution branch which corresponds to agents agreeing on the positive option.

## V. Dynamic Attention: Cascades and Tunable Sensitivity to Input

In this section we show how distributed state feedback dynamics in the attention parameters of the opinion dynamics (1) give rise to agreement and disagreement cascades with tunable sensitivity to distributed input. We show that the magnitude of the distributed input vector, and its orientation relative to the centrality eigenvector $\mathbf{w}_{max}$ ($\mathbf{w}_{min}$) when $\gamma > 0$ ($< 0$) provide control parameters for triggering cascades over the network. A single design parameter in the attention feedback dynamics can be used to tune the threshold above which inputs trigger a cascade.

As in [6] we define state feedback dynamics for the attention parameter $u_i$ of each agent $i$ to track the saturated norm of the opinions observed by agent $i$:

$$\tau_u \dot{u}_i = -u_i + S_u \left( x_i^2 + \sum_{k=1}^{N_a} (a_{ik} x_k)^2 \right). \qquad (7)$$

$S_u$ takes the form of the Hill activation function:

$$S_u(y) = \underline{u} + (\bar{u} - \underline{u}) \frac{y^n}{(y_{th})^n + y^n}, \qquad (8)$$

where threshold $y_{th} > 0$. We constrain $\bar{u}$ and $\underline{u}$ such that $\bar{u} > u_c \geq \underline{u} > 0$, with $u_c = u_a$ ($u_d$) when $\gamma > 0$ ($< 0$). As in [8], we define an *opinion cascade* as a network transition from a weakly to a strongly opinionated state, where in a *weakly (strongly) opinionated* state, the agents' attention is close to its lower (upper) saturation bound, i.e. $u \cong \underline{u}$ ($u \cong \bar{u}$). See Fig. 2 for an example of an opinion cascade in an agreement ($\gamma > 0$) and disagreement ($\gamma < 0$) regime.

**Assumption.** *$G$ is undirected.*

In vector form, coupled dynamics (1),(7) become

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \end{pmatrix} = -\begin{pmatrix} d\mathbf{x} \\ \mathbf{u} \end{pmatrix} + \begin{pmatrix} \mathbf{u} \odot \mathbf{S}\left((\alpha\mathcal{I} + \gamma A)\mathbf{x}\right) + \mathbf{b} \\ \mathbf{S}_u((\mathcal{I} + A)\mathbf{x}^2) \end{pmatrix} \qquad (9)$$

where $\mathbf{S}_u(\mathbf{y}) = (S_u(y_1), \ldots, S_u(y_n))$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{x}^2 = (x_1^2, \ldots, x_{N_a}^2)$, and $\odot$ is the element-wise product of vectors. The Jacobian of (9) at equilibrium point $(\mathbf{x}_s, \mathbf{u}_s)$ is

$$J_{(\mathbf{x},\mathbf{u})} = \begin{pmatrix} -d\mathcal{I} + (\mathrm{diag}\{\mathbf{u}_s\}(\alpha\mathcal{I} + \gamma A)) \odot K_1 & K_2 \\ (\mathcal{I} + A)\,\mathrm{diag}\{\mathbf{x}_s\} \odot K_3 & -\mathcal{I} \end{pmatrix}, \qquad (10)$$

$K_1 = \mathbf{S}'((\alpha\mathcal{I} + \gamma A)\mathbf{x}_s)\,\mathbf{1}^T$, $K_2 = \mathrm{diag}\{\mathbf{S}((\alpha\mathcal{I} + \gamma A)\mathbf{x}_s)\}$, $K_3 = 2\mathbf{S}_u'\left((\mathcal{I} + A)\mathbf{x}_s^2\right)\mathbf{1}^T$, and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{N_a}$. Let $G(\mathbf{y}, \mathbf{b})$ be the right hand side of (9) with $\mathbf{y} = (\mathbf{x}, \mathbf{u})$.

**Lemma V.1** (Stability of $\mathbf{x} = 0$)**.** *Consider* (9) *with* $\mathbf{b} = 0$. *The point* $(\mathbf{x}_s, \mathbf{u}_s) = (\mathbf{0}, \underline{u}\mathbf{1})$ *is an equilibrium point of the coupled dynamics. When either* $\gamma > 0$ *and* $\underline{u} < u_a$ *or* $\gamma < 0$ *and* $\underline{u} < u_d$, *it is locally exponentially stable.*

*Proof.* Plugging the state values into the coupled dynamics (9) easily verifies that $(\dot{\mathbf{x}}, \dot{\mathbf{u}}) = 0$ at $(\mathbf{x}_s, \mathbf{u}_s)$. Evaluated at this point, (10) simplifies to the block diagonal matrix

$$J_{(0,\underline{u})} = \begin{pmatrix} -d\mathcal{I} + \underline{u}(\alpha\mathcal{I} + \gamma A) & 0 \\ 0 & -\mathcal{I} \end{pmatrix}. \qquad (11)$$

When $0 < \underline{u} < u_a$ ($u_d$), (11) has $2N_a$ eigenvalues with negative real part, and the stability conclusion follows. □

**Lemma V.2** (Small Input Approximates Equilibrium Opinion)**.** *Let* $(\mathbf{x}_s, \mathbf{u}_s)$ *be an equilibrium of* (9) *with inputs* $\mathbf{b}$. *Let* $\underline{u} < u_c$ *where* $u_c = u_a$ *if* $\gamma > 0$ *or* $u_c = u_d$ *if* $\gamma < 0$.

*Define* $\mathbf{w} = \mathbf{w}_{max}$ *if* $\gamma > 0$ *or* $\mathbf{w} = \mathbf{w}_{min}$ *if* $\gamma < 0$. *Then*

$$\frac{\partial \|\mathbf{x}_s\|}{\partial |\langle \mathbf{w}, \mathbf{b}\rangle|}\bigg|_{\mathbf{b}=\mathbf{x}_s=0} > 0. \quad (12)$$

*Proof.* Since $\mathbf{x} = 0$ is an equilibrium of the system with $\mathbf{b} = 0$, $(\mathbf{x}_s, \mathbf{u}_s)$ can be approximated by the linearization

$$J_{(0,\underline{u})}\begin{pmatrix}\mathbf{x}_s\\ \mathbf{u}_s\end{pmatrix} + \begin{pmatrix}\mathbf{b}\\ \mathbf{0}\end{pmatrix} = \mathbf{0}. \quad (13)$$

$J_{(0,\underline{u})}$ is symmetric and invertible so its inverse has the same eigenvectors. Thus, to the first order, it holds that

$$\mathbf{x}_s = -J_x^{-1}\mathbf{b} = -\sum_{q=1}^{N_a}\frac{1}{\mu_q}\langle \mathbf{w}_q, \mathbf{b}\rangle \mathbf{w}_q \quad (14)$$

where $J_x$ is (4) with $u = \underline{u}$, $\mathbf{w}_q$ is an eigenvector of $A$ corresponding to $\lambda_q$, and $\mu_q = d + \underline{u}(\alpha + \lambda_q\gamma)$. Eigenvectors are orthogonal, so $\|\mathbf{x}_s\| = \sum_{i=1}^{N_a}\frac{1}{\mu_i^2}\langle \mathbf{w}_i, \mathbf{b}\rangle^2$. ☐

**Theorem V.3** (Saddle-Node Bifurcation)**.** *Consider* (9) *with a nonzero input vector* $\mathbf{b}$ *and define* $u_c = u_a$ *if* $\gamma > 0$ *and* $u_c = u_d$ *if* $\gamma < 0$. *Let* $\mathbf{w}_c = \mathbf{w}_{max}$ *or* $\mathbf{w}_{min}$ *respectively. Suppose* $u_{th} \ll 1$ *and* $\underline{u} < u_c$ *with* $|\underline{u} - u_c|$ *sufficiently small. There exists* $p > 0$ *such that when* $|\langle \mathbf{w}_c, \mathbf{b}\rangle| = p$ *there exists an equilibrium* $(\mathbf{x}_p, \mathbf{u}_p)$ *of* (9) *such that, if*

$$\langle \mathbf{w}_c, \mathbf{u}_p \odot \mathbf{v}_c^2 \odot \mathbf{S}''((\alpha\mathcal{I} + \gamma A)\mathbf{x}_p)\rangle > 0 \quad (15)$$

$$k_q\langle \mathbf{w}_c, \mathbf{u}_p \odot \mathbf{v}_c \odot \mathbf{v}_q \odot \mathbf{S}''((\alpha\mathcal{I} + \gamma A)\mathbf{x}_p)\rangle < 0 \quad (16)$$

*is verified for all* $q$ *at* $(\mathbf{x}_p, \mathbf{u}_p)$ *with* $\lambda_q \neq \lambda_c$ *an eigenvalue of* $A$ *with corresponding left (right) eigenvector* $\mathbf{w}_q$ $(\mathbf{v}_q)$*, where* $k_q = (\alpha + \gamma\lambda_q)/(d + \underline{u}(\alpha + \gamma\lambda_q))$*: (i) There exists a smooth curve of equilibria in* $\mathbb{R}^{2N_a} \times \mathbb{R}$ *passing through* $(\mathbf{x}_p, \mathbf{u}_p, p)$*, tangent to the hyperplane* $\mathbb{R}^{2N_a} \times \{p\}$*; (ii) There are no equilibria near* $(\mathbf{x}_p, \mathbf{u}_p, p)$ *when* $|\langle \mathbf{w}_c, \mathbf{b}\rangle| > p$ *and two equilibria near* $(\mathbf{x}_p, \mathbf{u}_p, p)$ *for each* $|\langle \mathbf{w}_c, \mathbf{b}\rangle| < p$*; (iii) The two equilibria near* $(\mathbf{x}_p, \mathbf{u}_p, p)$ *are hyperbolic and have stable manifolds of dimensions* $N_a$ *and* $N_a - 1$ *respectively.*

*Proof.* (10) depends continuously on the model parameters and on the state. So, by [13, Chapter II, Theorem 5.1] the eigenvalues and eigenvectors of (10) change continuously for $\|\mathbf{x}_s\|$ sufficiently small. Leaving the full development of the matrix perturbation theory for future work, we conjecture that if $\|\mathbf{x}_s\|$ is sufficiently small then the eigenvectors of (11) are a good approximation of the eigenvectors of (10). Since the origin of (9) with $\mathbf{b} = 0$ is stable by Lemma V.1 and because $\lambda_{min}$ and $\lambda_{max}$ are simple eigenvalues, if an eigenvalue of $J_{(\mathbf{x}_s,\mathbf{u}_s)}$ crosses zero for some $\|\mathbf{b}\|$ it must also be simple. This eigenvalue corresponds to a perturbation of $-d + \underline{u}(\alpha + \gamma\lambda_c)$ where $\lambda_c = \lambda_{max}$ or $\lambda_{min}$ respectively.

By Lemma V.2, if $\mathbf{b} \neq 0$ then at equilibrium $\|\mathbf{x}\| \neq 0$. Define $\tilde{\mathbf{v}}_c = (\mathbf{v}_c, \mathbf{0})$ and $\tilde{\mathbf{w}}_c = (\mathbf{w}_c, \mathbf{0})$. Let $g(z, \mathbf{b})$ be the Lyapunov-Schmidt reduction of (9) at an equilibrium $(\mathbf{x}, \mathbf{u})$ for sufficiently small inputs. We have

$$d^2G_{\mathbf{y}_p,\mathbf{b}_p}(\tilde{\mathbf{v}}_c, \tilde{\mathbf{v}}_c) = \sum_{j=1}^{N_a}\sum_{k=1}^{N_a}\frac{\partial^2(G)_i}{\partial x_j \partial x_k}(\mathbf{v}_c)_j(\mathbf{v}_c)_k\bigg|_{(\mathbf{y}_p,\mathbf{b}_p)} =$$

$$(\alpha + \lambda_c\gamma)^2\begin{pmatrix}\mathbf{u}_p\\ \mathbf{0}\end{pmatrix} \odot \begin{pmatrix}\mathbf{v}_c^2\\ \mathbf{0}\end{pmatrix} \odot \begin{pmatrix}\mathbf{S}''((\alpha\mathcal{I} + \gamma A)\mathbf{x}_p)\\ \mathbf{0}\end{pmatrix}. \quad (17)$$

The second derivative in the Lyapunov-Schmidt reduction is

$$g_{zz} = \langle \tilde{\mathbf{w}}_c, d^2G_{\mathbf{y}_p,\mathbf{b}_p}(\tilde{\mathbf{v}}_c, \tilde{\mathbf{v}}_c)\rangle = (\alpha + \lambda_c\gamma)^2 \times \sum_{i=1}^{N_a}(\mathbf{u}_p)_i(\mathbf{w}_c)_i^3 S''\left(\alpha(\mathbf{x}_p)_i + \gamma\sum_{\substack{k=1\\ k\neq i}}^{N_a} a_{ik}(\mathbf{x}_p)_k\right) > 0$$

by assumption (15). Additionally, the term $\langle \mathbf{w}_c, \mathbf{b}\rangle$ appears in $g(z, p)$ since $g_{b_i} = \left\langle \tilde{\mathbf{w}}_c, \frac{\partial G}{\partial b_i}\right\rangle = (\tilde{\mathbf{w}}_c)_i$.

Finally, we compute the coefficient of the cross-term $g_{z\hat{b}}$ in the Lyapunov-Schmidt reduction. For convenience, we express $\mathbf{b} = \sum_{q=1}^{N_a}\beta_q \mathbf{w}_q$ where each $\beta_q := \langle \mathbf{w}_q, \mathbf{b}\rangle$. Coefficients of the cross-terms $z\beta_q$ in $g(z, \mathbf{b})$ simplify to

$$g_{z\beta_q} = \left\langle \tilde{\mathbf{w}}_c, -d^2G_{\mathbf{y}_p,\mathbf{b}_p}\left(\tilde{\mathbf{v}}_c, J_{(0,u)}^{-1}E\left(\frac{\partial G}{\partial \beta_q}\right)\right)\right\rangle. \quad (18)$$

$E$ is a projection onto the range of $J_{(0,\underline{u})}$ and $J_{(0,\underline{u})}^{-1} : \mathbf{v}_c^{\perp} \mapsto \mathbb{R}^{N_a}$ is the inverse of to restriction of $J_{(0,\underline{u})}$ to the orthogonal complement to $\mathbf{v}_c$. We find that $J_{(0,u)}^{-1}E\left(\frac{\partial G}{\partial \beta_q}\right) = \frac{1}{\mu_q}(\mathbf{v}_q, \mathbf{0})$ and $g_{z\beta_q} = -(\alpha + \lambda_c\gamma)K_q$ where each $K_q$ is the quantity in (16). Since $g_{z\beta_q} > 0$ for all $q$, we conclude that the eigenvalue of the equilibrium is monotonically increasing with $| < \mathbf{w}_c, \mathbf{b} > |$. By continuity of eigenvalues of the perturbed Jacobian, it follows that the leading eigenvalue necessarily crosses zero as input is increased. By [14, Theorem 3.4.1] this singularity must be a saddle-node bifurcation point, with bifurcation parameter $\hat{b} = \langle \mathbf{w}_c, \mathbf{b}\rangle$ and properties outlined in the statement of the theorem. ☐

**Corollary V.3.1.** *The input magnitude* $\|\mathbf{b}\|$ *and its relative orientation* $\mathbf{b}\angle\mathbf{w}_c := \langle \mathbf{w}_c, \mathbf{b}\rangle/\|\mathbf{b}\|$ *can be used as controls to trigger a network opinion cascade.*

*Proof.* This follows by factoring out the magnitude of the input vector from the bifurcation parameter $\langle \mathbf{v}_c, \mathbf{b}\rangle$. ☐

Figure 3 illustrates the prediction of Corollary V.3.1, showing bifurcation diagrams with stable and unstable equilibria of the opinion dynamics in the agreement regime on a small network. The two diagrams illustrate the saddle-node bifurcation predicted by Theorem V.3 with $\|\mathbf{b}\|$ and $\mathbf{b}\angle\mathbf{w}$ as bifurcation parameters. Opinion cascades are activated when the bifurcation parameter passes the critical value. Although the predictions of the results in this section assume inputs are small, in simulation and through numerical continuation of the dynamics on different networks we observe that this result is quite robust. The existence of a saddle-node bifurcation, and therefore a threshold which differentiates between inputs which trigger a cascade and ones which do not, persists across network structures and for large inputs.

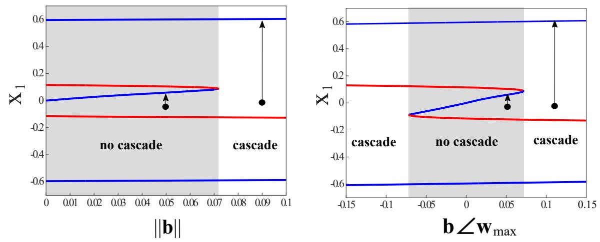


Fig. 3: Blue (red) lines track the first coordinate of the stable (unstable) equilibrium solutions of the coupled dynamics (9) on a 3-agent undirected line graph. Parameters: $u_{th} = 0.1$, $\gamma = 1$, $n = 3$, $\mathbf{b} = \|\mathbf{b}\| \cdot |\mathbf{b}\angle \mathbf{w}_{max}| \cdot \mathbf{w}_{max}$; left: $|\mathbf{b}\angle \mathbf{w}_{max}| = 0.1$, right: $\|\mathbf{b}\| = 0.1$ Bifurcation diagrams generated using MatCont [12].

A consequence of Theorem V.3 is that also for opinion cascades the node centrality indices are the key determinant of the effect of inputs on the coupled attention-opinion dynamics (9): The smaller the angle between the input vector and the agreement or disagreement centrality vector, the smaller the needed input strength to trigger an agreement or a disagreement opinion cascade. Figure 4 numerically illustrates our theoretical prediction. The transition line from the red (no cascade) to the white (cascade) regions correspond to the threshold, i.e., the saddle node bifurcation predicted by Theorem V.3, at which the opinion cascade is ignited. It shows, for different network topologies and agreement and disagreement opinion cascades, that as the angle between the input vector and the centrality vector decreases, the norm of the input needed to trigger a cascade gets smaller. In the cascade region of the simulations in Figure 4, the centrality eigenvector accurately predicts the sign distribution among the nodes. Rigorously proving this is subject of future work.

The cascade threshold is implicitly defined by the design parameter $y_{th}$ in the attention saturation function (8). In future work we will explore how the sensitivity of the group to distributed input can be tuned with this parameter.

## VI. Final Remarks

We have derived and proved a systematic method for designing distributed inputs to control opinion formation and opinion cascades for both agreement and disagreement among distributed agents that communicate over a network. Future directions include expanding the analysis presented here to multi-option cascades using the general formulation of nonlinear multi-option opinion dynamics of [6].

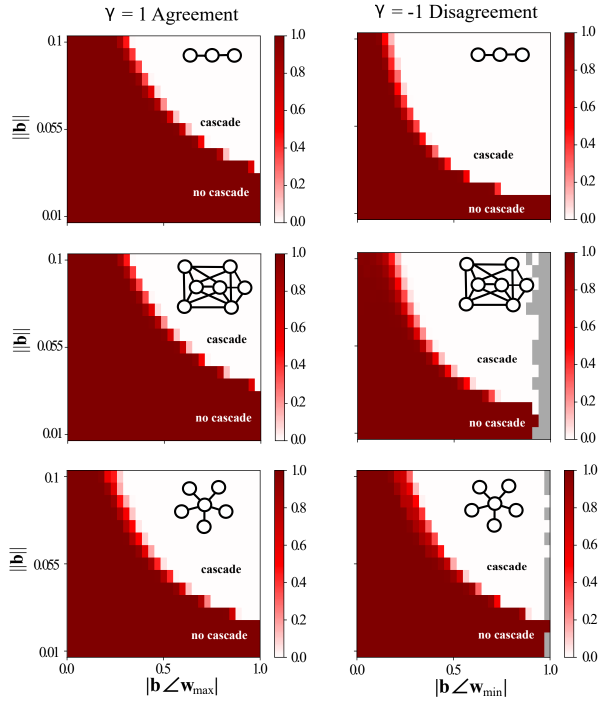


Fig. 4: Heatmaps with color corresponding to proportion of simulations in the given parameter range that did not result in a network cascade by $t = 500$. Dark red corresponds to no cascades, white to there always being a cascade. Grey squares are bins with no datapoints. Each plot corresponds to $1.5 \times 10^5$ distinct simulations on an undirected graph shown in the diagram. Simulation parameters: $\tau_u = 10$, $u_{th} = 0.2$, $\underline{u} = u_a - 0.01$ for $\gamma = 1$ (left plots) and $\underline{u} = u_d - 0.01$ for $\gamma = -1$ (right plots). For each simulation, inputs $b_i$ were drawn from $\mathcal{N}(0, 1)$ and the input vector $\mathbf{b}$ was normalized to a desired magnitude. There were 10000 simulations performed at each constant input magnitude, with 15 magnitudes sampled uniformly spaced between 0 and 0.1. The initial conditions for each simulation were $x_i = 0$, $u_i = 0$ for all $i = 1, \ldots, N_a$.

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